Proving completeness by orthogonal decomposition.

This section contains a small excursion into some elementary abstract data type theory, needed in the sequel, where we will prove that a certain axiom system (Σ, E) completely axiomatizes a certain algebra $\mathfrak{A}$. By this we understand that $\mathfrak{A}$ is a minimal Σ-algebra (i.e. every element of $\mathfrak{A}$ is denoted by a ground Σ-term), and

$$(\Sigma, E) \vdash t = s \quad \Leftrightarrow \mathfrak{A} \models t = s.$$

Rephrased, we can also say: $I(\Sigma, E) \cong \mathfrak{A}$. Here $I(\Sigma, E)$ is the closed term model of (Σ, E), and $\cong$ denotes isomorphism.

In this section we explain the method used for proving the completeness of (Σ, E) for $\mathfrak{A}$. It was used in [BKO 86] for proving completeness of some axiomatization of finite acyclic processes with synchronous communication under failure semantics. Here we will explain the method in more detail, and also prove it correct.

The idea is to decompose the task of proving "(Σ, E) completely axiomatizes $\mathfrak{A}$" into some simpler completeness proofs, by taking in some sense 'orthogonal' projections of $\mathfrak{A}$ as well as corresponding projections of (Σ, E). (See Figure 1.)

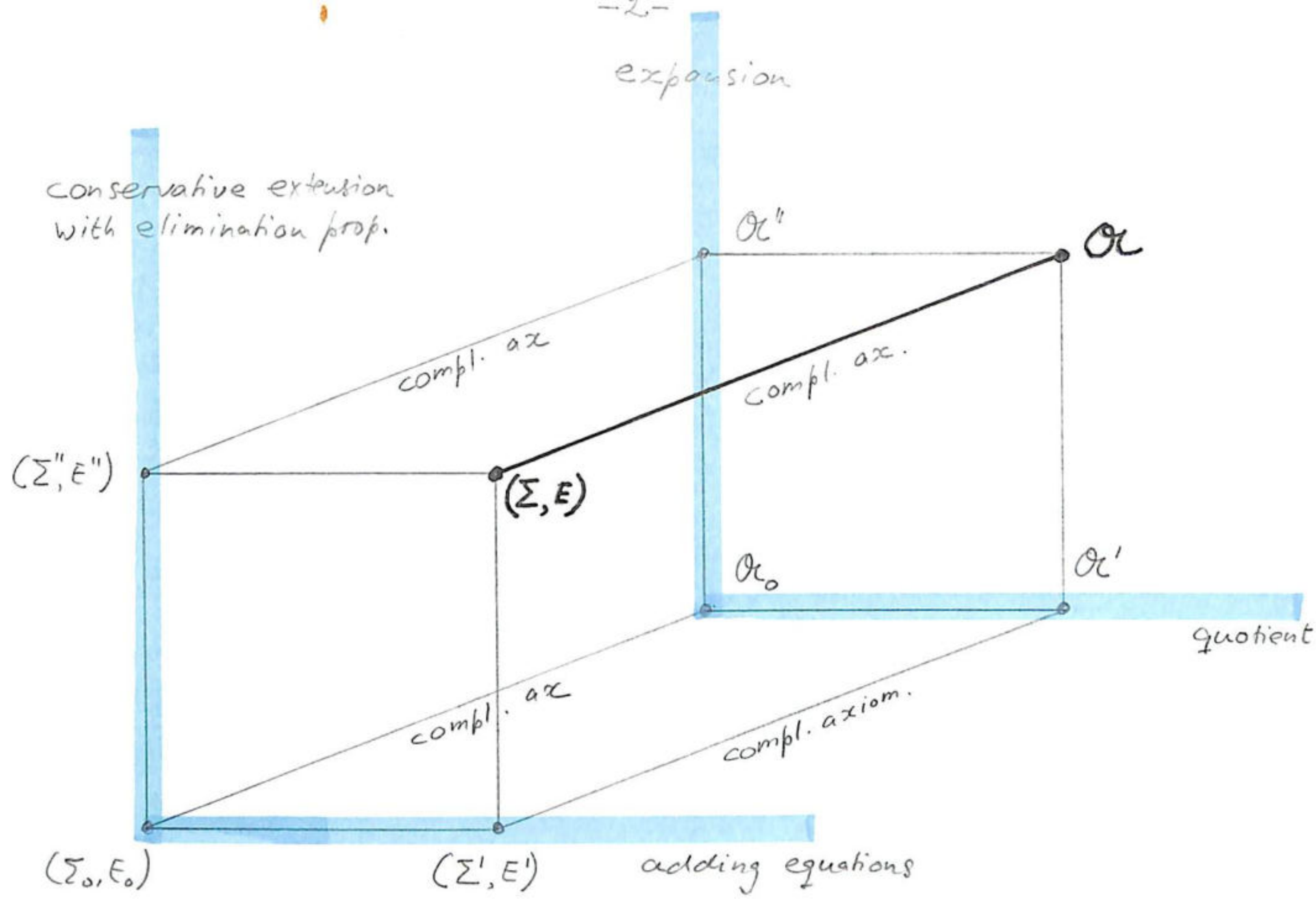


Figure 1.

As it will turn out, the completeness proof $(\Sigma, E) \longrightarrow \mathfrak{A}$ will be reduced in this way to the three easier completeness proofs $(\Sigma_0, E_0) \rightarrow \mathfrak{A}_0$, $(\Sigma'', E'') \rightarrow \mathfrak{A}''$, $(\Sigma', E') \rightarrow \mathfrak{A}'$, plus a proof that (Σ'', E'') is a conservative extension of (Σ_0, E_0) with the elimination property, plus another verification that will be discussed later.

We will now describe the method in detail. Let $\mathfrak{A} = \langle A, F_i \ (i \in I) \rangle$ be an algebra (a model), and $\mathfrak{A}_q = \mathfrak{A}/_\equiv$ a <u>quotient algebra</u>. So $\equiv$ is a congruence (w.r.t. the F_i) on A. We will write

$$\mathfrak{A}/\equiv \; = \langle A/\equiv \,, \; F_i/\equiv \ (i \in I) \rangle$$

with obvious definitions of $A/\equiv$, $F_i/\equiv$. Note that we do <u>not</u> work modulo isomorphism : $A/\equiv$ is the set of $\equiv$-congruence classes of A.

On the other hand, let $\mathfrak{A}_e$ be an <u>expansion</u> of $\mathfrak{A}$, in the terminology of Model Theory; so:

$$\mathfrak{A}_e = \langle A, F_i \, (i \in I), G_j \, (j \in J) \rangle .$$

We will be interested in such triples $(\mathfrak{A}_e, \mathfrak{A}, \mathfrak{A}_q)$.

<u>DEFINITION</u>. Let $(\mathfrak{A}_e, \mathfrak{A}, \mathfrak{A}_q)$ be a triple of algebra's. Then it is an <u>orthogonal triple</u> of algebra's if $\equiv$ (where $\mathfrak{A}_q = \mathfrak{A}/\equiv$) is also a congruence for the new operators $G_j \, (j \in J)$ in $\mathfrak{A}_e$. We denote the orthogonality as in Figure 2.

$$\begin{tikzcd}
\mathfrak{A}_e & \\
\mathfrak{A} \arrow[u, "\text{expansion}"] \arrow[r, "\text{quotient by } \equiv"'] & \mathfrak{A}_q \ (= \mathfrak{A}/\equiv)
\end{tikzcd}$$

<u>Figure 2.</u>

Now it is easy to see that an orthogonal triple $(\mathfrak{A}_e, \mathfrak{A}, \mathfrak{A}_q)$ uniquely determines an algebra $\mathfrak{A}_{eq}$ obtained by expanding $\mathfrak{A}_q$; or what amounts to the same, by quotienting $\mathfrak{A}_e$.

$$\begin{tikzcd}
\mathfrak{A}_e \arrow[r, "\text{quot}"] & \mathfrak{A}_{eq} \\
\mathfrak{A} \arrow[u, "\text{exp}"] \arrow[r, "\text{quot}"'] & \mathfrak{A}_q \arrow[u, "\text{exp}"']
\end{tikzcd}$$

<u>Figure 3.</u>

The result of this "push-out" (can it be described in category theory phraseology?) is

$$\mathfrak{A}_{eq} = \langle A/\equiv,\ F_i/\equiv\ (i \in I),\ G_j/\equiv\ (j \in J) \rangle$$

where $(F_i/\equiv)([a_1]_\equiv, \dots, [a_n]_\equiv) = [F_i(a_1, \dots, a_n)]_\equiv$, etc.

On the level of the syntax (theories, axiom systems, specifications (Σ, E)) we have an analogous decomposition.

$$\begin{tikzcd}
(\Sigma \cup \Delta, E \cup D) \arrow[r] & (\Sigma \cup \Delta, E \cup D \cup F) \\
(\Sigma, E) \arrow[u, "\begin{array}{l}\text{Conservative}\\ \text{extension}\\ \text{with}\\ \text{elim. prop.}\end{array}"] \arrow[r, "\begin{array}{l}\text{compatibility of } F\\ \text{w.r.t. the vertical extension}\end{array}"] & (\Sigma, E \cup F) \arrow[u]
\end{tikzcd}$$

We call $((\Sigma \cup \Delta, E \cup D), (\Sigma, E), (\Sigma, E \cup F))$ where the first is a conservative extension with elim. prop. of the second, a <u>triple of theories</u>.

<u>DEFINITION</u>. The triple of theories as above is an <u>orthogonal</u> triple of theories if the horizontal extension of (Σ, E) is compatible with the vertical one, in the following sense (here $\vec{t}$ is $t_1, \dots, t_n$, etc.):

if $(\Sigma, E \cup F) \vdash \vec{t} = \vec{t'}$,
then $(\Sigma \cup \Delta, E \cup D \cup F) \vdash \underline{G}_j(\vec{t}) = \underline{G}_j(\vec{t'})$.

Here $\underset{\vee}{G}_j$ ($j \in J$) are the 'new' operator symbols in the signature extension $\Sigma \cup \Delta$.

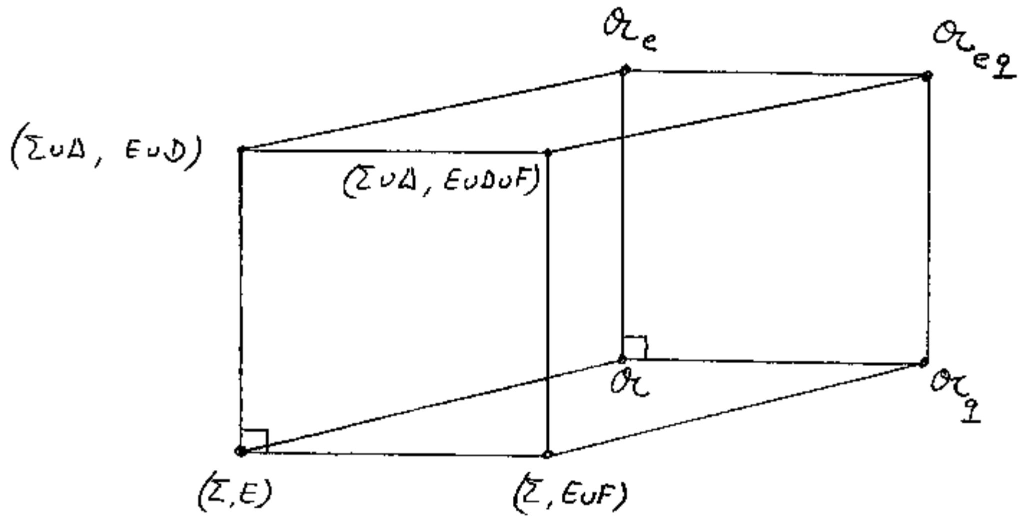
We can now state the

DECOMPOSITION LEMMA.

Let $(\mathfrak{A}_e, \mathfrak{A}, \mathfrak{A}_q)$ be a triple of algebra's, and let $((\Sigma\cup\Delta, E\cup D), (\Sigma, E), (\Sigma, E\cup F))$ be a corresponding *) triple of theories.

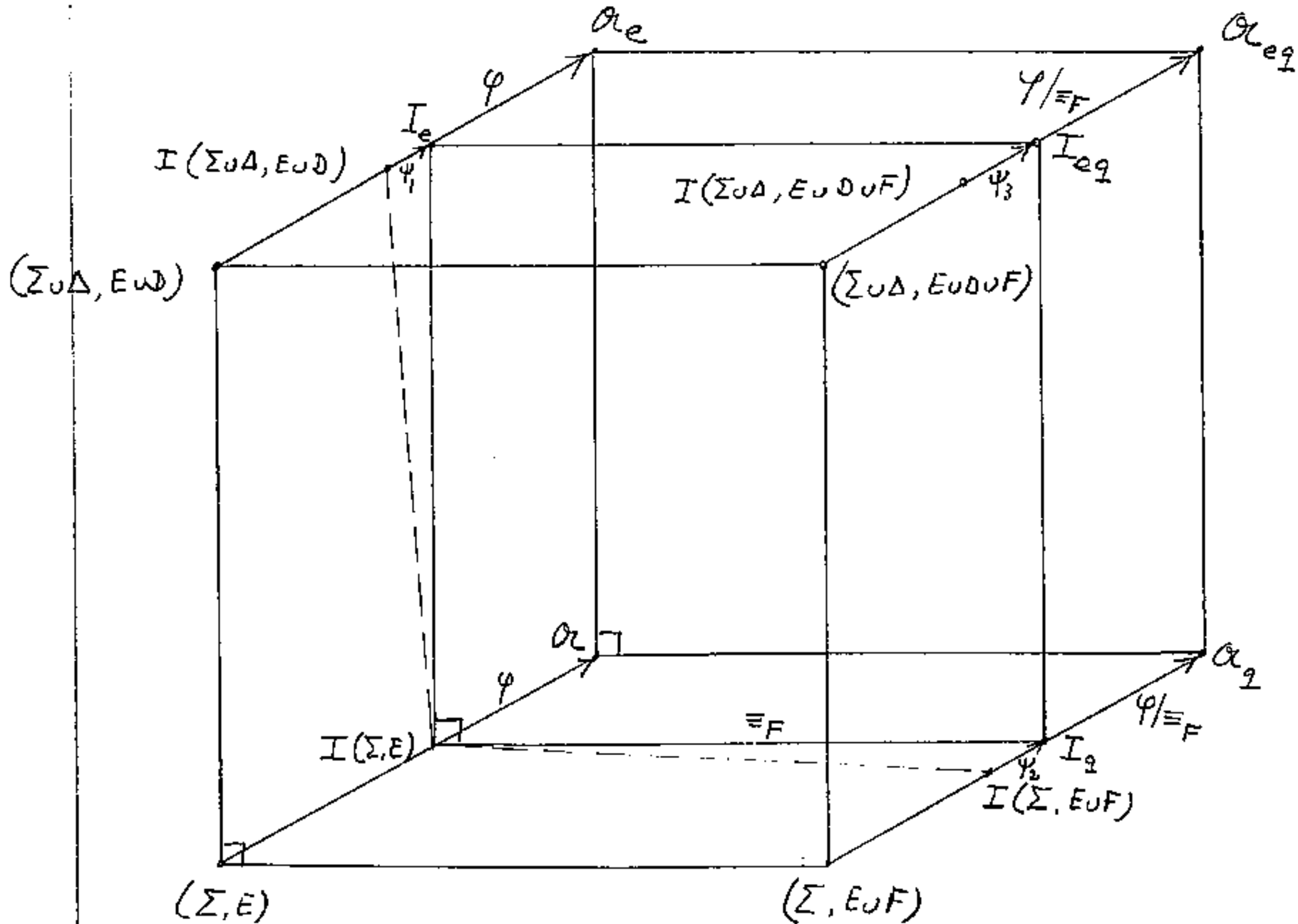
Then if either triple is orthogonal, the other one is too; and in that case $\mathfrak{A}_{eq}$ is defined and:

$(\Sigma\cup\Delta, E\cup D\cup F)$ completely axiomatizes $\mathfrak{A}_{eq}$.



*) 'Corresponding': i.e. $(\Sigma\cup\Delta, E\cup D)$ completely axiomatizes $\mathfrak{A}_e$, (Σ, E) compl. axiom. $\mathfrak{A}$, and $(\Sigma, E\cup F)$ compl. axiom. $\mathfrak{A}_q$.
(As a hidden assumption, we suppose that the interpretations involved, are "the same"!)

PROOF. Consider the figure



Let the triple of theories $((\Sigma\cup\Delta, E\cup D), (\Sigma, E), (\Sigma, E\cup F))$ be given. Consider the corresponding three algebra's, the closed term models

$$I(\Sigma\cup\Delta, E\cup D), \quad I(\Sigma, E), \; I(\Sigma, E\cup F).$$

This is in our definition of 'triple', not yet a triple of algebra's, because strictly speaking $I(\Sigma\cup\Delta, E\cup D)$ is not an expansion of $I(\Sigma, E)$, since the universe of the first consists of other elements than that of the second; and likewise $I(\Sigma, E\cup F)$ is not a quotient algebra of $I(\Sigma, E)$ in our strict sense.

But it is easy to take slight variations I_e, I_q of $I(\Sigma \cup \Delta, E \cup D)$, $I(\Sigma, E \cup F)$ respectively, so that $(I_e, I(\Sigma, E), I_q)$ <u>is</u> a triple of algebra's.

Indeed, let's define the desired isomorphisms:

$\psi_1 : I(\Sigma \cup \Delta, E \cup D) \longrightarrow I_e$

$\psi_2 : I(\Sigma, E \cup F) \longrightarrow I_q$.

An element in $I(\Sigma \cup \Delta, E \cup D)$ has the form $[t]_{=_{E \cup D}}$ where $t \in Ter_0(\Sigma \cup \Delta)$ ('0' for ground terms), and $=_{E \cup D}$ is provable equality using $E \cup D$, so

$$[t]_{=_{E \cup D}} = \{ t' \in Ter_0(\Sigma \cup \Delta) \mid E \cup D \vdash t = t' \}.$$

Because of the elimination property, we can take t in fact $\in Ter_0(\Sigma)$. Doing this, we define:

$$\psi_1([t]_{=_{E \cup D}}) = [t]_{=_{E \cup D}} \restriction \Sigma$$

where the restriction $\restriction \Sigma$ removes all terms not in $Ter_0(\Sigma)$.

In fact, by the conservativity of the extension $(\Sigma \cup \Delta, E \cup D)$ over (Σ, E), we have that

$$[t]_{=_{E \cup D}} \restriction \Sigma = [t]_{=_E}.$$

As for ψ_2, this isomorphism simply partitions $=_{E \cup F}$ - classes (the elements of $I(\Sigma, E \cup F)$) into $=_E$ - classes:

$$\psi_2([t]_{E \cup F}) = \{ [t']_E \mid t' =_{E \cup F} t \}.$$

The congruence $\equiv_F$, leading from $I(\Sigma, E)$ to $I_q = I(\Sigma, E)/\equiv_F$, is given by:

$$[t]_E \equiv_F [t']_E \qquad \text{if} \quad t =_{E\cup F} t'.$$

We now have isomorphisms

$$\varphi : I_e \longrightarrow \mathfrak{A}_e$$
$$\varphi : I(\Sigma, E) \longrightarrow \mathfrak{A}$$
$$\varphi/\equiv_F : I_q \longrightarrow \mathfrak{A}_q .$$

Orthogonality of $(I_e, I(\Sigma,E), I_q)$ is equivalent to that of $(\mathfrak{A}_e, \mathfrak{A}, \mathfrak{A}_q)$.
Orthogonality of $(I_e, I(\Sigma,E), I_q)$ is equivalent to that of $((\Sigma\cup\Delta, E\cup D), (\Sigma, E), (\Sigma, E\cup F))$, which is the compatibility requirement stated above.

Assuming this orthogonality property, we finally have the existence of I_{eq} and $\mathfrak{A}_{eq}$.
It is obvious that there is the isomorphism

$$\varphi/\equiv_F : I_{eq} \longrightarrow \mathfrak{A}_{eq} .$$

and a little exercise shows that there is an isomorphism

$$\psi_3 : I(\Sigma\cup\Delta, E\cup D\cup F) \longrightarrow I_{eq} .$$

Hence there is the isomorphism

$$(\varphi/\equiv_F)\circ\psi_3 : I(\Sigma\cup\Delta, E\cup D\cup F) \longrightarrow \mathfrak{A}_{eq},$$

showing that $(\Sigma\cup\Delta, E\cup D\cup F)$ is indeed a complete axiomatization of $\mathfrak{A}_{eq}$. $\square$