Chapter 4
Tree Ordinals

(in preparation)

4.0. Introduction. Tree ordinals constitute an economic way to work with ordinals and their arithmetic. The economy is that we do not work with the actual ordinals, but with representations of them that are a sort of 'thinned out' versions. These are known in Proof Theory as *fundamental sequences*, which are ω-long sequences (or streams) having as elements natural numbers (finite ordinals) or again fundamental sequences (standing for infinite ordinals). The nesting of such fundamental sequences is only finitely deep, leading to a countably branching, well-founded tree.

To represent a countable branching is not a priori possible in first order term rewriting, at least not in the usual version; but we can just as well work with infinite sequences which can be obtained by iterated pairing, thus staying in the finitely branching framework of first order (possibly infinitely deep) terms.

Before getting technical, let us describe the aim of this mountain walk. We indeed want to climb a mountain, one that is called ε_0, and an even more awesome one called Γ_0, an ordinal that plays an important role in the theory of termination of term rewriting. (See Gallier [xx], Dershowitz [xx].) The aim is to describe these big countable ordinals by means of first order infinitary term rewriting. (Infinitary lambda calculus would do just as wel, but we do not need that). Our mountain walk will give us a good impression of these large countable ordinals, but we will also encounter several key notions in infinitary term rewriting. So, let's get technical.

4.1. The alphabet will be 0, unary S (successor), and binary pairing P(,). We also write P as infix ":" . Apart from the finite terms, we can now make many infinite terms. One is S(S(S…., with as S^ω, encountered in Chapter 1. For some reason this is not a term that we want, though. We will later return our steps and return to this issue. Other infinite terms are the stream of *zeros* 0:0:0:0:..., the Thue-Morse stream 1:0:0:1:0:1:1:0:... However, interesting as especially the last one is, it is not one of our intended tree ordinal terms.

4.2. We also have the *numerals* or finite ordinals (natural numbers) 0, S(0),…, $S^n(0)$,... Note that the subterm relation coincides with the usual partial order < on natural numbers.

4.3. Now we define the set of terms $\mathcal{TO}$, tree ordinals.

(i) $0 \in \mathcal{TO}$

(ii) $t \in \mathcal{TO} \Rightarrow S(t) \in \mathcal{TO}$

(iii) $t_0 \subseteq t_1 \subseteq t_2 \subseteq \ldots \subseteq t_n \subseteq \ldots$ is a chain of terms in $\mathcal{TO}$, then the infinite term $t_0 : t_1 : t_2 : \ldots \in \mathcal{TO}$.

In a figure, using the P-notation, this term is the tree

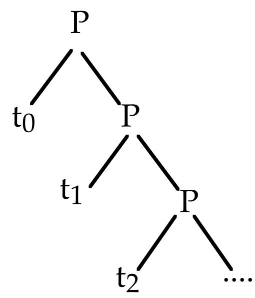

Figure 4.1

We will designate the special stream of natural numbers $0 : 1 : 2 : \ldots : n : \ldots$ where n is $S^n(0)$ by ω. Note that $\omega \in \mathcal{TO}$, but $S^\omega \notin \mathcal{TO}$.

4.4. The set of terms $\mathcal{TO}$ is already far-reaching in its notational strength, they designate all countable ordinals, in the following sense:

Let Ω be the set of countable ordinals, or in other words, Ω is the first uncountable ordinal.

4.4.1. DEFINITION. We define $[\![\ ]\!] : \mathcal{TO} \to \Omega$ by:

(i) $[\![0]\!] = 0$

(ii) $[\![St]\!] = [\![t]\!] + 1$

(iii) $[\![t_0{:}t_1{:}t_2{:}\ldots]\!] = \lim_{n\to\infty} [\![t_n]\!]$

Now we have by transfinite induction that the range of $[\![\ ]\!]$ is all of Ω. Note that $[\![\omega]\!] = \omega$, where we take the confusion (overloading) between the constant ω and the ordinal ω for granted.

4.4.2. EXAMPLE.

1. $[\![S(S(S(\omega)))]\!] = \omega + 3$.
2. ω^2 is designated by $\omega : S(\omega): S^2(\omega) : S^3(\omega) : \dots : S^n(\omega) : \dots$
3. ω^3 is designated for example by $\omega : \omega^2 : S(\omega^2): S^2(\omega^2): \dots$
4. $\omega^{\omega^2} = \omega^\omega : \omega^{\omega+1} : \omega^{\omega+2} : \omega^{\omega+3} : \dots$
5. $\omega^{\omega+1} = \omega^\omega : (\omega^\omega + 1) : (\omega^\omega + 2) : \dots$

At this point it becomes convenient to resort to a two-dimentional notation using trees that are infinitely branching–but note that these are, strictly speaking, not infinitary first-order terms!

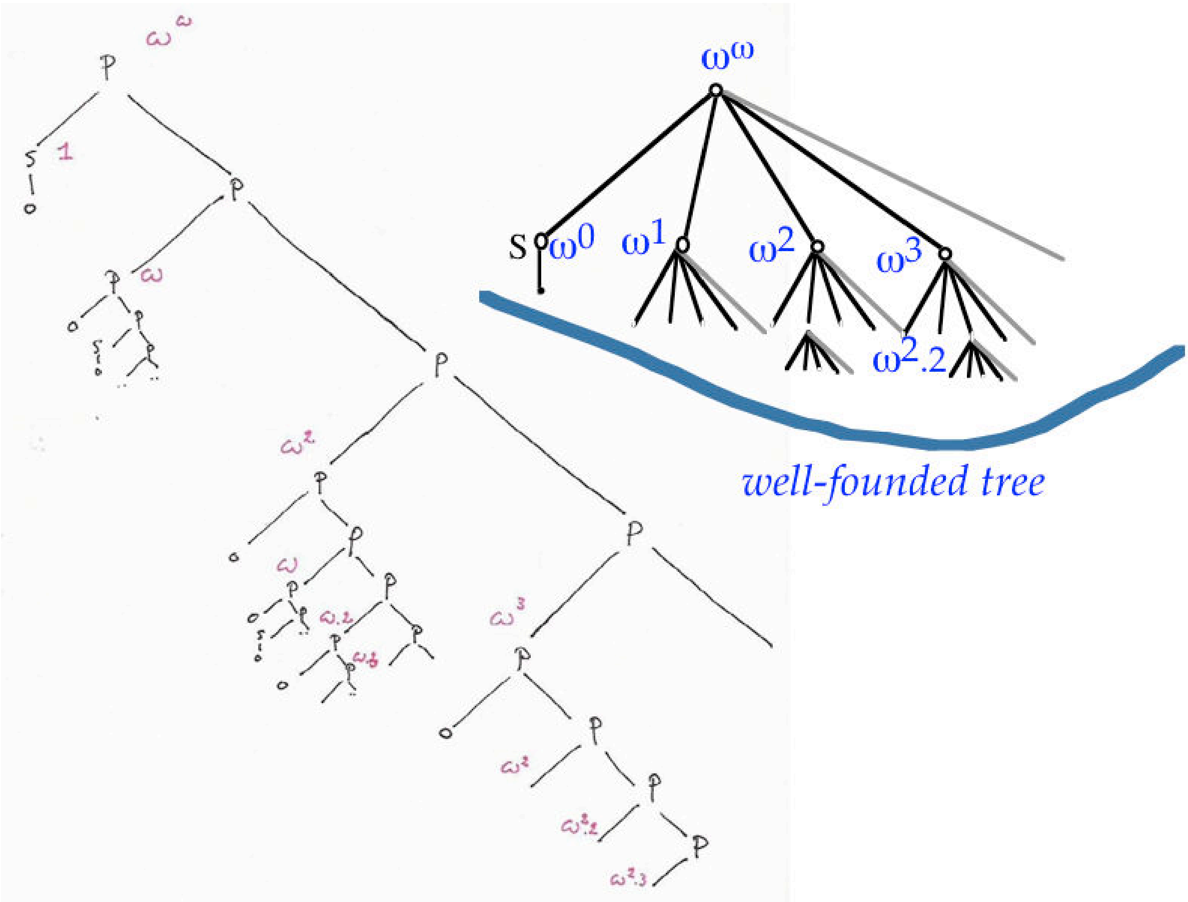

Figure 4.2

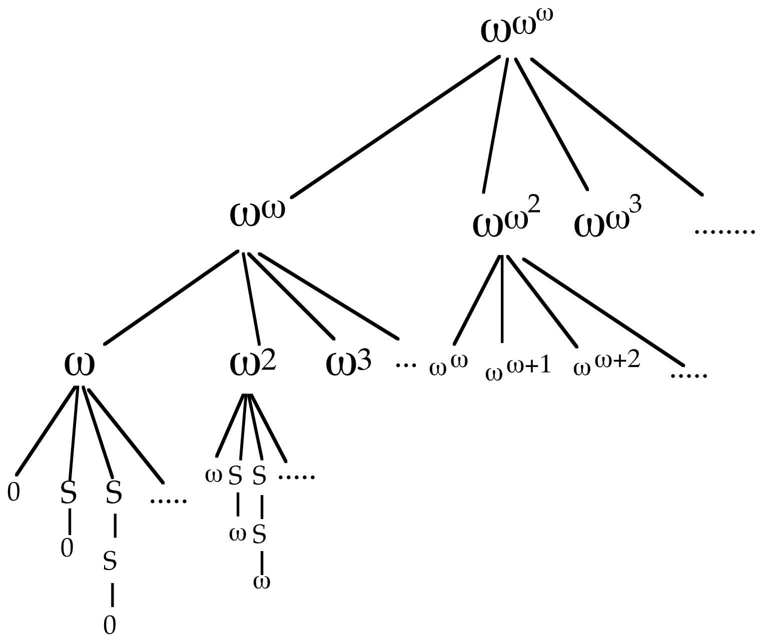

Figure 4.3

4.5. Here we encounter an interesting point to be elaborated as an exerxise. The trees as above contain enormously many repetitions, i.e. identical subtrees. In term graph rewriting this situation of identical subtrees is an explicit concern and it is the starting point for an operation to render such trees more economic: this is the operation called *collapsing* (of identical subtrees).

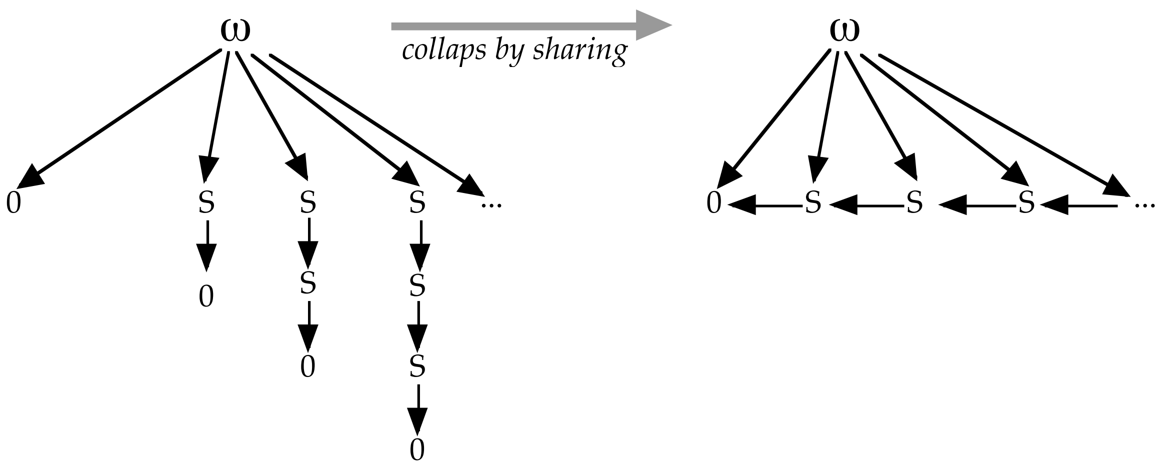

Figure 4.5

The result of the maximally collapsing operation yields in a sense the real ordinal α designated by t, ⟦ t ⟧ = α, but not quite: it is a non-transitive subgraph of graph(α). Adding transitivity (edges induced by transitivity) yields the full ordinal α.

Note that the ordinals ∈ Ω are by no means uniquely designated by the t ∈ $\mathcal{TO}$. E.g. ⟦0 : 2 :4 : 6 : 8 : …⟧ = ⟦0 : 1 : 4 : 9 : 16 : 25 : …⟧ = ω.
Also, ⟦ $t_0 : t_1 : t_2 : \ldots$ ⟧ = ⟦ $t_n : t_{n+1} : t_{n+2} : \ldots$ ⟧, only the tail 'counts'.

4.5.1. EXAMPLE. Let t = 0 : 2 :4 : 6 : 8 : …

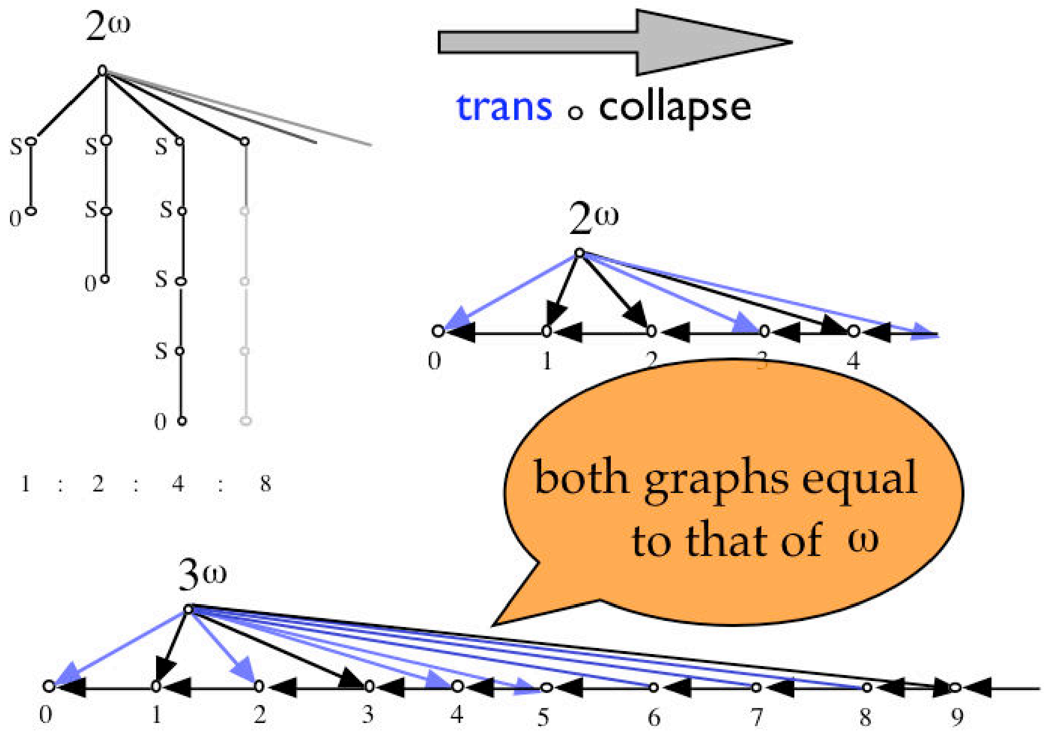

Figure 4.6

So tree ordinals are a kind of sub-transitive, meagre versions of ordinals. And leaving away most of the fat, i.e. the transitivity, thinning them out, makes them fit in and manageable in the restricted space of infinitary first-order (or also infinitary lambda) rewriting.

4.6. *Computing with ordinals.*
Now that we have represented the countable ordinals in a fat-free way as tree ordinals, we would like to do ordinal arithmetic: addition A, multiplication M exponentiation E, and a new operation stacking (or building) B. Let us just give the rewrite rules that enable (implement) these operations. It is a straightforward extension of Dedekind's TRS for only A, M, taking into account that we now have a third construction case, namely the formation of a sequence $t_0 : t_1 : t_2 : \ldots$. We also employ the familiar definition borrowed from functional programming for the natural numbers, i.e. ω.

Ordinal arithmetic is not so simple. For instance, identities such as these three:

(i) $(\omega^{\omega}.2 + \omega^3.4 + \omega^2) + (\omega^3.3 + \omega^2.2 + 1) = \omega^{\omega}.2 + \omega^3.7 + \omega^2.2 + 1$

(ii) $(\omega^6.3 + \omega^2.4 + 2) + (\omega^4.5 + \omega^2) = \omega^6.3 + \omega^4.5 + \omega^2$

(iii) $(\omega^{\omega+2}.3 + \omega^{\omega} + \omega + 7) . (\omega^{\omega+1}.2 + \omega^{\omega} + 3) = \omega^{\omega 2+1}.2 + \omega^{\omega 2} + \omega^{\omega+2}.9 + \omega^{\omega} + \omega + 7$

are at first sight not at all obvious.[1]

[1] The Dedekind TRS for A, M, E, B is certainly not meant to perform such calculations–for a TRS that does perform such calculations, see Castélan [xx], Oudshoorn [xx]. The difference is analogous to performing natural number arithmetic either in de decimal system, or in the unary system. Working as in the three equations above with Cantor normal forms, compares to the decimal system; we are at present concerned with the 'unary system', with different intentions.

Dedekind TRS for A,M,E,B	
1.	$\omega \to N(0)$
2.	$N(x) \to x : N(S(x))$
3.	$A(x, 0) \to x$
4.	$A(x, S(y)) \to S(A(x, y))$
5.	$A(x, y{:}z) \to A(x, y) : A(x, z)$
6.	$M(x, 0) \to 0$
7.	$M(x, S(y)) \to A(M(x,y), x)$
8.	$M(x, y{:}z) \to M(x, y) : M(x, z)$
9.	$E(x, 0) \to S(0)$
10.	$E(x, S(y)) \to M(E(x,y), x)$
11.	$E(x, y{:}z) \to E(x, y) : E(x, z)$
12.	$B(x, y, 0) \to x$
13.	$B(x, y, S(z)) \to E(y, B(x, y, z))$
14.	$B(x, y, u{:}z) \to B(x, y, u) : B(x, y, z)$

Table 4.1

4.6.1. EXAMPLE.

(i) $\omega + 1 = \omega + S(0) = S(\omega + 0) = S(\omega) = S(0{:}1{:}2{:}3{:}\ \ldots)$

(ii) $1 + \omega = 1 + (0{:}1{:}2{:}3{:}\ \ldots) = (1{+}0, 1{+}1, 1{+}2, \ldots) = (1,2,3, \ldots) \approx (0{:}1{:}2{:}3{:}\ \ldots) = \omega$.
So $1 + \omega = \omega \neq \omega + 1$, as we should have.

(iii) $2^\omega = 2^{(0:1:2:3:\ \ldots)} = 2^0 : 2^1 : 2^2 : 2^3 : \ldots = 1 : 2 : 4 : 8 : \ldots$; $3^\omega = 2^{(0:1:2:3:\ \ldots)} = 3^0 : 3^1 : 3^2 : 3^3 : \ldots = 1 : 3 : 9 : 27 : \ldots \approx 1 : 2 : 4 : 8 : \ldots$, so indeed $2^\omega = 3^\omega = \omega$.

4.6.2. EXAMPLE.

- $\omega^2 = \omega.\ (0{:}1{:}2{:}3{:}\ \ldots) = (\omega.0 : \omega.1 : \omega.2 : \ldots) = (0 : \omega : \omega.2 : \ldots)$
- $\omega + \omega^2 = (\omega + 0 : \omega + \omega : \omega + \omega.2 : \omega + \omega.3 : \ldots) = (\omega : \omega.2 : \omega.3 : \ldots) \approx \omega^2$

- For B(ω, ω, ω) we calculate that B(ω, ω, ω) = $\omega^{B(\omega,\, \omega,\, \omega)}$. Does that mean that
- B(ω, ω, ω) is ε_0?

Figure 4.7

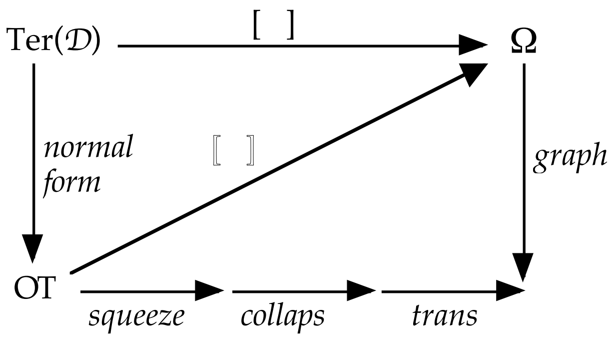

Figure 4.7. describes the relationship between the tree ordinals and the real ordinals in some detail. The (finite) terms from the TRS $\mathcal{D}$ are directly interpreted as ordinals by [ ], with A, M, E, B as the corresponding operations on ordinals. Normal forms with respect to the TRS $\mathcal{D}$ are the infinite terms ∈ $\mathcal{TO}$. (*Correction:* TO in the figure must be $\mathcal{TO}$.) 'Squeeze' is the conversion from the P-notation to infinitely branching terms. 'Collaps' and 'trans' are described above. 'Graph' associates to a real ordinal its 'transition graph'. The diagonal is the interpretation ⟦ ⟧: $\mathcal{TO} \to \Omega$.

4.7. Preliminary analysis of the Dedekind TRS $\mathcal{D}$.

Richard Dedekind
1831-1916

$\mathcal{D}$ is an orthogonal constructor TRS. From the orthogonality we have confluence (CR). It is not SN, for ω admits an infinite reduction.

Now the point view of infinitary rewriting. A bit surprisingly, $\mathcal{D}$ is not CR^{∞}; for there are two different collapsing rules: A(x,0) → x and B(x, y, 0) → x. Now if we make an infinite tower of the two collapsing contexts, A(□ , 0) and B(□ , y, 0), we can infinitely reduce this collapsing tower to the two infinite terms displayed in Figure 4.8, that can only reduce to themselves and therefore form a counterexample to CR^{∞}.

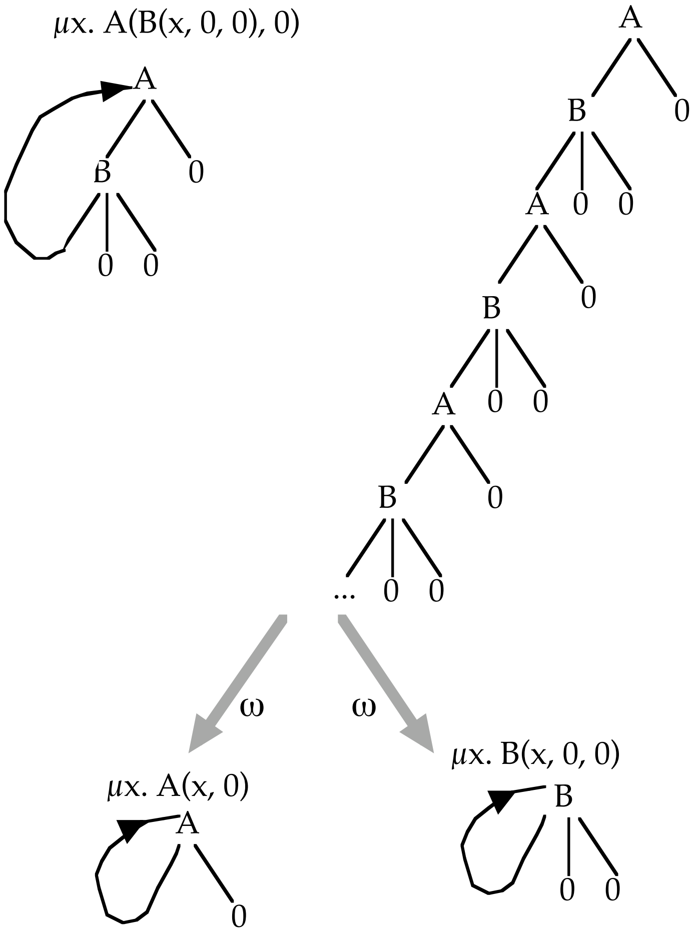

Figure 4.8

4.6.1. REMARK.

(i) Note that without B we do have CR^∞!

(ii) Actually, terms like these μx. A(x,0) are pathological, and not intended. But with this observation, we touch at a sensitive issue, to be discussed later: how to specify precisely the intended domain, with the exclusion of 'garbage' terms, using a notation for mixed inductive/coinductive specifications.

(iii) *Todo: raise the issue of SN^∞ and productivity for this TRS.*

Rewrite rules for Veblen matrix and Γ_0	
1.	$\phi(0, x) \to \omega^x$
2.	$\phi(S(x), 0) \to \sigma(x, 0, \omega)$
3.	$\phi(S(x), S(y)) \to \sigma(x, S(\phi(S(x), y)), \omega)$
4.	$\phi(S(x), y{:}z) \to \phi(S(x), y) : \phi(S(x), z)$
5.	$\phi(x{:}y, 0) \to \phi(x, 0) : \phi(y{:}0)$
6.	$\phi(x{:}y, S(z)) \to \xi(x{:}y, S(\phi(x{:}y, z)))$
7.	$\phi(x{:}y, z{:}u) \to \phi(x{:}y, z) : \phi(x{:}y, u)$
8.	$\sigma(x, y, 0) \to y$
9.	$\sigma(x, y, S(z)) \to \phi(x, \sigma(x, y, z))$
10.	$\sigma(x, y, z{:}u) \to \sigma(x, y, z) : \sigma(x, y, u)$
11.	$\xi(x{:}y, z) \to \phi(x, z) : \xi(y,z)$
12.	$\gamma(y, 0) \to y$
13.	$\gamma(y, S(z)) \to \phi(\gamma(y, z), 0)$
14.	$\gamma(y, z{:}u) \to \gamma(y, z) : \gamma(y, u)$

Table 4.2

REMARK.
(i) The term $\gamma(0, \omega)$ 'is' the ordinal Γ_0.
(ii) The stacking function B, was defined by

$B(x,y,0) \to x$
$B(x,y,S(z)) \to E(y, B(x,y,z))$
$B(x,y, u{:}z) \to B(x,y,u) : B(x,y,z)$

Note the resemblance with σ; if we use σ′, a variant of σ with x,y switched, then σ′ is exactly analogous to the definition of B as based on E. Apart from the permutation of its first two arguments, the function σ arises from ϕ exactly the same way as the function B arises from E.

(iii) The TRS is very likely productive. A proof has yet to be given.

(iv) Ariya Isihara has built a tool to compute with the TRS above. Pointing and clicking at an operator in a node, performs a reduction of the tree at that point.

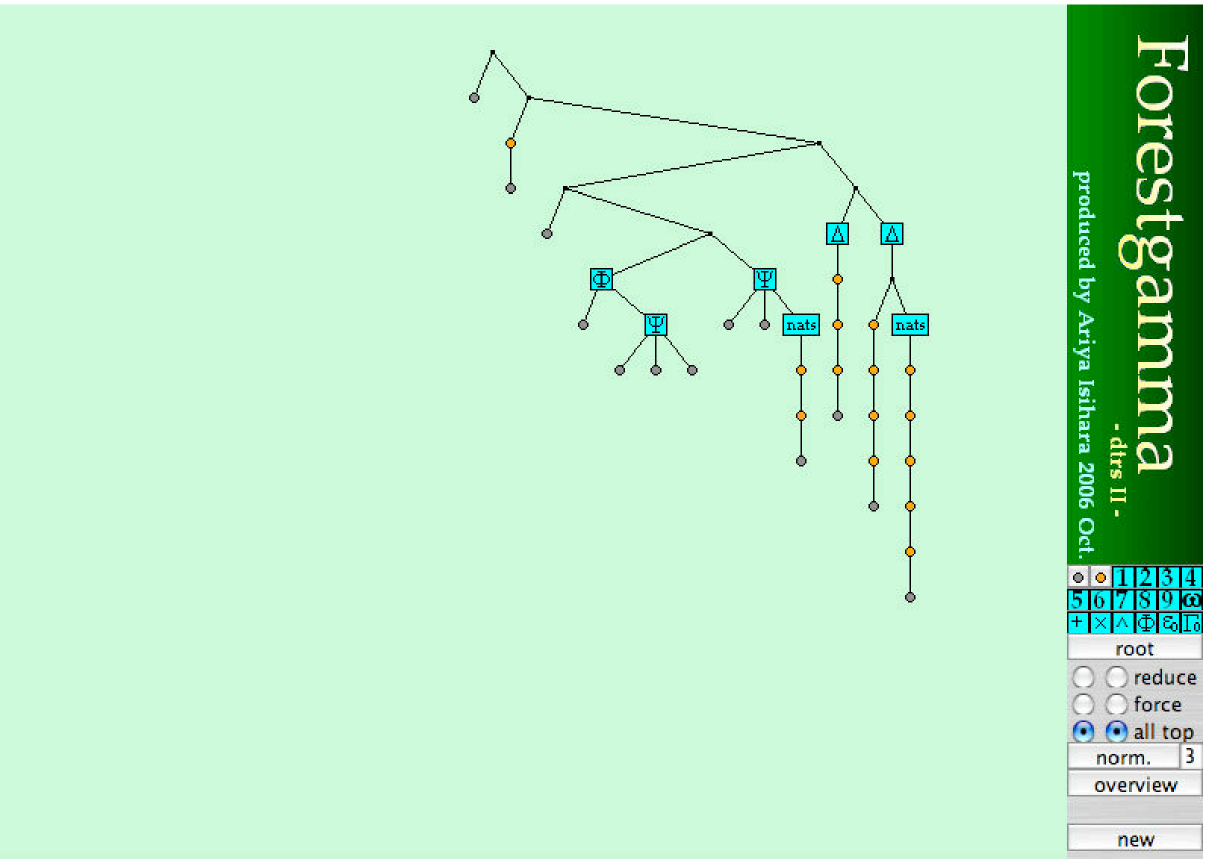

Figure 4.9. Snapshot in computation of Γ_0 by tool of Ariya Isihara

	0	**1**	**2**	**3**	η_3	ε_{η_3}	$\phi\omega 0 + 1$	$\phi\omega 1$
0	$\omega^0 = 1$	ω	ω^2	ω^3	ω^{η_3}	$\omega^{\varepsilon_{\eta_3}}$ $\phi 0(\phi 23)$	$\phi 0(\phi\omega+1)$	
1	ε_0	ε_1	ε_2	ε_3	ε_{η_3} $\phi 1(\phi 23)$		$\phi 1(\phi\omega+1)$	
2	η_0	η_1	η_2	η_3 $\phi 23$	η_{η_3}		$\phi 2(\phi\omega+1)$	
3	ζ_0	ζ_1	ζ_2					
ω	$\phi\omega 0$							

Table 4.3. The Veblen matrix

Note that in the Veblen matrix the same ordinal may appear at different positions. E.g. the three orange field contain the same ordinal, that has different notations. And the ordinals in the rightmost bluish column are all identical. So one could ask how the 'isolines' in the matrix look like; ordinals at the same isoline are equal in value. The blue lines in the following sketch are isolines. Ordinals increase in the directions of the heavy arrows.

Oswald Veblen 1880-1960

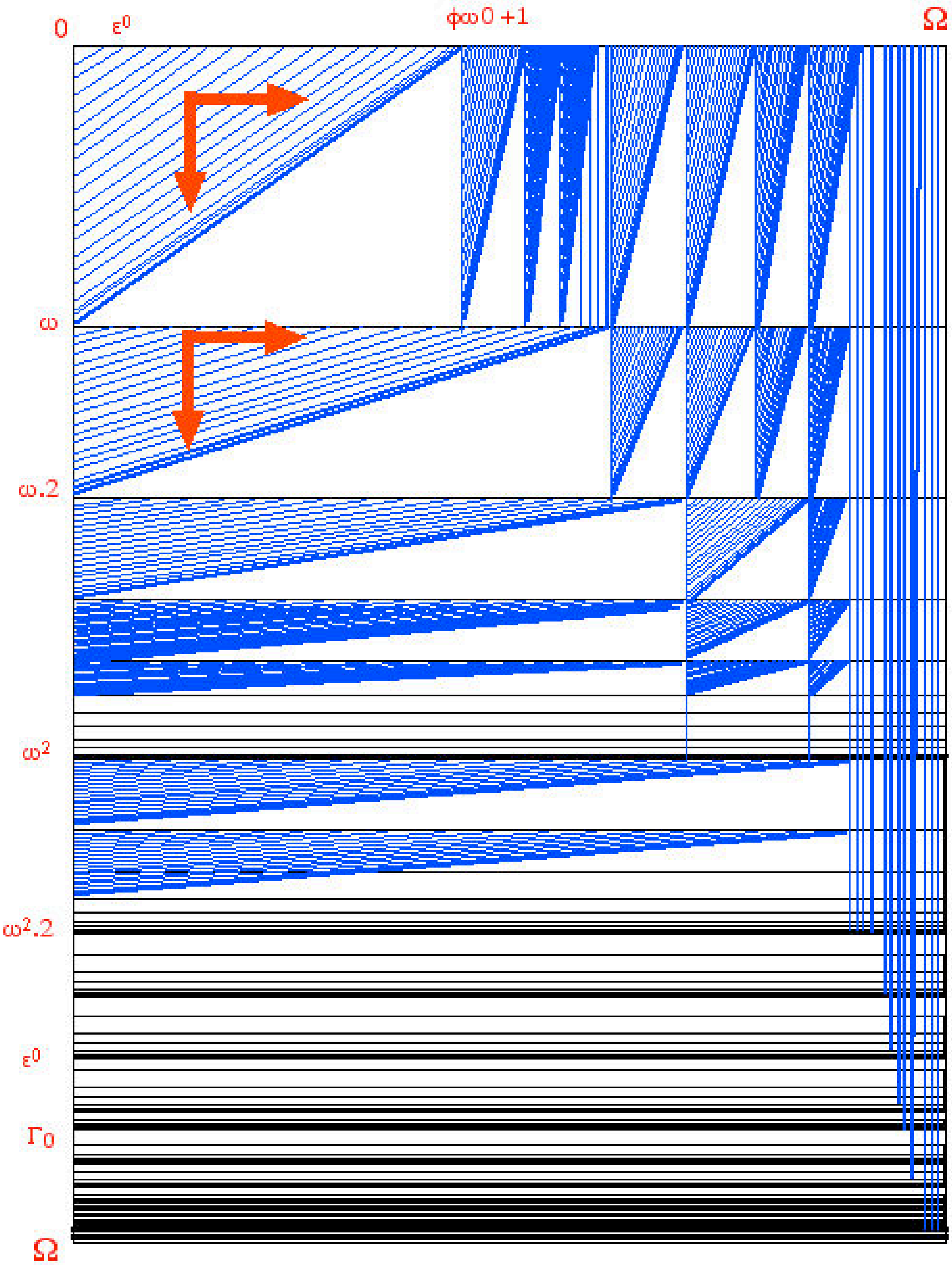

Figure 4.10