A Tale of Two Jewels
for Jonathan Seldin
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Abstract

This informal story is dedicated to Jonathan Seldin, with congratulations on the occasion of his 80th anniversary. The focus of interest is formed by the two polyhedra known as permutohedron and associahedron, and their embedding relation. The embedding and the corresponding lattice homomorphism is well-known, but we arrive to it in a new way, as far as we know, consisting of exploiting the resemblance between the classical Yang-Baxter Equation (YBE), and a 'degenerate' form of that equation, known as the (quantum) pentagon equation or relation (PE). The latter can be seen as an abstraction of the YBE by replacing one of the generator symbol occurrences by τ, the famous silent step in Milner's CCS, adopted also in the process theory ACP. We discuss some well-known phenomena around this embedding of associahedron into permutohedron, including a glimpse of homotopical completion concerning the monoid presentations of these polyhedra.

Contents

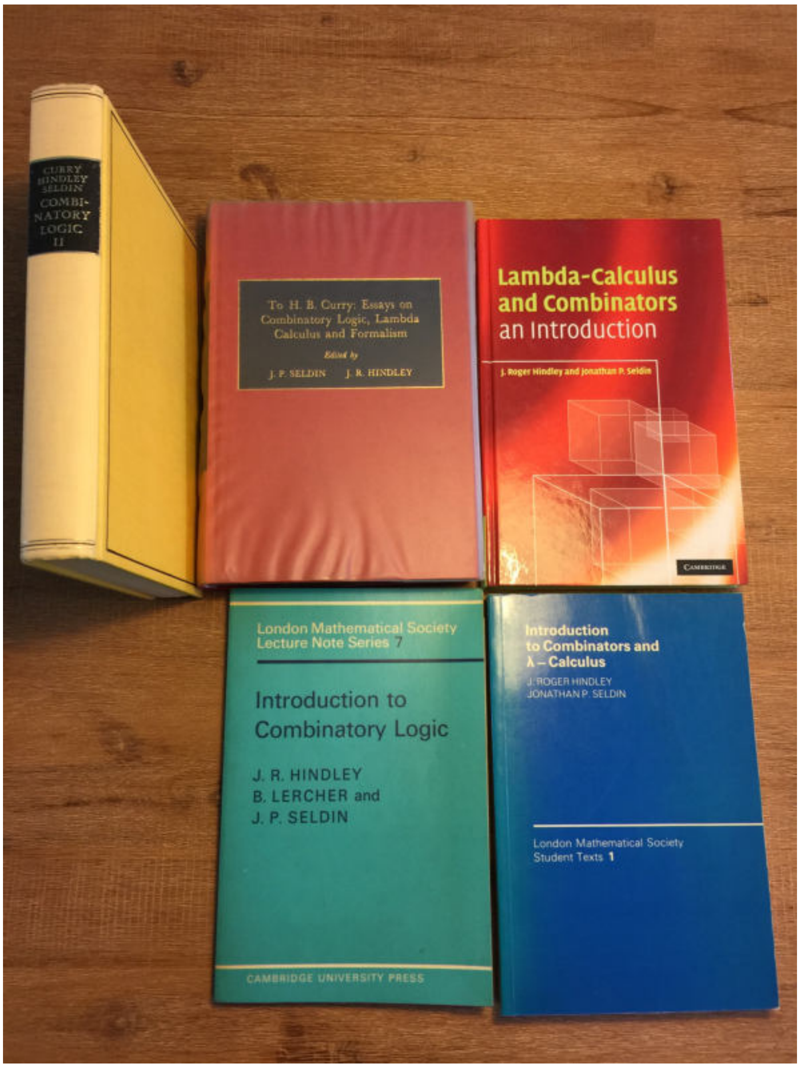


Figure 1: Books on Lambda Calculus and Combinatory Logic co-authored by Jonathan Seldin [41], [42], [43], [40]. An extensive bibliography of books on combinators can be found in Wolfram [2021], [74]. *(All displayed books grace the author's book shelves, and were and are an inspiration.)*

Introduction

This informal tale concerns the emergence of two objects of beauty, the permutohedron and the associahedron, and how they are related, by a lattice homomorphism. The permutohedron arises in the study of permutations as the name suggests, and the associahedron in the study of the associativity equation $(xy)z = x(yz)$. Because braids are an enhanced version of permutations, by passing 'over 'and 'under' of strands, the permutohedron is also instrumental to visualize simple positive braids, a notion that already caught the attention of Gauss. Also the associahedron goes back in time a long way, namely to Euler who was interested in triangulations of regular n-gons, at least in their number, later known as Catalan numbers, ubiquitous in the area of combinatorics.

What makes the scrutiny of these two jewels and their relation so rewarding and captivating is that here we have a focal point where a lot of disciplines meet, from logic to mathematics to computer science. Indeed, we encounter universal algebra, word problems, finite state transducers, combinatorics, Catalan numbers, geometrical group theory, Garside theory, pattern avoiding permutations, Dyck languages, algorithms on trees, homological rewriting, term rewriting, confluence and termination problems, critical pair completion. Enough to fill a number of interesting teaching courses on all levels, basic to advanced!

When writing this paper we were surprised to find exactly the same formula as we will encounter for the associahedron in papers in fields totally different from ours, to wit incidence geometry about Desargues configurations and quantum theories about anyons. The equation is called there the *(quantum) pentagon relation* or *pentagon equation*. It has been noted in some papers that this equation seems a degenerate form of the classical YBE for braids, that we will discuss extensively and intensively. A historical nice to know item is that the pentagon relation already occurred a century ago in the work of Veblen, as the *Veblen flip*, see Doliwa [22].

As to the picture-oriented style of rendering this story, which is more a *graphic novel* than a paper, most of the story is told in the extended figure captions. As observed frequently (Henrik Ibsen, Confucius, Napoleon Bonaparte, Turgenev, Alan Watts and many more), *a picture is worth a thousand words.* [1]

Contribution of this paper

Almost all matters of our story are very well-known in the impressive theory about braids and Tamari lattices, including the permutohedron and associahedron. The book [57] is a real treasure chest. Maybe new in our paper is the attention for the colored braid representation, the well-known YBE in the α_{ij} version, not the

[1] On the other hand, according to Edsger Dijkstra: "A picture may be worth a thousand words, a formula is worth a thousand pictures". (It would follow that a formula amounts to a million words.) And John McCarthy stated: "As the Chinese say, 1001 words is worth more than a picture."

Artin 'relative' notation, and the relation with its degenerate form of the (quantum) pentagon equation PE.

1 The Cube equation in λ-calculus

Because this story is dedicated to Jonathan Seldin we like to have the λ-calculus as point of departure, in particular starting in volume I of Combinatory Logic of Curry and coauthors [16], for Volume II including Jonathan. In Volume I starting on page 113 the analysis of λ-calculus is concerned with the behaviour of *residuals*, also known in Bethke, Klop, de Vrijer [6] as *descendants* of λ-terms and β-redexes. The theorem is stated by Curry as follows, on p. 120:

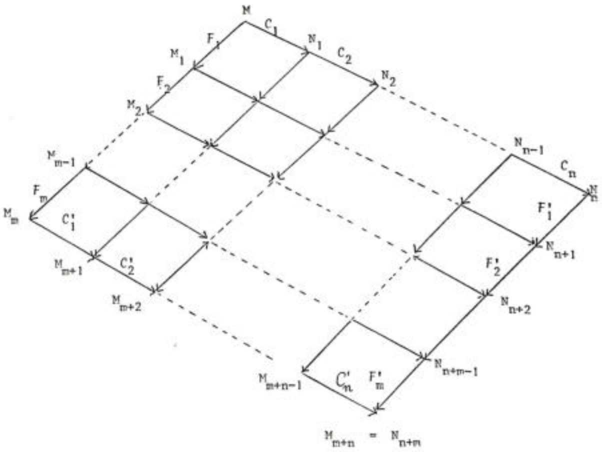
Theorem 1.1. *Let* R, S, Q *be three* β*- or* δ*-redexes in* M*. Then the residuals of* Q *after a complete reduction relative to* R *and* S *are the same no matter whether* R *or* S *is contracted first.*

The proof takes two densely worded pages 120, 121. Jean-Jacques Lévy captured the essence of this theorem picturally in the form of a cube after introducing his notion of projection, see the historical picture 2. Some more history of 40 years ago concerning the Cube Equation CE is in Figure 3.

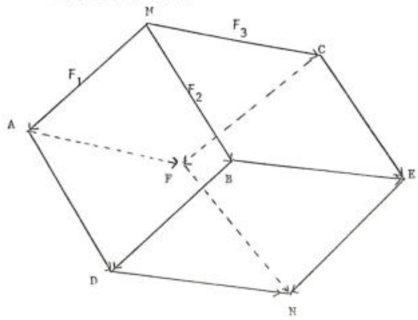
Explaining the statement of CE is easy, see Figure 4; it just amounts to two ways of evaluating the repeated projection along the red and the blue route in the figure. Whether it holds for a rewrite system is another matter. It holds for λ-calculus, Combinatory Logic, in general for first-order orthogonal term rewriting systems. It holds by definition for Van Oostrom's abstract residual systems. Whether it holds for monoid presentations such as for braids, or for generalized braids, is something to verify meticulously. [2] As we will see it holds for the two jewels in this story, the *dramatis personae* permutohedron and associahedron. The CE is the key unlocking much syntactic information as obtained by the theory of residual systems, an off-spring of the culture of λ-calculus and its twin Combinatory Logic, further developed by Melliès and van Oostrom. And, see Figure 4, it figures prominently in Garside theory, under a different name, to wit, coherence property [3], or the Cube-condition.

[2]This is done in part by an admirable recent (2021) paper by Zantema and van Oostrom, soon to be published, titled: *The paint pot problem and common multiples in Artin-Tits monoids.*

[3]Caveat: this is not the coherence as in Mac Lane's coherence theorems; the latter notion is adressed in the contribution of Huet in this book.



2.2.1. Lemme du cube : Soient F_1, F_2, F_3 des ensembles de radicaux dans une expression M. Si $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ sont les trois réductions $M \xrightarrow{F_1} A$, $M \xrightarrow{F_2} B$, $M \xrightarrow{F_3} C$, on a la figure suivante :

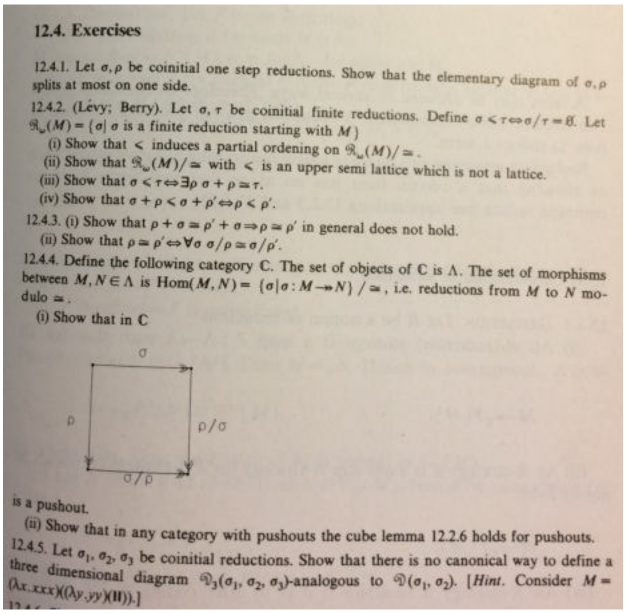


où on a les réductions suivantes :

$$(\mathcal{D}_2/\mathcal{D}_1) : A \to D \qquad (\mathcal{D}_2/\mathcal{D}_3) : C \to E$$
$$(\mathcal{D}_1/\mathcal{D}_2) : B \to D \qquad (\mathcal{D}_3/\mathcal{D}_1) : A \to F$$
$$(\mathcal{D}_3/\mathcal{D}_2) : B \to E \qquad (\mathcal{D}_1/\mathcal{D}_3) : C \to F.$$

$$((\mathcal{D}_3/\mathcal{D}_2)/(\mathcal{D}_1/\mathcal{D}_2)) = ((\mathcal{D}_3/\mathcal{D}_1)/(\mathcal{D}_2/\mathcal{D}_1)) : D \to N$$
$$((\mathcal{D}_1/\mathcal{D}_3)/(\mathcal{D}_2/\mathcal{D}_3)) = ((\mathcal{D}_1/\mathcal{D}_2)/(\mathcal{D}_3/\mathcal{D}_2)) : E \to N$$
$$((\mathcal{D}_2/\mathcal{D}_1)/(\mathcal{D}_3/\mathcal{D}_1)) = ((\mathcal{D}_2/\mathcal{D}_3)/(\mathcal{D}_1/\mathcal{D}_3)) : F \to N$$

Figure 2: The Cube Equation and the presupposed notion of projection originated in the PhD.-thesis (1978) of Jean-Jacques Lévy. The equality in the three Cube formulae is Lévy's projection equivalence. See also Huet (1994) for a formal development of residual theory, geared towards employment in computer verification. *With consent of Jean-Jacques, compiled and copied by present author.*



12.4. Exercises

12.4.1. Let σ, ρ be coinitial one step reductions. Show that the elementary diagram of σ, ρ splits at most on one side.

12.4.2. (Lévy; Berry). Let σ, τ be coinitial finite reductions. Define $\sigma < \tau \Leftrightarrow \sigma/\tau = \emptyset$. Let $\mathfrak{R}_\omega(M) = \{\sigma \mid \sigma$ is a finite reduction starting with $M\}$

(i) Show that $<$ induces a partial ordening on $\mathfrak{R}_\omega(M)/\simeq$.

(ii) Show that $\mathfrak{R}_\omega(M)/\simeq$ with $<$ is an upper semi lattice which is not a lattice.

(iii) Show that $\sigma < \tau \Leftrightarrow \exists \rho\ \sigma + \rho \simeq \tau$.

(iv) Show that $\sigma + \rho < \sigma + \rho' \Leftrightarrow \rho < \rho'$.

12.4.3. (i) Show that $\rho + \sigma \simeq \rho' + \sigma \Rightarrow \rho \simeq \rho'$ in general does not hold.

(ii) Show that $\rho \simeq \rho' \Leftrightarrow \forall \sigma\ \sigma/\rho \simeq \sigma/\rho'$.

12.4.4. Define the following category **C**. The set of objects of **C** is Λ. The set of morphisms between $M, N \in \Lambda$ is $\mathrm{Hom}(M,N) = \{\sigma \mid \sigma : M \twoheadrightarrow N\}/\simeq$, i.e. reductions from M to N modulo $\simeq$.

(i) Show that in **C**

is a pushout.

(ii) Show that in any category with pushouts the cube lemma 12.2.6 holds for pushouts.

12.4.5. Let $\sigma_1, \sigma_2, \sigma_3$ be coinitial reductions. Show that there is no canonical way to define a three dimensional diagram $\mathfrak{D}_3(\sigma_1, \sigma_2, \sigma_3)$-analogous to $\mathfrak{D}(\sigma_1, \sigma_2)$. [*Hint.* Consider $M = (\lambda x.xxx)((\lambda y.yy)(\mathbf{I}\mathbf{I}))$.]

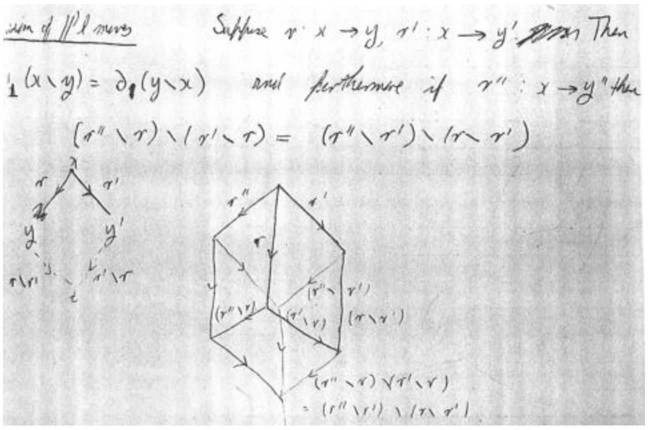


Figure 3: Regarding the crucial Cube Equation some more history: Page 323 of Barendregt [4] contains two very interesting exercises in particular 12.4.4 connecting the CE with category theory, subsequently around that time verified in detail in a 12-page unpublished handwritten note by Gordon Plotkin. The proofs in that manuscript refer repeatedly to the corresponding proofs in the Ph.D.-thesis of Jean-Jacques Lévy. For a recent (2022) account by Lévy of his construction at that time of the labeled λ-calculus, codifying the notion of residuals, see his [52]. *Fragments from Barendregt's 1984 book with Henk's consent and a manuscript note of Plotkin, copied by present author, included with Gordon's consent.*

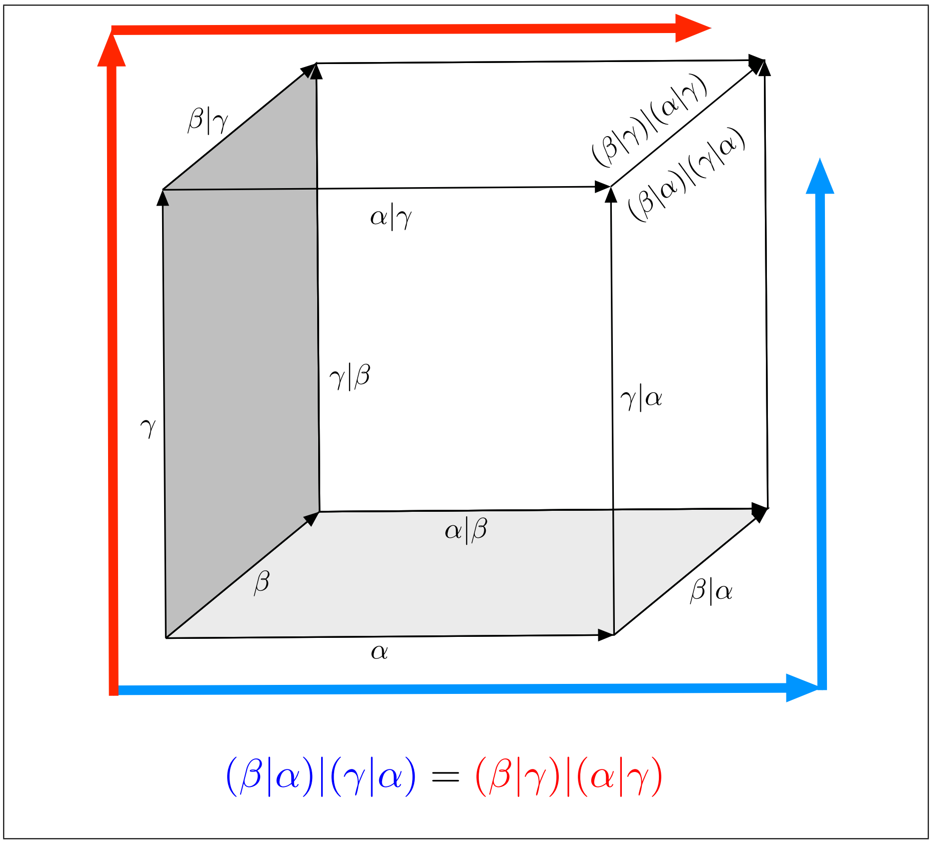


Figure 4: The Cube Equation CE. A fundamental theorem of residuals in λ-calculus in Curry [16]; made explicit as in this figure using his notion of projection by Jean-Jacques Lévy in his Ph.D.-thesis, subsequently used by Melliès and Van Oostrom for their axiomatic theory of (abstract) residual systems. It is included in Barendregt [4], p.315, 12.2.6 as the Cube Lemma. It is also a key property in Garside monoid theory, where it occurs frequently, see location pointers mentioned in Endrullis-Klop [28]. In Dehornoy [17] p.59, Figure 2.1 it is called the Coherence property. In Dehornoy et al. [20] p.67, Prop. 4.16 it is called the (θ-)Cube condition. *Picture by author.*

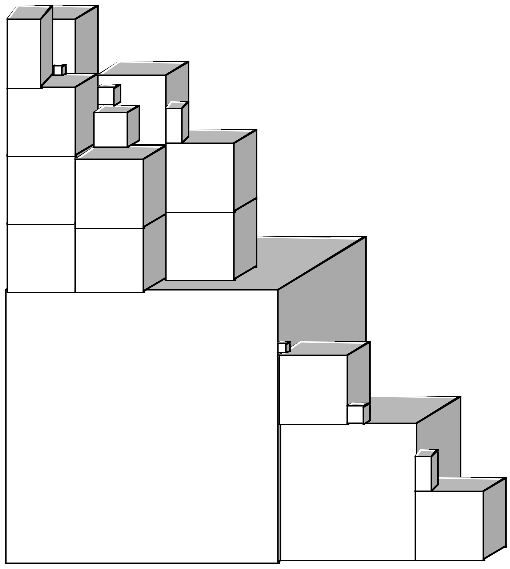


Figure 5: Newman's Lemma in three dimensions: spatial tiling with elementary cubes (e.c.'s) for a terminating (SN) reduction relation yields confluence (convergence): the tiling process will result in a big completed cube. In analogy of a 2-dimensional e.d. (elementary diagram) an e.c. has three single steps as initial steps and arbitrarily many converging steps. This lemma is useful in in establishing the Cube Equation (CE) for suitable monoid presentations, such as identified in Garside theory. The CE is valid for λ-calculus, orthogonal rewrite systems, and also for the permutohedron and the associahedron presentations; but e.g. not for the Artin-Tits monoid as in Figure 45. *Picture by author.*

Btw, Figure 4 fails to say what equality is meant: it is in λ-calculus Lévy-equivalence, in monoid presentations the monoid equality as given by the equations. We will encounter below the example of the associahedron equality given by the Pentagon Equation; there it is literal syntactic equality. Another bytheway: Lévy-equivalence is actually a homotopic notion, as will be clarified somewhat in a later part of this story. It refers to transforming a given reduction by nudges to another reduction.

2 The Yang-Baxter Equation YBE

There are numerous ways in which the Yang-Baxter Equation appears, in very different contexts, from presentations of permutations and simple positive braids, to complicated settings in quantum theory. The following pictures give an impression. First a naive way of an 'intuitive derivation' of the YBE, in Figure 6. Next, in Figure 7, a more sophisticated way of manifesting the YBE, as the hexagon in a the threedimensional cube C_3. This has already the flavour of homotopical rewriting: the two-step reductions are rewritten by the red and blue arrows. Third, an even more sophisticated rendering of YBE, by filling up the faces of the cube C_3. This yields the YBE as an equation between cells, also with the flavour of homotopical rewriting. A glimpse of such rewritings will be discussed in a later section.

By the way, the naive rendering of YBE has a nice generalization, mentioned in a math forum discussion by the 2022 Abel prize recipient topologist Dennis Sullivan, who called it a *trick from chemistry*. See Figure 9.

Remark 2.1. A comment on these two different but equivalent notations as in Figure 6 is in order. The majority of the literature containing the YBE about braids favours the asymmetrical Artin notation like $aba = bab$. The advantage is that multiplication of words amounts to just concatenation. This is so because the underlying category for this monoid has only a single object. For the α_{ij}-notation that simple feature is lost, we have to take into account when multiplying words, the 'states'. It is akin to a 'typed ' framework; multiplication is only possible if the state of the source coincides with the state of the target of the first word. So we have just as in typed λ-calculus:

$$w : \sigma \to \tau,\ v : \tau \to \rho \implies (w \cdot v) : \sigma \to \rho$$

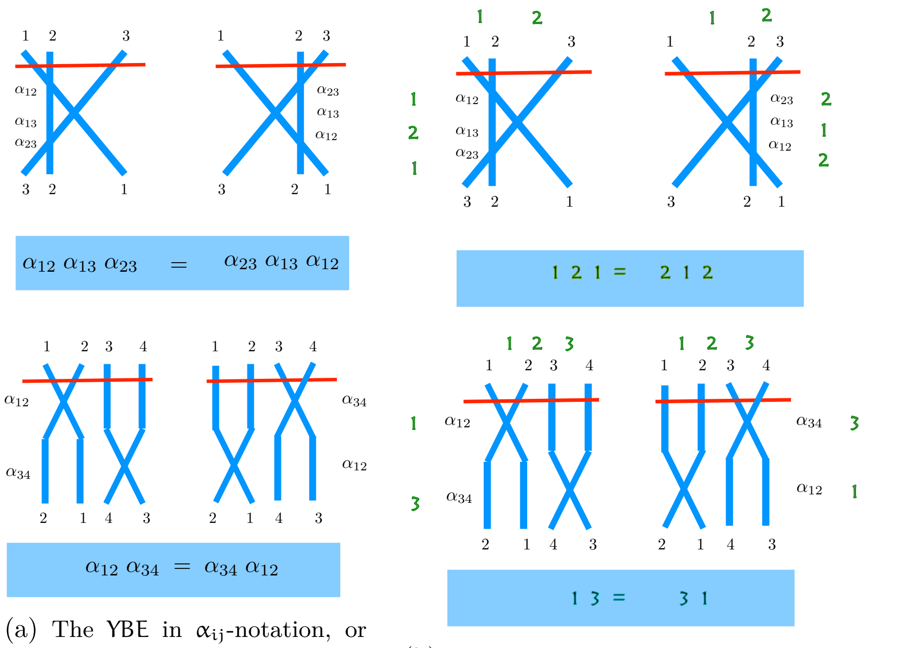
Suggestively, we also could write employing left- and right superscripts:

$${}^{\sigma}w^{\tau} \cdot {}^{\tau}v^{\rho} = {}^{\sigma}(w \cdot v)^{\rho}$$

So there is a kind of 'cut-rule' applied. This state-dependent framework is based on categories with several objects; see Dehornoy [20], Ch. 1, The category context, page $29 - 39$ for more explanation about this framework, including presentations of categories, that in that marvellous encyclopedic book on Garside theory is generally used.

Question 2.2. Is there a version of Tietze moves able to deal with such state-dependent presentations, or in other words, categories with presentations as just referenced?

For term rewriters, the two notation styles, Artin-style versus α_{ij}-style, have an important consequence: in the α_{ij}-notation it is entirely



(a) The YBE in α_{ij}-notation, or colored notation: the strands have a color $1, 2, 3$.

(b) YBE in Artin-notation: the gaps between the strands are numbered in green $1, 2$.

Figure 6: The Yang-Baxter Equation (YBE) arises in many contexts where three objects are permuted to their reverse order. (a) The lower figure permutes two disjoint pairs of objects, which amounts to commutativity. The red line is a time-line: sliding it downwards, the successive crossings manifest themselves in the order as in the equation. The upper figure is sometimes called the star-triangle relation. Note the resemblance with one of the Reidemeister moves to manipulate knots. (b) The Yang-Baxter Equation (YBE) in another notation, originally due to Artin. Here the green numbers denote the gaps between the objects $1, 2, 3, 4$. This notation is commonly used in literature about (simple positive) braids and in homotopical rewriting. *Picture by author.*

clear how symbols in the *lhs* propagate to those in the *rhs* of the equation. This is not so in the Artin notation. This facilitates some proofs in the basic set-up of simple positive braids, e.g. that so-called *simplicity* of such braids is preserved by projections, as defined in Endrullis-Klop [28]. How important such tracing of symbols is, is witnessed by Lévy's labeled λ-calculus, see [52].

There are more term rewriting advantages for the α_{ij}-notation. For one thing, it is an absolute notation, whereas the Artin-notation is relative: in a braid word occurrence of a generator designating a crossing, its meaning, which strands are crossing, depends of its place in the word. A symbol 1 here means something different than a 1 in another place. Not so for the α_{ij}-notation. Another benefit of the α_{ij}-notation, called 'colored' braids notation in Endrullis-Klop [28], is that the presentations for braids or permutations yield decreasing diagrams. In recent homotopy rewriting theory this technique of decreasing diagrams of De Bruijn and Van Ooostrom [58], Endrullis [31] , [30] has been used by Ivan Yudin [76] for homotopical completion of Hecke-algebras, leading to Zamolodchikov-cells, a notion that we will encounter later on in this story for the permutohedron.

By contrast, the Artin presentation with $121 = 212$ etc., is a standard example of corresponding elementary diagrams that do *not* yield decreasing diagrams. So we then do not have automatically the confluence property 'for free', such as decreasing diagrams deliver.

There is yet another reason for a rewriters preference for the α_{ij}-notation, a reason which pertains to the present paper. It is not only directly linked to the equational presentation of the permutohedron, but surprisingly, also to that of the associahedron; and by combining these, also to the equational presentation of the permutoassociahedron. This is in fact the main observation of this paper, namely that the (quantum) pentagon equation PE is an abstraction of the YBE, which yields a (well-known) embedding of associahedron into permutohedron, more precisely of A_{n+1} into P_n.

`https://math.stackexchange.com/users/232583/`
`dennis-sullivan`

See the Endnote [1]

3 The permutohedron P_4

Just as we have seen the hexagon, which is P_3, the permutohedron of order 3, arising from the cube C_3 in Figure 7, or also simply by the transpositions from 123 to its reverse 321 in Figure 8, and also in Figure 49, we obtain P_4 by drawing the transpositions of 1234 to 4321. It can

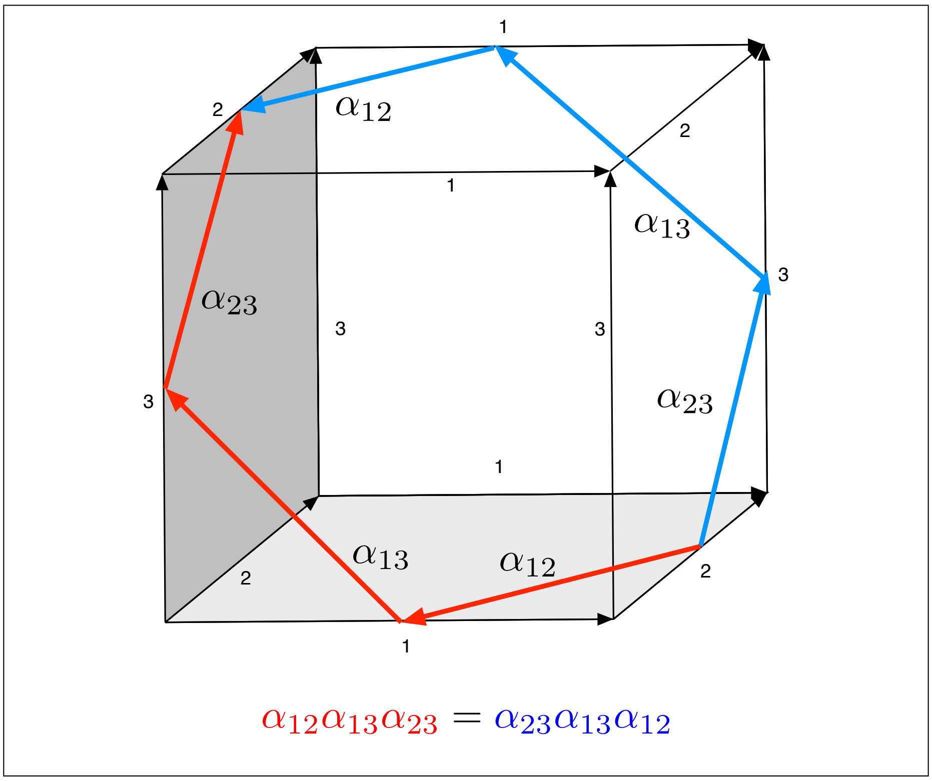


Figure 7: The hexagon is the permutohedron of order 3, P_3, residing in the cube C_3, displaying the Yang-Baxter Equation YBE. The red and blue arrows suggest the consecutive flipping steps, but that is more clearly visible in the visualisation of Figure 8, which is essentially the same. *Picture by author.*

also be viewed as arising by filling in the faces of the the tessaract C_4, the 4-dimensional cube. Figure 32 sketches how that is done; the first few steps are the colored faces there. Analogous renderings work for all n. And thus a chain is arising of permutohedra of increasing order, with P_n embedded in its successor P_{n+1} having as a limit the infinite permutohedron P_∞, as Figure 57 suggests.There is much to explore in this gem-like object. For one thing, it is a truncated octahedron, as Figure 12 shows. An even more amazing fact is that P_4 is able to fill up, tessellate, the 3-dimensional space; and for every n P_{n+1} tessellates n-dimensional space. Noteworthy is Figure 14, displaying a walk around

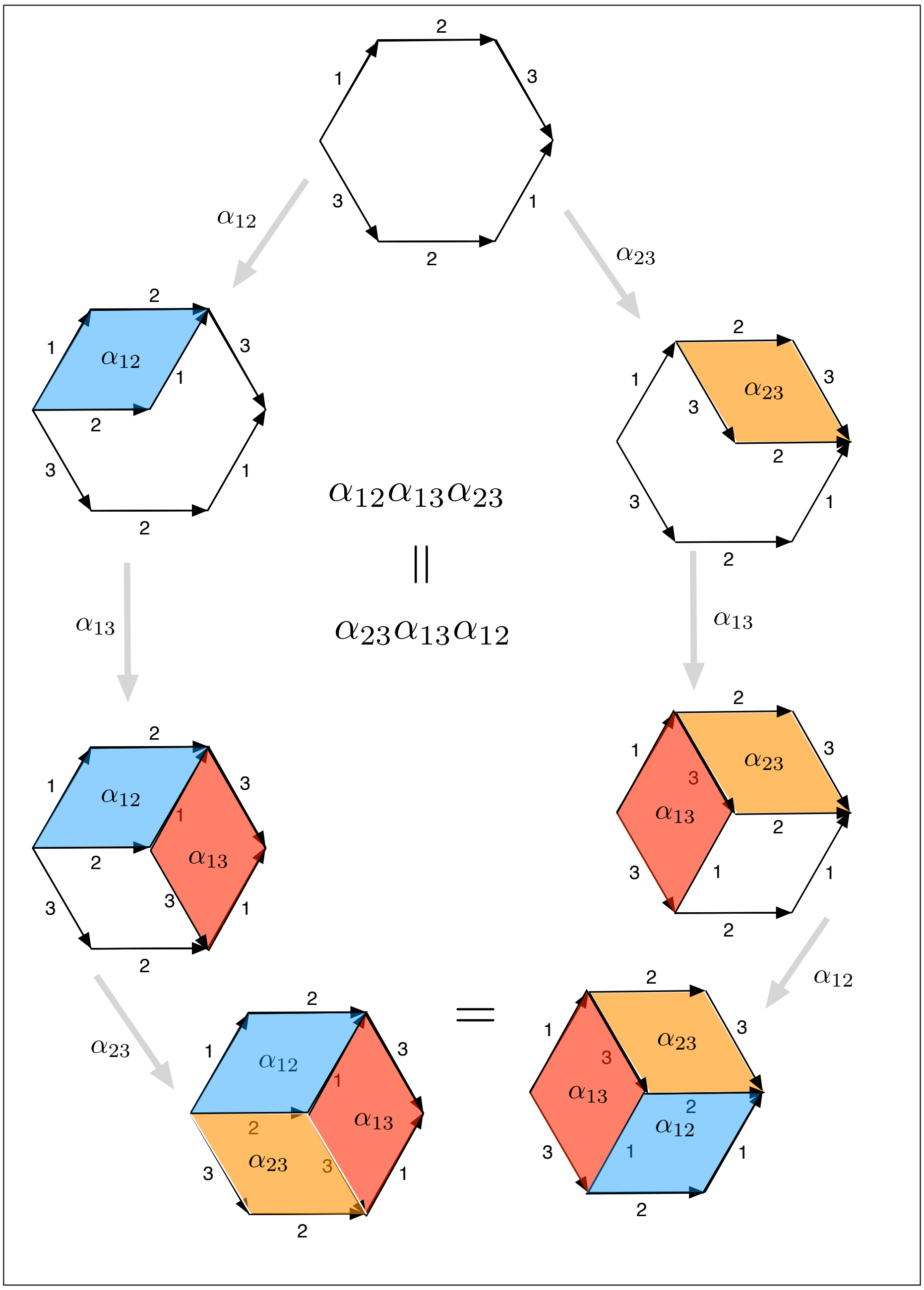


Figure 8: The hexagon of hexagons, again displaying the YBE. The equation at the bottom is an equation between words composed of cells, a typical endeavour of homotopic rewriting. Compare with Figure 7. *Picture by author.*

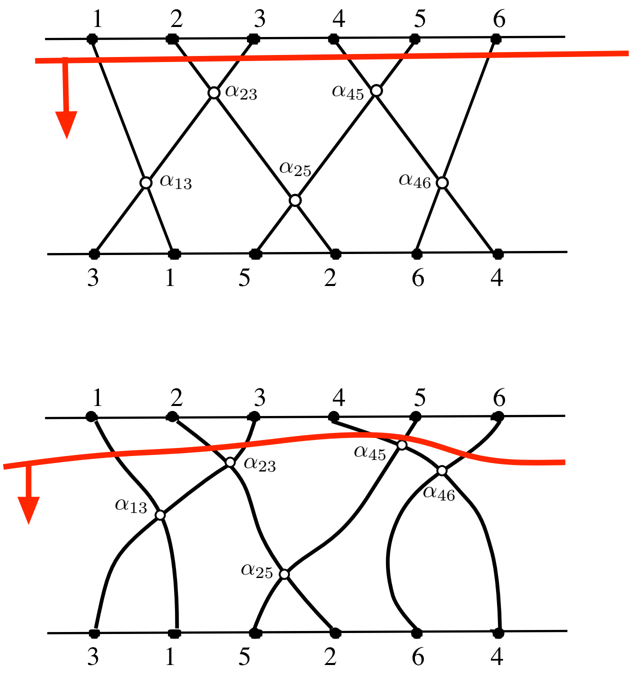


Figure 9: *The chemical trick.* If the red time-line is moved downwards, the varous cross-points that are encountered indicate the transpositions that constitute the permutation from upper sequence 123456 to lower sequence 315264. In this example $\alpha_{23}\alpha_{45}\alpha_{13}\alpha_{25}\alpha_{46}$. In fact, the red line needs not to be strictly horizontally moved downwards, we can do it in a slanted way. Even more, it can be drawn somewhat curved. Also the straight lines from upper side to lower side can be continuously deformed in a somewhat curvy way, as long as we take care that each pair still intersects only once. The different α_{ij}-words so obtained, which are in fact paths on the permutohedron P_6 will be equivalent anyway by the YBE-equations. This observation works also for the alternative Artin gap-notation of the YBE, by looking at the gap 'above' a cross-point, the usual Artin-style of denoting positive braids. *Picture by author.*

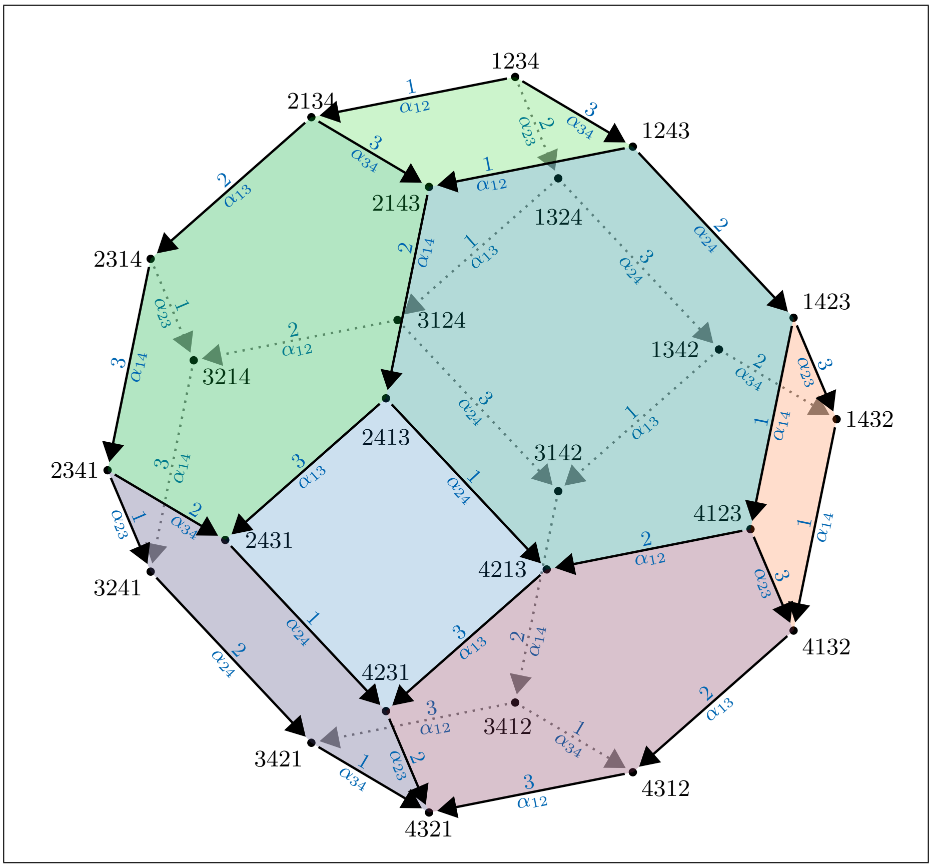


Figure 10: P_4, the permutohedron of order 4, with dual labeling of the edges, in Artin notation using the three gaps $1, 2, 3$ and in colored braid notation, using the six transpositions $\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}$. This polyhedron can be viewed in various ways: as an abstract reduction system, which is confluent and terminating, as a lattice, as a finite state transducer translating the Artin notation into the α_{ij} notation and vice versa. *Picture tikzed by J. Endrullis.*

the P_4 sphere by repeatedly flipping a face. This walk around has a homotopical flavour, and embodies in fact a homotopical completion of the monoid that gave rise to P_4, known as the Zamolodchikov cycle, to which we will come back in Figures 30 and 31.

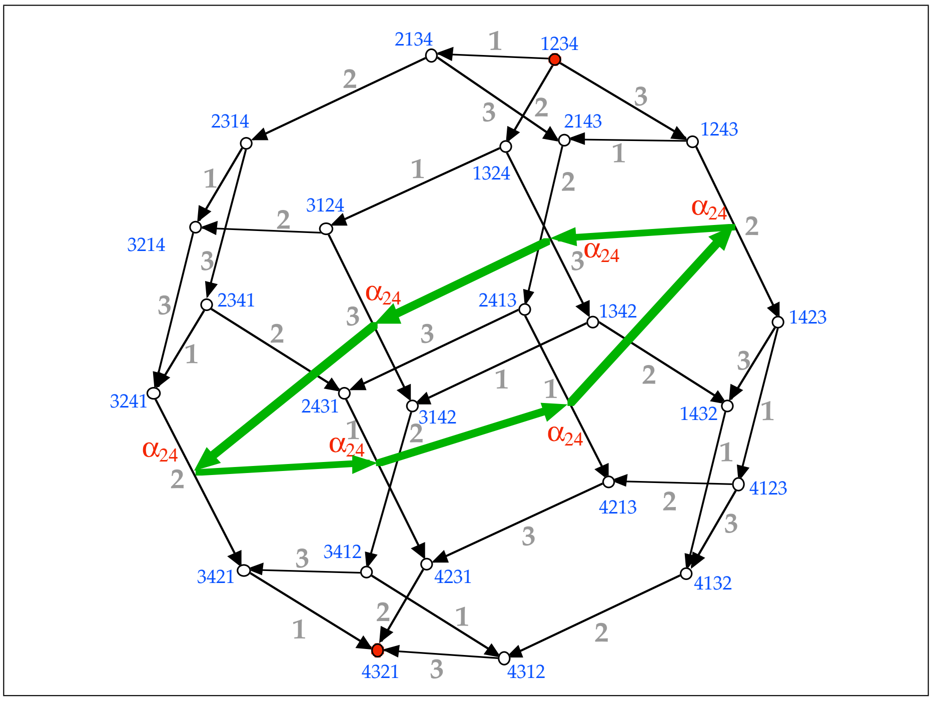


Figure 11: A cyclic walk around the P_4-globe. On the permutohedron P_4 the edge labels propagate in an 'orthogonal' fashion around the globe. Note that this holds for the α_{ij}-notation, but not for the Artin gap-notation with the three generators $1, 2, 3$. The walk as displayed in green can be seen as one half of the Zamolodchikov-cycle that will be displayed later. *Picture by author.*

4 The associahedron

Now that we have surfed around the permutohedron in a dozen homotopic moves by flipping faces, it is time for our second jewel in this story. Looking around on the web for the permutohedron, we soon encounter a less well-known jewel, a polyhedron known as the associahedron, with 9 faces consisting of 3 squares and 6 pentagons. It can be rendered in several geometric ways as the next figures will show. It arises from the basic equation for associativity $(xy)z = x(yz)$, applied on terms generated by constants $a_1, \dots, a_n$ and application. In fact, we use the associativity rule as a rewrite rule $(xy)z \to x(yz)$. Now starting for $n = 5$, and constants a, b, c, d, e from the initial term $a(b(c(de)))$ with association to

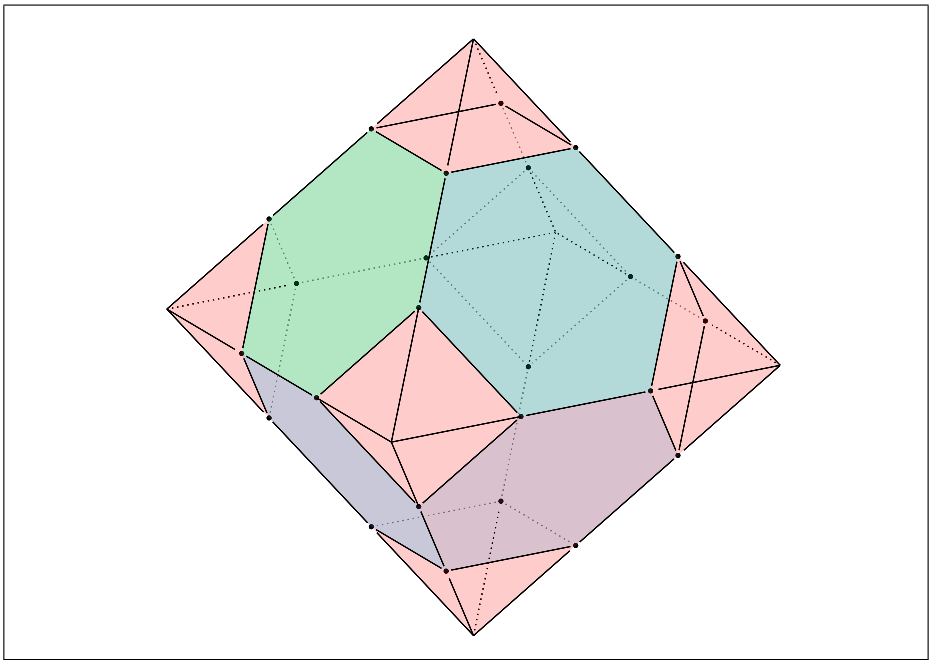


Figure 12: The skeleton of the permutohedron P_4 can also be obtained as a truncated octahedron, as is well-known. This figure is convenient in visualizing how the faces of P_4 are adjacent to each other; every of the 6 squares is adjacent to 4 of the 8 hexagons, these 4 are connected in a ring HHHH around the square. Every hexagon is surrounded by a connected ring of 6 faces, of the form SHSHSH, for S a square, H a hexagon. So this figure gives a good impression of the language of S, H-words formed when travelling over the globe P_4. (In turn, the octahedron itself is the graph displaying how the 6 faces of the 3D-cube C_3 are adjacent.) *Picture tikzed by J.Endrullis.*

the right as top vertex, we get as reduction graph with the normal form $(((ab)c)d)e$ as bottom vertex $\perp$ the skeleton of the polyhedron A_5, the *associahedron of order* 5. It is a gem-like object, first discovered and studied by Dov Tamari [70]. It turned out to be surprisingly ubiquitous. The number of such parenthesized expressions or terms on n generators is the equally ubiquitous Catalan number, whose definition can be found everywhere, in particular in the books by Richard Stanley [67]. The sequence of Catalan numbers starts with $1, 1, 2, 5, 14, 42, 132, 429, \ldots$.

These numbers were almost 300 years ago also determined by Leon-

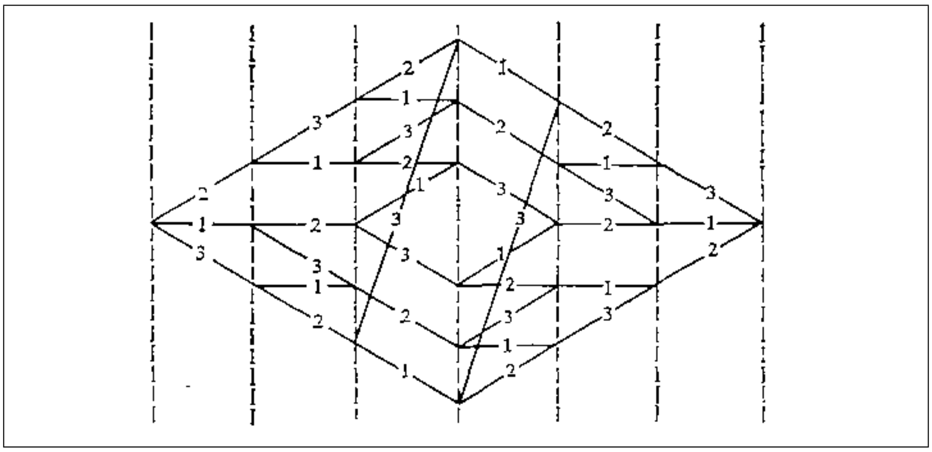


Figure 13: Frank Garside in his seminal paper [36] on the braids group gave (essentially) this drawing of the permutohedron P_4. The edge labels $1, 2, 3$ are the usual Artin notation for braids, enumerating the gaps between the strands of the braids. This notation is also present in the permutohedron P_4 in Figure 19, there next to the alternative colored braid notation employing the α_{ij}-generators. *(Figure copypasted from Garside [36], page* 247, *Fig.7.)*

hard Euler in another context, namely in how many parts one can divide a regular n-gon in pieces by non-intersecting diagonals. Actually, the two counting problems turned out to be the same, there is a now very well-known isomorphism between such n-gon triangulations and parenthesized terms, or binary trees with the constants as end points, as in the Figures 16, 18, 22, 23. What is more, not only their number is the same, also their structure under a certain move is the same. For the n-gon triangulations this is the *diagonal flip*, see the figures just mentioned. For the corresponding binary trees that is the *tree rotation*, which is just an application of the associativity rewrite step on the binary trees that are the term trees. The pictures make this clear without formal definition, Btw, tree rotations are important in data algorithms.

Now to analyze this situation further, it is necessary to have some notation on the polygons and the trees. For the polygons this is done as in the figures, the constants $a, b, c, d, \ldots$ are placed on the edges of the polygon, the upper edge is mostly left blank and serves as the result edge when we determine the term that corresponds with the polygon diagonalisation. Figures 20, 23 give examples. In between the letters at

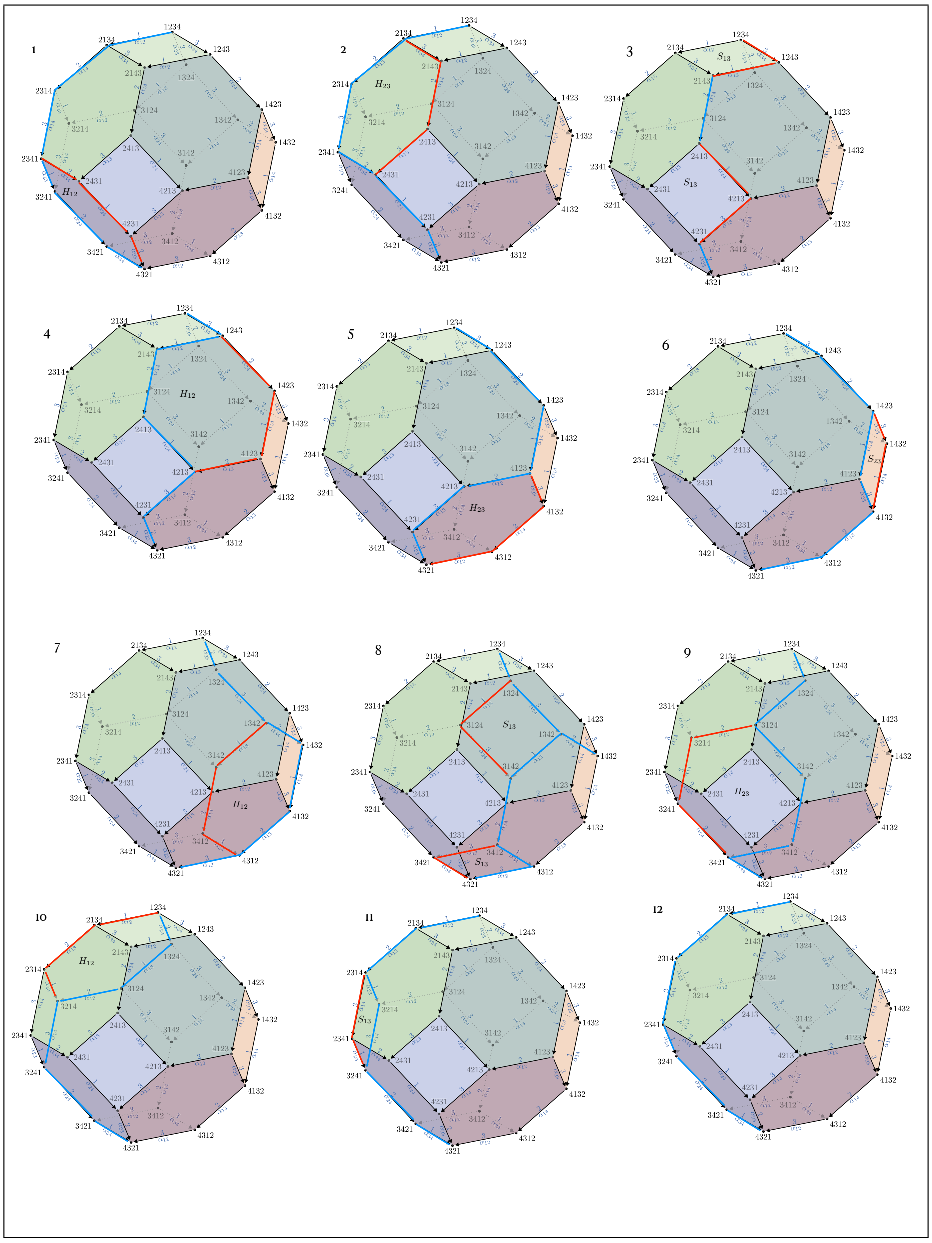


Figure 14: A walk around the permutohedron in 12 face flips. Blue is before the flip of that face, still cool; red is after the flip, as a still warm afterglow of the flipped part.The original starting blue path from North pole to South pole is in the last snapshot 12 restored. Each flipped face contains its name, in the same notation as in Figure 31. *Picture by author, on the underlying permutohedron tikzed by J. Endrullis.*

the sides we place natural numbers at the corner points of the polygon. These numbers give rise to what we call the *canonical numbering* of the binary tree that is determined by the cake cutting. It arises by letting the polygon corner numbers 'percolate upwards' to the nodes of the inscribed binary trees. In fact, the node numbers are then the *inorder numbering* of the binary trees, a fact coming back in the theorem of Knuth 7.1 that we include below. This numbering will be the key for the embedding of the associahedron into the permutohedron, but we are not yet on that point. See footnote [4] .

5 Cutting the cake and flipping diagonals equals rotating the tree

Before we capture the moves and the equations that constitute the second jewel in this story, the associahedron, we must understand that there are different 'isomorphic' views on these moves or actions. They are depicted in Figure 16. See also Figure 18.

The Figures 35, 36 and 37 give a closer view on the relations between these equivalent views, and how the objects are related by the moves.

We now turn to the equations that govern this associahedron structure, in analogy to the YBE that gave rise to the permutohedron P_n. It turns out that the crucial equation for the present associahedron is very close to the YBE! We only have to cross out the middle symbol in the *rhs* of YBE. See Figure 21. This suggests that we can embed the associahedron into the permutohedron. Here is how that can be done.

6 Embedding the associahedron into the permutohedron

First we use the canonical numbering of the tree to give a notation to the tree rotation moves. Such a tree rotation step consists in flipping

[4]Dehornoy [61] uses another node numbering of the binary trees, using 01-sequences that increment by 0 or 1 when going to the left or right. Employing this notation he arrives at a new presentation for the Thompson monoid, i.e. for the associahedron, that includes *quasi-commutation equations*. For the canonical node numbering that we use here, these extra equations disappear.

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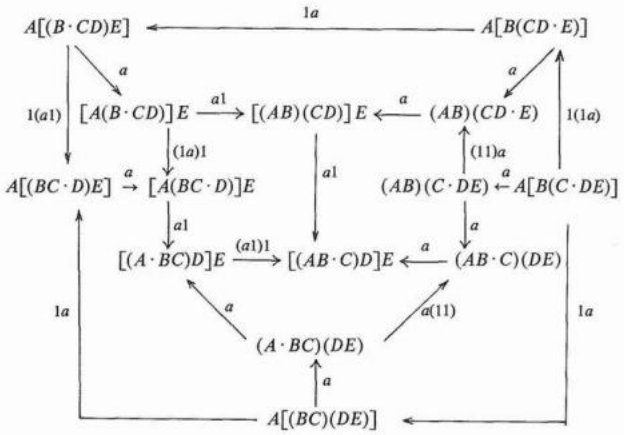
For $n = 5$ the proof may be visualized in the following graph, where AB is short for $A \otimes B$ and $A \cdot BC$ is short for $A(BC)$:



Each complementary region for the graph is either a pentagon (an instance of [3.5]) or a square (an instance of naturality). The whole graph may be regarded as the skeleton of a 3-cell; the regions are its faces and correspond to products $ABCDE$ with two pairs of parentheses omitted, while the edges correspond to products with one pair of parentheses omitted. Stasheff [10] has shown that the corresponding graph for every n gives an $(n-2)$-cell.

Figure 15: The associahedron A_5 occurs in category theory in Mac Lane 1963 [53]), demonstrating the coherence of $\otimes$ for $n = 5$. We leave the small puzzle to decipher Mac Lane's notation for the edges, the arrows are directed just opposite to ours, to the puzzle-minded reader. See also Huet's lecture from 1987, page 6, included in this book. *Picture plus caption copied from cited reference.*

one edge from going left to going right, thereby swapping the numbers of the nodes at start and end of the flipped edge. So it makes sense to notate such a tree rotation flip step by α_{ij}. The figures contain several examples. Now this reduction relation is easily seen to be WCR, weak Church-Rosser. That means we have elementary diagrams, nowadays often called cells, that have two diverging steps completed by some converging sequences of steps to a common reduct. Such e.d.'s constitute an equation by equating the two reductions of that cell. The surprise is that we then get the following equation as in Figure 21,

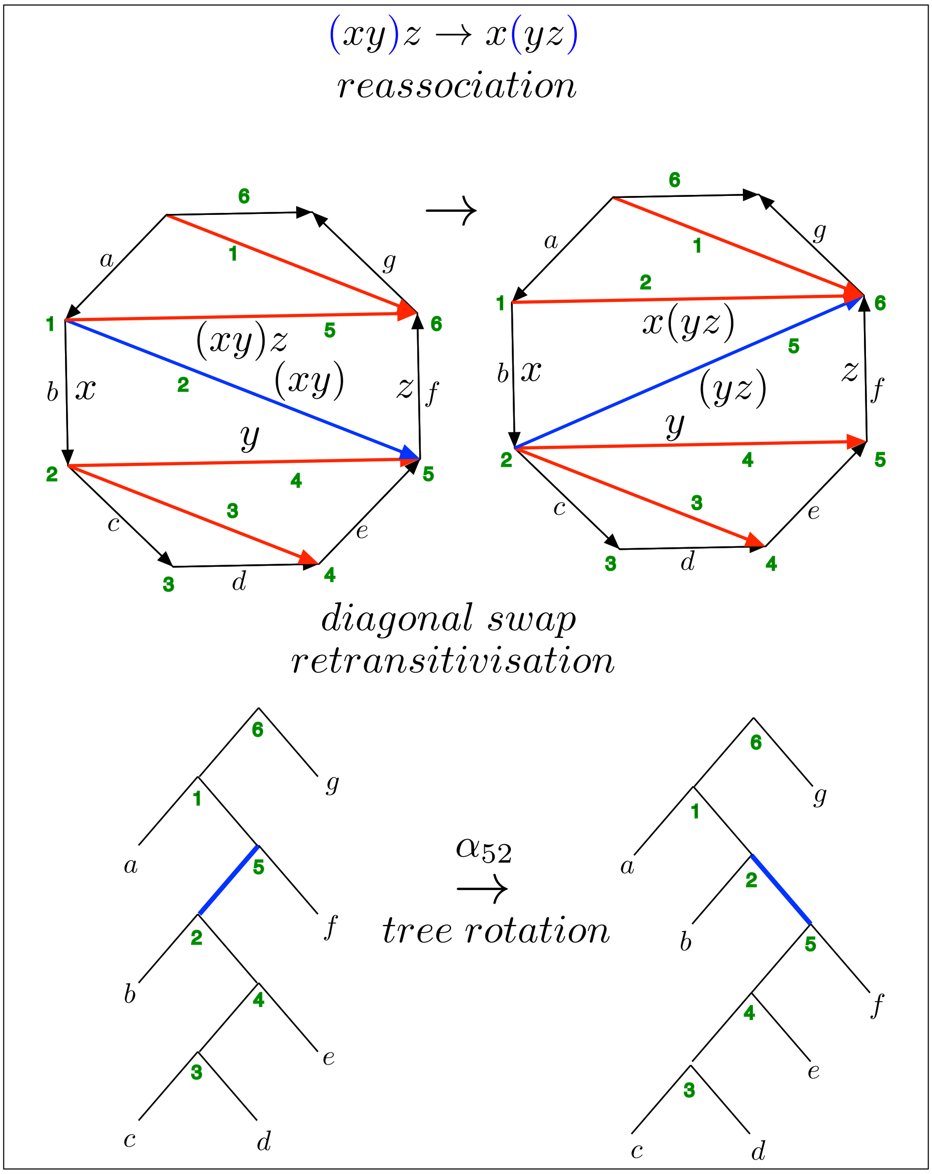


Figure 16: Four different views of the same action. Note the blue bracket pair in the equation at the top. The word 'retransitivisation' is explained in Section 11. *Picture by author.*

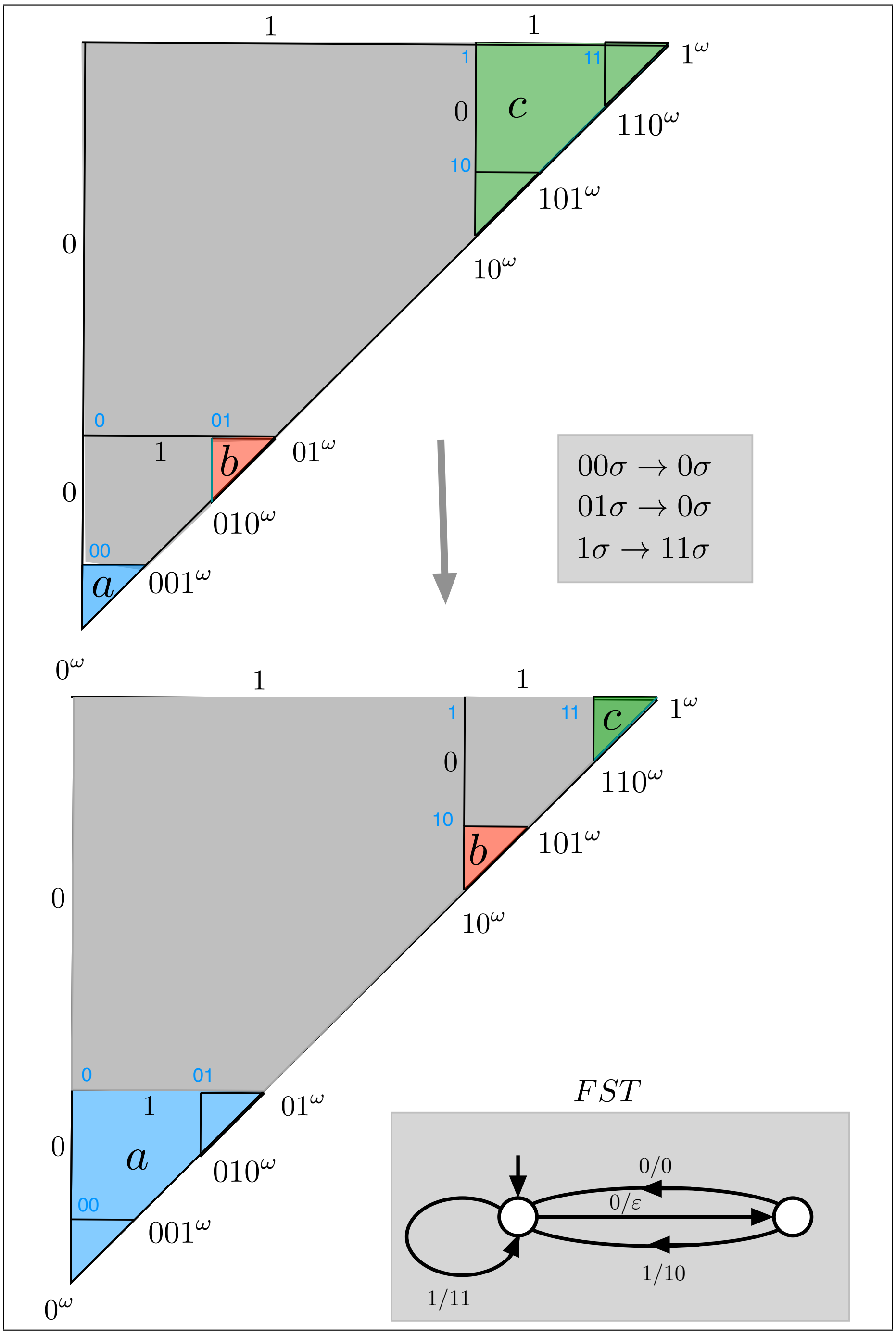


Figure 17: Tree rotations work on the Cantor space $\mathcal{C}$ of 01-streams, and thereby can be seen as elements of the Thompson group $\mathcal{F}$, see Dehornoy [18] and Clay [14]. The figure shows the action on $\mathcal{C}$ of the tree rotation α_{21}, corresponding to the rewrite step $(ab)c \to a(bc)$, which is according to Dehornoy one of the two generating basis elements of $\mathcal{F}$. This tree rotation can also be rendered as the FST, finite state transducer, depicted in the lower grey window. *Picture by author.*

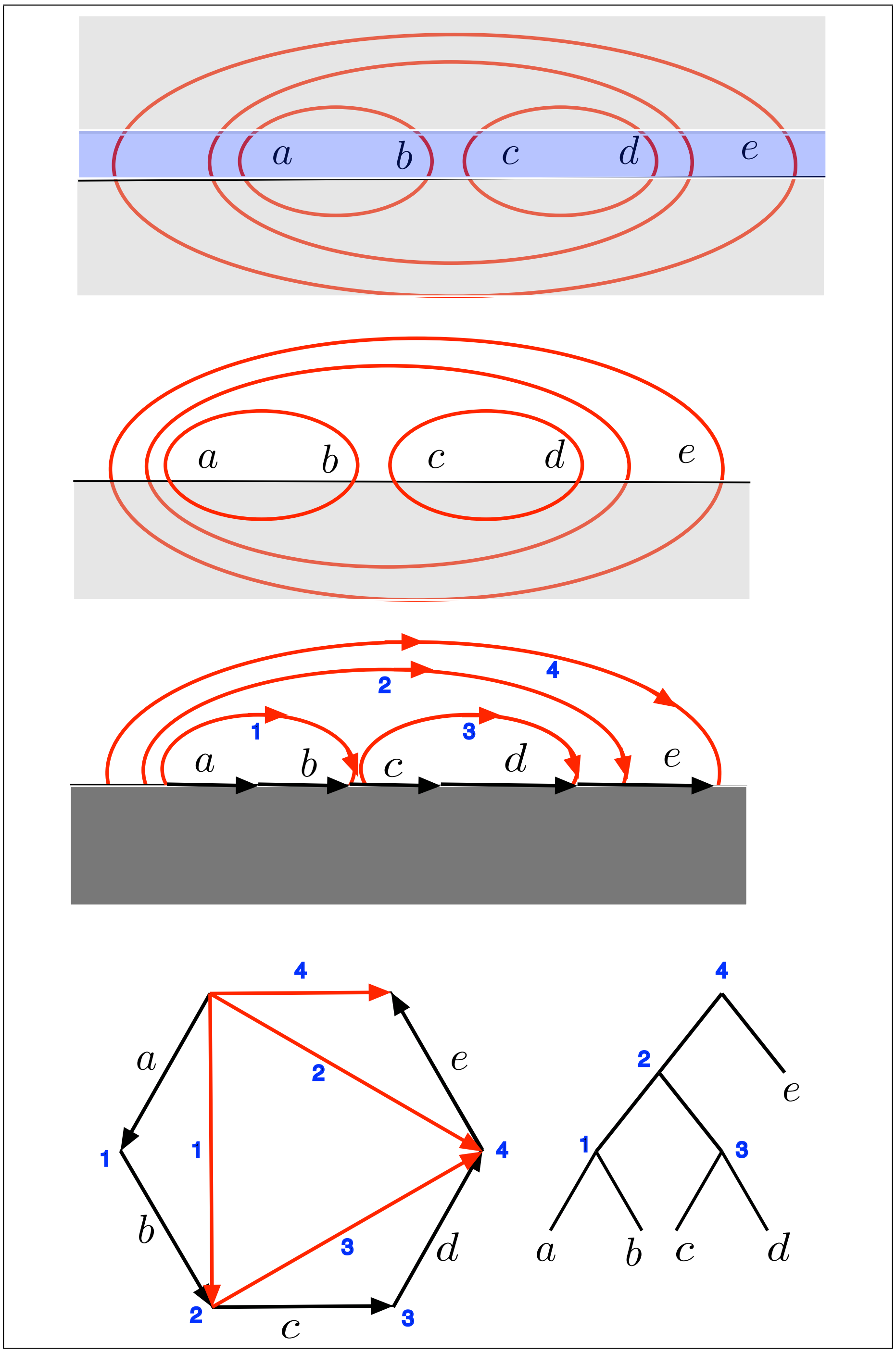


Figure 18: From parentheses to diagonals in a regular polygon. In the blue strip is the parenthesized term, ((ab)(cd))e), extended to a 'linear' Venn-diagram. The upper half is the same as the diagonalisation of the polygon, as can be seen by by rolling up the line with abcde. The half Venn-diagram is also isomorphic to the binary tree. Going to the left in the tree the node numbers are decreasing, to the right increasing. The moves in the three formats are isomorphic too: first a *shift* of a pair of brackets, second a diagonal *flip*; third a tree *rotation*. *Picture by author.*

that is very reminiscent to the YBE of before; except it has not the hexagon form $WWW = WWW$ as the YBE, but instead the pentagon form $WWW = WW$. It seems a degenerate form of the YBE. It is also rather ubiquitous and well-known as the (quantum) pentagon relation or equation. We will call it PE. Given the resemblance with the YBE, it is a plausible conjecture that the polyhedron connected to this PE, will be embeddable in the YBE-polyhedron, being the permutohedron. We have endeavoured to realize such an embedding guided by the form of the YBE and PE: comparing these equations which in our notation, based on the canonical tree node numbering, employ the same 'action' labels α_{ij}, we know precisely which edges α_{ij} should be collapsed, suppressed. With the intuition of *process algebra* à la CCS or ACP in the back of our mind, we could view this as an *abstraction* by replacing the symbol α_{ij} by the silent move τ of Robin Milner in his CCS.

It is well-known in the theory of Tamari lattices and polyhedra that indeed there are many ways to embed the A_5 into the permutohedron P_4, but we want something extra, namely satisfying the notation of the steps, which is in both cases α_{ij}. We have done so, 'manually', by starting to embed the faces at the top of both structures, then successively adding adjacent faces, thereby noting what steps should be collapsed, to satisfy the collaps in the equation PE, where the middle symbol in the *rhs* is collapsed. We then arrived at the Figure 19, where the 10 collapsed edges have been marked in red. Now it turned out that this embedding is exactly the same as the one in the authoritive paper on Tamari lattices by Nathan Reading [60], which convinced us of the canonicity of this embedding, that according to the theory is also a *lattice homomorphism*, and gives a *lattice congruence* for the equivalence classes of the collapsed edges.

Summarizing, we have devised a *process simulation* of the associahedron process into the permutohedron process satisfying the preservation of paths in the former modulo the τ-steps. If the embedding is named $\varphi : A_5 \to P_4$, we have:

Theorem 6.1. *Let* s, t *be vertices of* A_5*. Let* $\twoheadrightarrow_{A_5}$ *denote paths in* A_5*, and* $\twoheadrightarrow^{\tau}_{P_4}$ *paths in* P_4 *modulo* τ*, i.e. possibly containing* τ*-steps. Then*

$$s \twoheadrightarrow_{A_5} t \implies \varphi(s) \twoheadrightarrow^{\tau}_{P_4} \varphi(t)$$

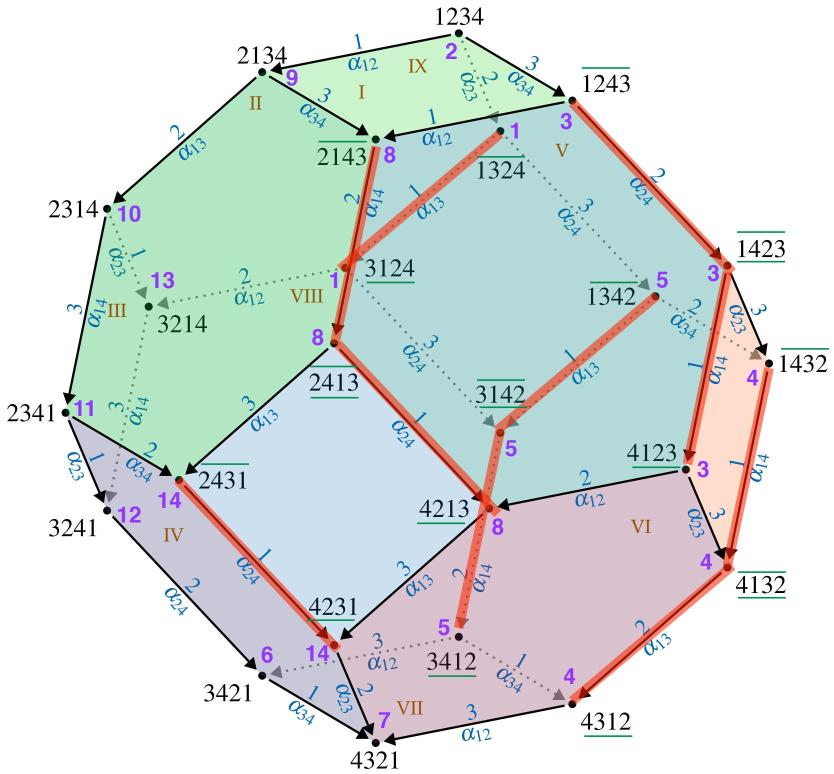


Figure 19: Collapsing the 10 red edges in the permutohedron P_4 yields the associahedron A_5. Green underlined are permutations containing the pattern 312; they are minimal in the red fibres. Green overlined are permutations containing the pattern 132; they are maximal in the red fibres. Nodes in the middle of the boomerang-shaped fibres have over- and underlining, they contain both patterns. The purple bold numbers $1, \ldots, 14$ refer to the nodes of the associahedron A_5 that is embedded. The capital Roman digits I, ..., IX refer to the 9 faces of A_5 mapped to that face of P_4. The typical boomerang shape of the red fibres, equivalence classes of the embedding, is witnessing that the embedding is in fact a lattice homomorphism, and that the equivalence is a lattice congruence; see N. Reading [61], p.298, 299, Fig. 5, Local forcing requirements. *Picture by J. Endrullis and author.*

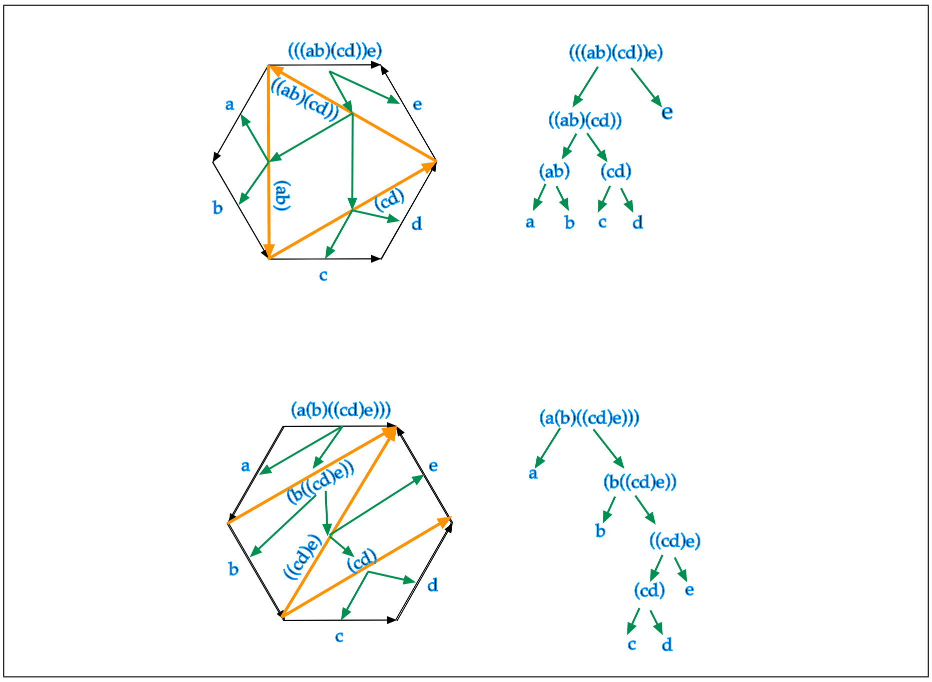


Figure 20: A regular hexagon with non-intersecting diagonals is tantamount to a parenthesized expression of five letters a, b, c, d, e. Each orange diagonal stands for a subterm; the whole term is on the upper edge, the 'result' edge. *Picture by author.*

7 Puzzles of Noyes Chapman and Donald Knuth

The permutohedron is all about permutations, or what is the same, about simple positive braids. Thinking about permutations we remember the classical 15-puzzle of Noyes Chapman, that needs no explanation. There are also analogous puzzles for sale, namely $(n^2 - 1)$-versions for $n = 3, 5, 6, 7$, materially realized in wood or plastic. The strategy for their solution to restore the initial position $1, \ldots, n^2 - 1$ is well-known, by a repeated cyclic rotation of the numbers. For $n = 4$ there is a nice app available in the App store, if you have the right operating system on your iphone. (And not the obsolete one of the author.)

For our other jewel in this story there is another noteworthy puzzle. More interesting mathematically, at least less well-known, is Donald Knuth's *stack permutation puzzle*, that is directly pertaining to the as-

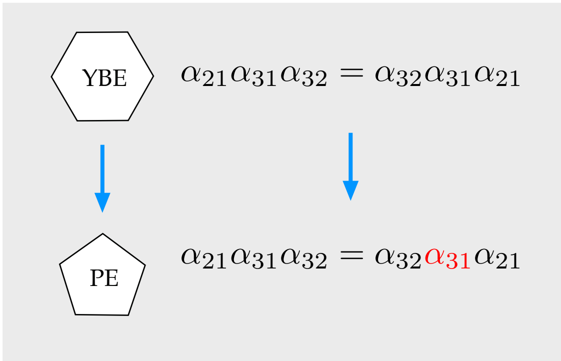


Figure 21: From hexagon to pentagon, yielding the embedding of associahedron into the permutohedron. The Yang-Baxter Equation YBE for the hexagon and the permutohedron yields the Pentagon Equation PE for pentagon and associahedron, by crossing out the middle generator in the righthand-side. This equation called (quantum) pentagon relation or equation is well-known in quantum theory, but also in incidence geometry, where it is called the Veblen flip. (See Adam Doliwa et al. [22].) *Picture by author.*

sociahedron and its embedding in the permutohedron. We looked in vain in the App store whether someone had already realized that puzzle as an app, but that is still an open opportunity for an industrious app-designer. The puzzle and corresponding theorem of Knuth refers to *pattern-avoiding permutations*, a subject pursued and well-studied in the art of combinatorics. There is even a sequence of satellite conferences or workshops dedicated to permutations during already two decades. In the case of Knuth's puzzle the patterns that are avoided are 312 and its inverse 231. The first pattern is avoided if the permutation at hand does not have a non-contiguous subword 'high-low-medium'. For instance the permutation 423 or 5672435 is not 312-avoiding; the offending pattern is painted in textcolor red. Note that the pattern needs not to be consecutive. See Nathan Reading's paper [61] about their emergence in the canonical embedding of associahedron into the permutohedron, well-known in the theory of Tamari-lattices and polyhedra, that we reconstructed (but only for the case $n = 4$) along the route of the equational rendering of the two jewels, to wit the Yang-Baxter Equation YBE and its degenerate form, the quantum pentagon equation

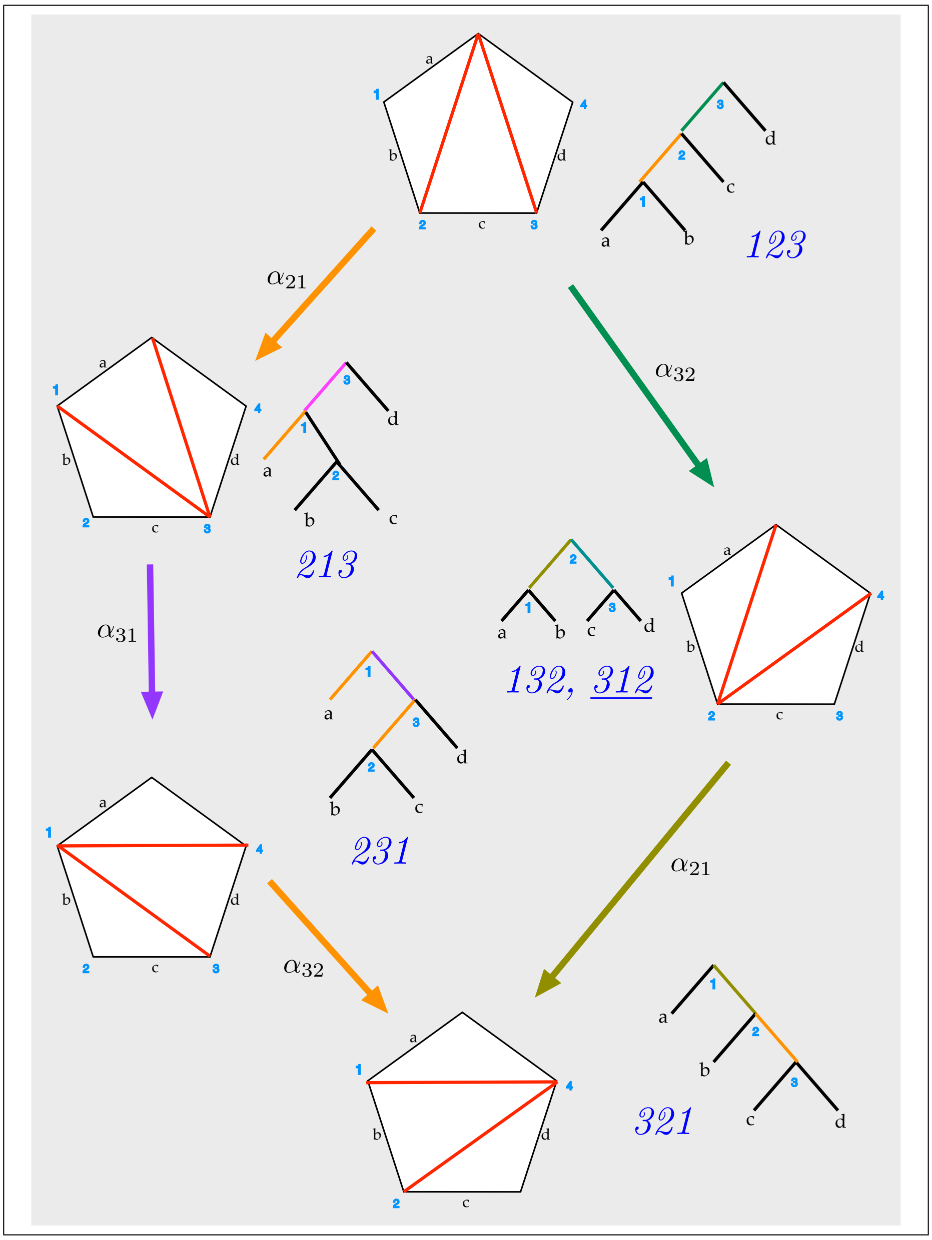


Figure 22: The pentagon A_4 of pentagons. The blue node numbers in the trees are corresponding in an obvious way to the way the pentagon cake is cut by the diagonals. The blue italics permutations $123, 213, \ldots$ are a record of how the cake is cut, in what order, or equivalently, how the tree is constructed. In one case there are two ways; of which one, 312, has the peculiarity that it contains a 'wrong' pattern, it is not 312-avoiding. *Picture by author.*

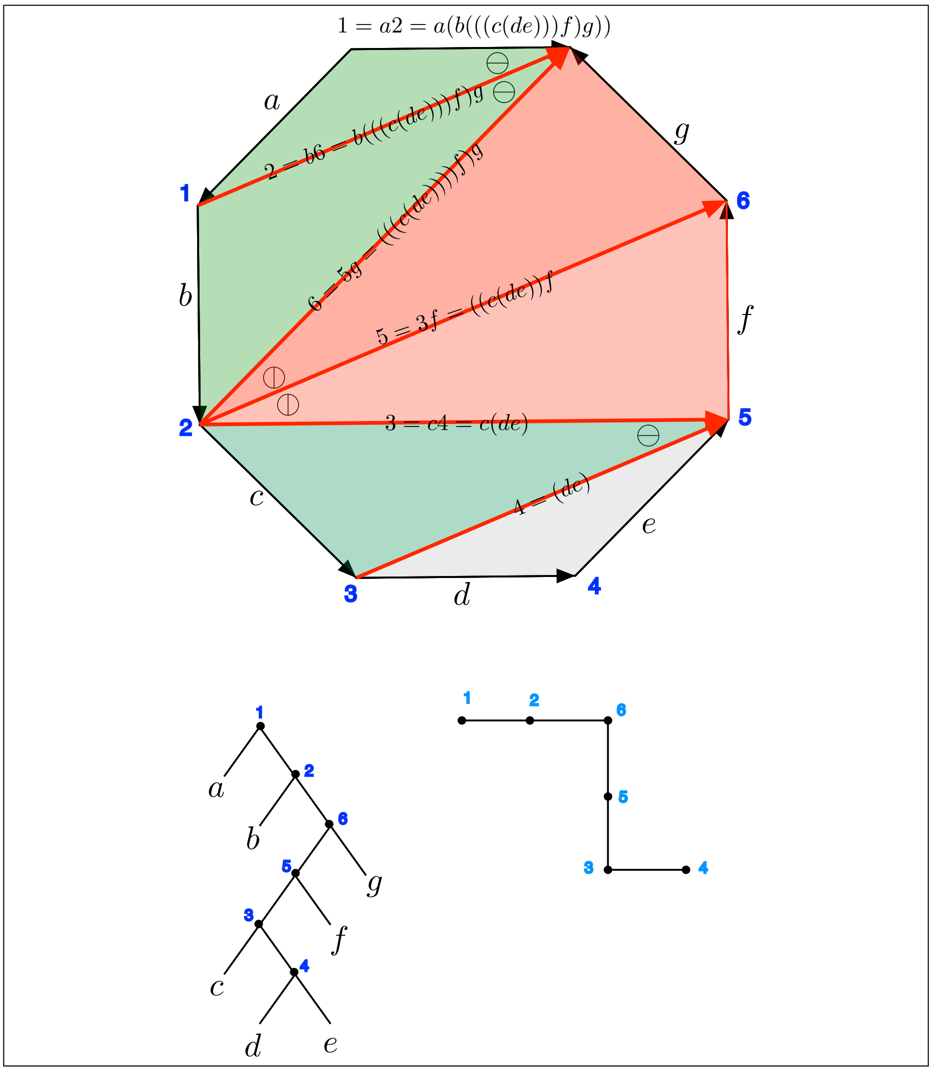


Figure 23: A regular octagon with non-intersecting diagonals and the ensuing parenthesized terms on generators $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}$. Note the direction of all arrows; all 'cells' are commuting, in other words all paths are homotopy equivalent. The binary tree is displayed corresponding to the bracketed term; its node numbers are inherited from the octagon vertices. Note that left is decreasing, right is increasing. Next to the binary tree is the condensed tree, by removing the constants (terminal symbols) $\mathsf{a}, \dots, \mathsf{d}$ and edges to them. It is drawn horizontal-vertical. The vertical edges are the 'redexes' with respect to tree rotation, or diagonal flip or bracket shift. The condensed tree can be even given without the node labels; these can only be in a unique way restored satisfying the h-v- restraint of increasing-decreasing. The form of the condensed tree is easily recognizable in the cuttings of the octogonal cake. The green pieces are the horizontal edges, the red parts the vertical redexes. The grey part down is an artefact by cutting away to the condensed tree. The condensed tree is a snapshot of an intermediate stage of sorting vertical sequence $6, \dots, 1$ to horizontal ordered sequence $1, \dots, 6$. Another example of this sorting process is in Figures 25, 26. *Picture by author.*

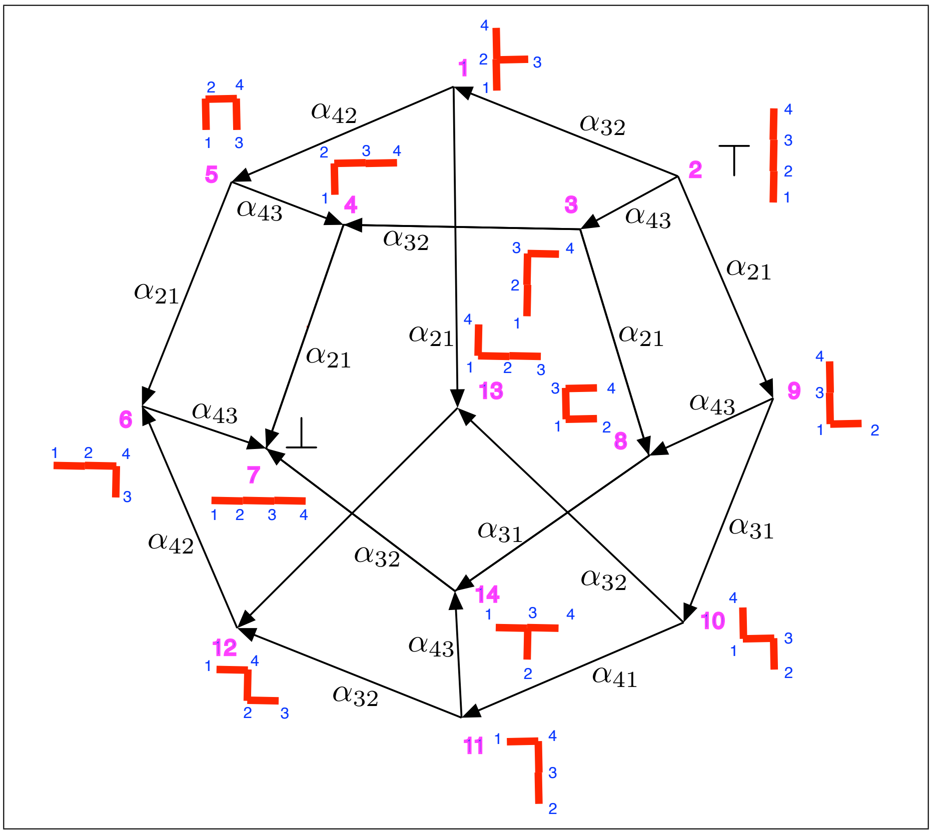


Figure 24: The associahedron A_5 as a matchstick game. The binary trees on a, b, c, d, e can be written in a condensed form, by omitting the constants and the edges leading to these end nodes. Even the node numbers can be omitted (not in this Figure), because the h-v-invariant, stating that horizontally numbers increase and vertically numbers decrease, forces a unique possibility of restoring the numbers $1, 2, 3, 4$ at the nodes. Thus we have a game of abstract forms. Btw, note that the forms are generated only by steps to the right and steps down; so some mirror images do not exist. Note that the associahedron is a two-dimensional sorting process from the vertical decreasing position at the top to the horizontal increasing position at the bottom $\perp$, vertex 7. Remarkably, the sorting is faster than the permutohedron does in 6 steps; here it is only $3, 4, 5$ steps, one can choose. It would be interesting to sort out this comparison for general n. See also the bigger example in Figure 25. *Picture by author.*

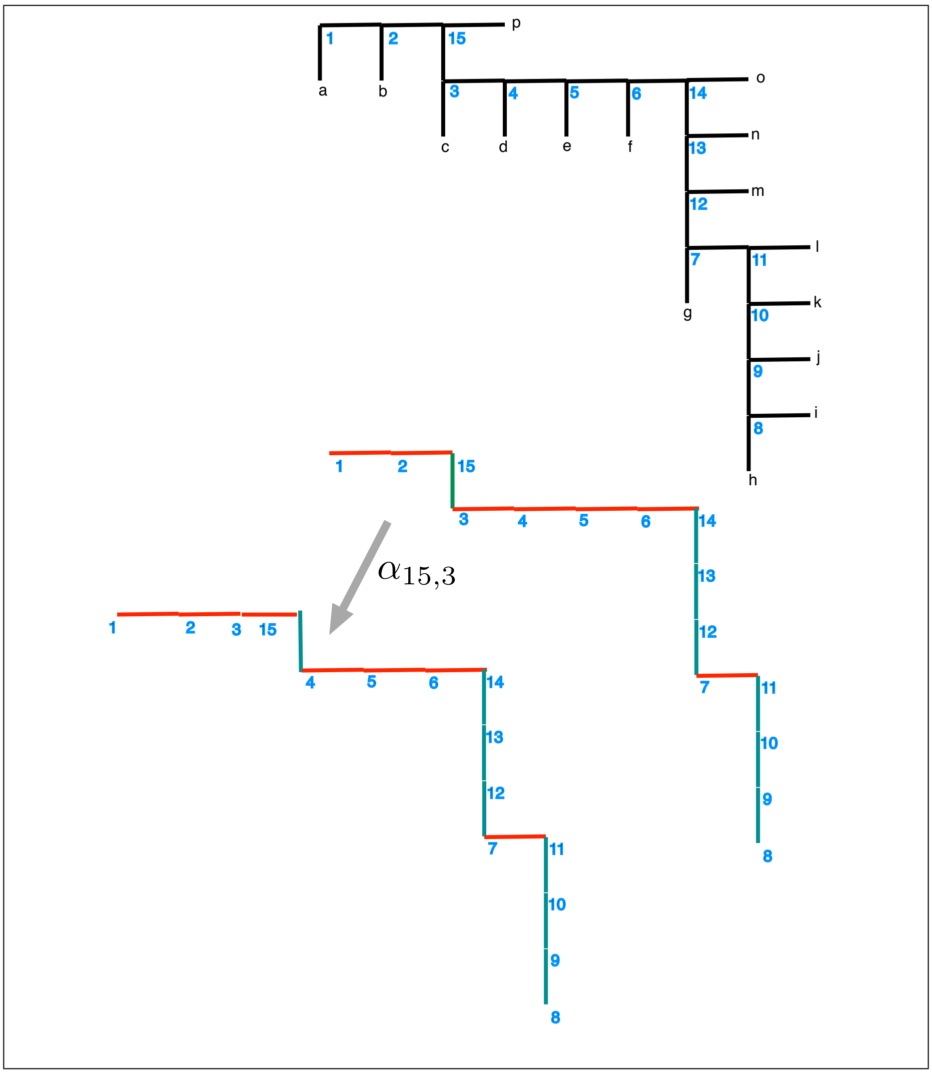


Figure 25: A tree (a vertex) on the associahedron, in this Figure A_{16}, corresponds with a snapshot of how far the sorting of the initial numbers $15, \ldots, 1$ has proceeded. In horizontal direction the numbers are already sorted in the correct increasing order; in the vertical stretches they are still in the wrong, decreasing order. Each redex contraction flips one wrong pair to its correct order. This horizontal-vertical property (h-v-property) of horizontal increasing, vertical decreasing is an invariant under tree rotation moves. *Picture by author.*

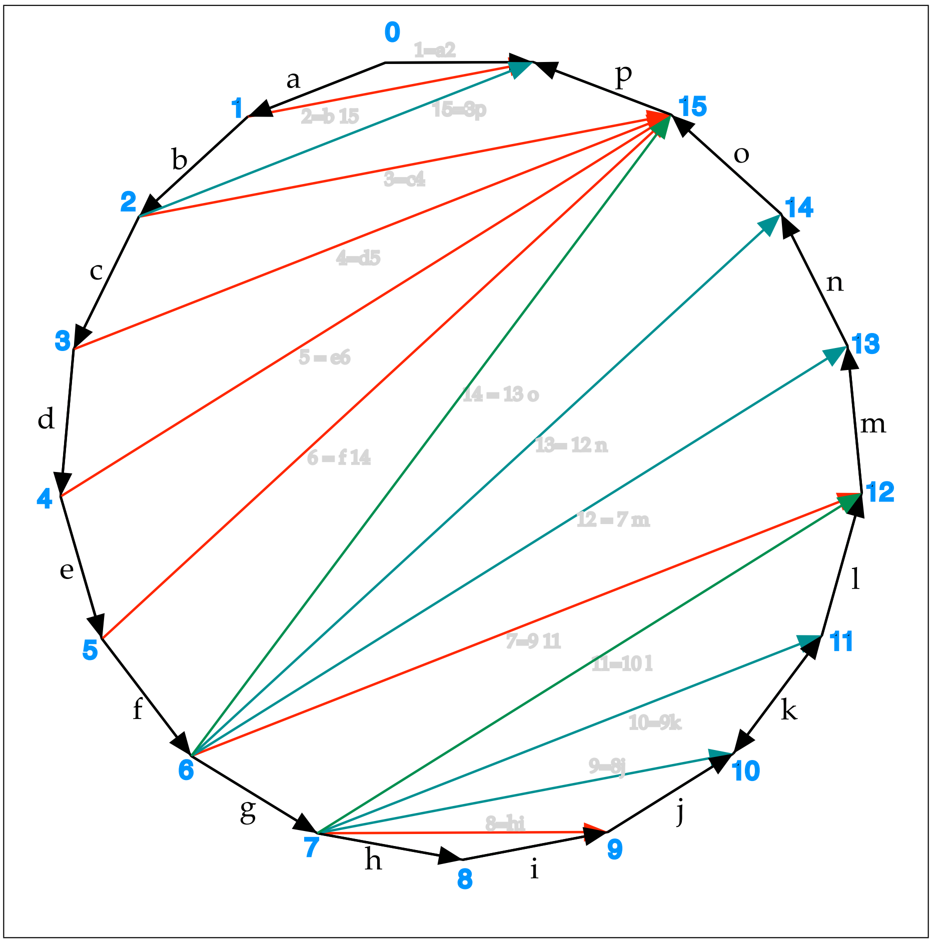


Figure 26: Cutting the 17-gon cake for Carl Friedrich Gauss. The cake cutting corresponds with Figure 25, where this cutting is presented as binary tree, first in full, middle in condensed form, below after the indicated redex contraction $\alpha_{15,3}$. The 7 green diagonals are the 'redexes', i.e. flippable; they correspond with the 7 green down edges in Figure 25, middle tree. See also Figure 23. *Picture by author.*

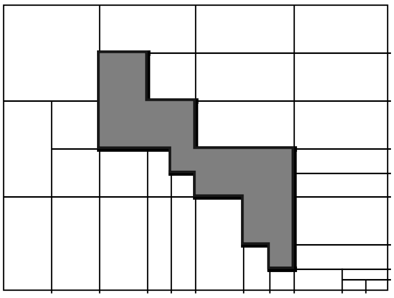


Figure 27: In reduction diagrams of orthogonal rewrite systems such as λ-calculus, Combinatory Logic or other orthogonal term or string rewriting systems, a homotopy feature is already present, in particular corresponding to the crucial notion of Lévy-equivalence. Namely, parallel reduction paths inside a reduction diagram, composed of cells that are otherwise known as elementary reduction diagrams (e.d.'s) in Klop 80 or Terese 03, the two paths are Lévy-equivalent, and indeed homotopy equivalent with respect to flipping the e.d.'s. *Picture by author, copied from p.118, Figure 4.21 in Terese 03.*

(a) All marbles at the sphere are in their place

(b) The hole in the sphere without marble

Figure 28: Spherical fidget: an analogon to the 15-puzzle of Noyes Chapman.

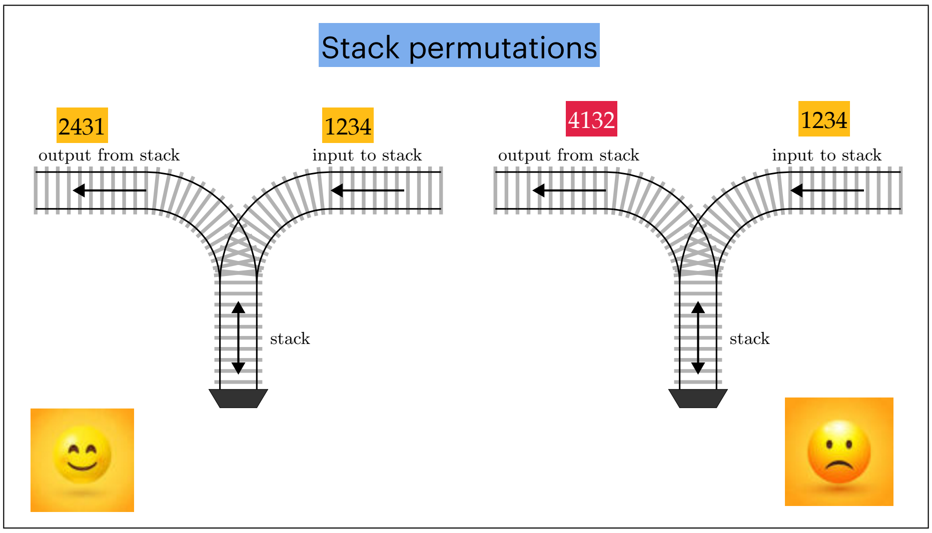


Figure 29: Some permutations can be realized by shunting as in a railway station, using a stack with the usual *pop* and *push* actions; but some permutations are not stack realizable. *(The basic railway figure, inspired from a similar one in Knuth [50], Section* 2.2.1*, p. 236, is tikzed by J. Endrullis; the adornations are from the author.*

PE.

An example of two instances of Knuth's puzzle, one positive, one negative, is depicted in the Figure 29. Here as in Knuth [49], the puzzle is depicted in the format of a train-shunting-puzzle. The figure is inspired by the one in Knuth [50], redrawn in the wonderful tool *tikz* (but not by the present author). See Footnote [5] .

The notion of 312-avoiding permutation is also present in Knuth's theorem in [49], p.239, section 2.2.1, Exercise 5 as follows. We quote Knuth's wording verbatim, except for the phrase about 312-avoiding.

Theorem 7.1. *(Knuth [49])*

1. *Show that one can obtain the permutation* $p_1 p_2 \ldots p_n$ *from* $12 \ldots n$

[5] *Caveat:* in Figure 29 the upward arrow in the stack might be slightly misleading, by suggesting that the stack content may be moved upwards back to the input. That is not allowed, it may be moved upwards but only as a *pop* action to the left, the output area. So we must not move cars against the right-to-left arrow displayed in the input area. Otherwise we could compose every permutation! Come to think of it, that would not be a bad idea for the railway shunting supervisor...

using a stack $\iff$ *there are no indices* $i < j < k$ *such that* $p_j < p_k < p_i$. *In other words, the permutation is* 312*-avoiding.*

2. *p.329, Exercise* 6*: Suppose that a binary tree has* n *nodes which are* $u_1u_2 \dots u_n$ *in preorder and* $u_{p_1}u_{p_2} \dots u_{p_n}$ *in inorder. Show that the permutation* $p_1p_2 \dots p_n$ *can be obtained by passing* $12 \dots n$ *through a stack , in the sense of the previous part 1 above.*

3. *Conversely, show that any permutation* $p_1p_2 \dots p_n$ *obtainable with a stack corresponds to some binary tree in this way.*

Proof. For the notions of *preorder* and *inorder* for binary trees see the cited book on page 317.

1. 1. See Solution in Knuth's book on page 533.

2. 2. See solution in Knuth's book, page 559.

3. 3. See solution in Knuth's book on page 560.

□

Rumination. How about other orders on permutations than the present one, often called the *weak order* on permutations? One other ordering is depicted in Forcey [34]. The permutohedron then obtained has the same skeleton, but maintaining the α_{ij} notation, a different labeling of the edges; the structure then obtained is not isomorphic to the weak order, not even bisimilar, when these structures are viewed as processes. There are other Bruhat-orders on permutations that could be considered; we do not know if these have been studied or compared as to what polyhedra they generate. Given the impressively rich culture in combinatorics and geometry of groups we would expect that there is knowledge about that digressing path.

Another thought in that direction: what happens with the permutohedron if we consider cyclic permutations, of which the end coincides with the beginning? So they are strings of pearls as it were. Also these are of course well-known in combinatorics. Then the P_4 would collapse considerably; its 24 vertices $1234, \dots, 4321$ would reduce to the following. (We postpone their determination to a leasure moment in the future.)

Fantasizing is free. What if the swaps are as on the spherical fidget puzzle that grand-daughter Charlène showed the author? What is the polyhedron then arising, with as vertices the different configurations of the marbles. Also this is a sorting puzzle, with a topological flavour. See footnote [6] and footnote [7] .

8 A glimpse of the homological/homotopical perspective of rewriting

Consider C_3, the Cube in 3 dimensions, with its edges oriented as in Figure 7. We can view this structure as an Abstract Reduction System (ARS) with the corner points of C_3 as objects and the arrows labeled $1, 2, 3$ as reduction or rewrite relations. This ARS is complete, that is, confluent (CR) and terminating (SN). Cube C_3 can be seen in forward direction of its arrows as a process changing the initial object 1 to the final object 8 , the single normal form, in various ways, that actually can be enumerated as the permutations $123, 213, 132, 231, 321$. We note that the three reductions $1, 2, 3$ are commuting relations, satisfying the squares as in Figure 7. Figure 32 is similar but one dimension higher, displaying the tessaract.

There is also another reading of these elementary diagrams, when we observe the change of the sequence of steps, or the words if you want. This perpendicular reading is the starting point of a homological/homotopical view and development of rewriting, corresponding to 2-categories where the objects are paths.

In logic, mathematics and computer science we are accustomed to apply 'reflection' or application of a notion on itself, and so we have sets and sets of sets and sets of sets of sets, even *ad infinitum.* Likewise we have functions describing change, but also describing the change of change and so on, in the form of derivatives. And we have next to

[6]This puzzle is one of a class of *sliding puzzles*, played on arbitrary finite graphs. The nontrivial analysis of that family has been considered by some leading experts in graph theory and combinatorics. See [73], [75] for an analysis of positions for which a solution exists, and on which graphs.

[7]The mechanism realizing this spherical sliding puzzle is surprisingly simple; it consists of an elastic sponge inside that absorbs the marble that is pushed inwards, and pushes it out to fit in the adjacent open hole.

morphisms in category theory morphisms of morphisms and so on. So it is not surprising that we have starting with term or string rewriting also rewriting of rewriting and rewriting of rewriting and so on. In the French school of homotopical rewriting these higher levels of rewriting have been captured in a nice notation, using the single arrow ($\rightarrow$) for rewriting on the ground level, double arrow ($\Longrightarrow$) for the first level, triple arrow ($\Rrightarrow$) for the second level, and even four-fold arrow which I don't know how to latex. It is displayed in Figure 53, copied with kind consent of its authors.

The permutohedron and related structures obtained by monoid presentations lend themselves for discussion of a relatively recent perspective in term rewriting, namely the perspective of *homotopical* or *homological* rewriting. In the world of typed lambda calculi and their impact on proof verification there is the emergence of homotopical type theory, boosted by the univalent axiomatics programme of the late Vladimir Voevodsky. We cannot not dwell on that subject, if only out of ignorance, but look at the restricted area of term rewriting, in particular string rewriting.

The consideration of homotopy and homology pertaining to string and term rewriting dates back some three decades, with the emergence of higher dimensional rewriting, connected also with higher order categories. See Burroni [8]. A milestone here was the work of Craig Squier, with his notion of Finite Derivation Type (FDT); see Squier [66]. The last decade has seen some fascinating developments of several French rewriters pursuing this direction. For the present author these subjects are viewed from a distance with awe and admiration, like one can admire and be impressed by images of the Himalayas without having the experience of actually having climbed them, nor the ability.

See footnote [8] .

See endnote [2] .

[8]In the current literature this combination CR & SN is called *complete* or *convergent*.

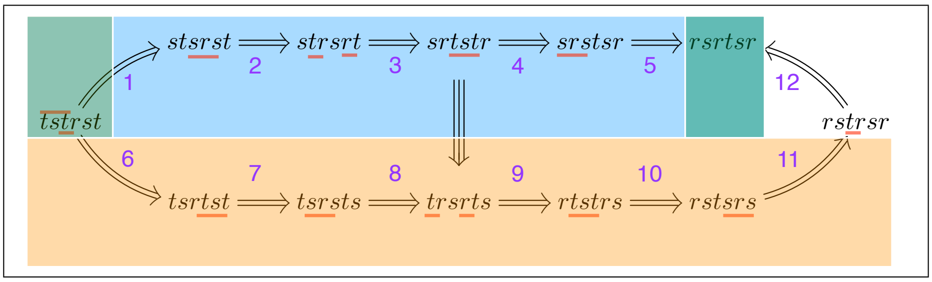


Figure 30: The Zamolodchikov cycle on the permutohedron P_4, presented in [39] as the homotopical completion of B_4^+, the monoid presentation underlying the permutohedron P_4. The oval cycle is from [39], the colors and underlining are ours to match the figure with Figure 31. The green fields contain respectively the begin term with overlapping redexes, and the end term of the two diverging reductions. The symbols r, s, t are in Figure 31, respectively, $3, 2, 1$. Note in steps 3 and 9 the simultaneous contraction of the two squares. Note also that we have here an instance of Knuth-Bendix completion of the initial overlap. (In fact, just as in the YBE).

9 The permutoassociahedron

There is a large treasure chest of generalized polyhedra, cyclohedra and related structures. The important reference is the Tamari volume [57]. Another jewel mentioned as a 'mystery' source in that comprehensive source book (p. 124, Ceballos-Ziegler [10]) is the *permutoassociahedron*, which can be viewed as a common extension of the two main jewels in this note, as the name indicates. Just as in the associahedron we consider in this structure parenthesized terms involving generators a, b, c, d, but in addition one is allowed to commute consecutive pairs of the letters. Thus we have as defining equations

$$\begin{array}{|ll|} (xy)z & = x(yz) \\ ab & = ba \\ ac & = ca \\ ad & = da \\ bc & = cb \\ bd & = db \\ cd & = dc \end{array}$$

Note that we do not have commutativity on all terms, only the pairs

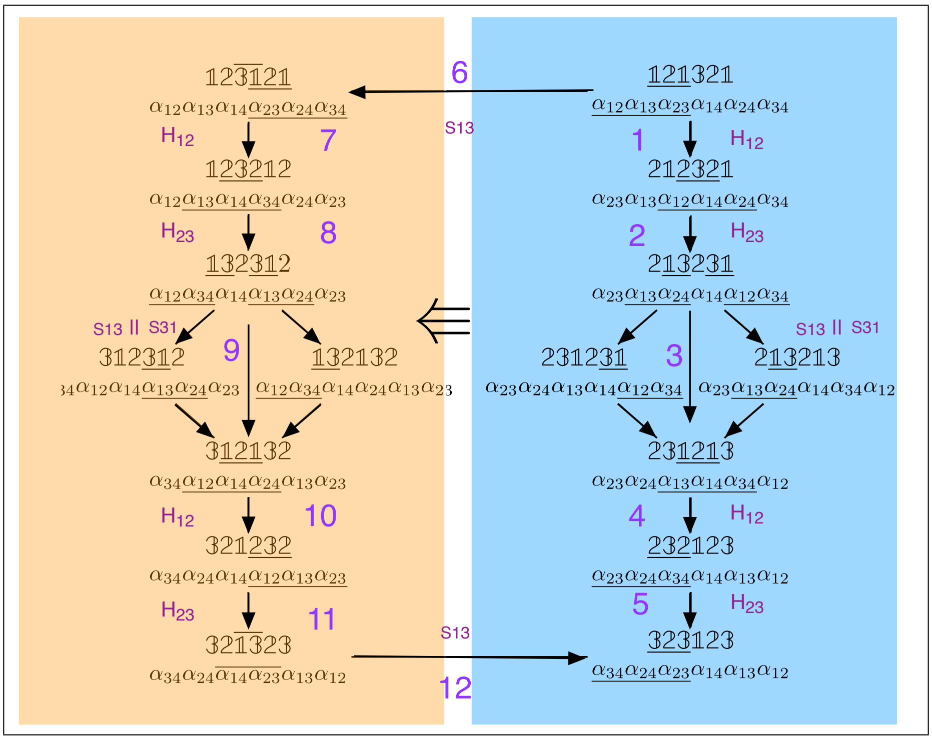


Figure 31: This Figure of the Zamolodchikov cycle on the permutohedron occurred basically in [28], without colors and the homotopy triple arrow, here also with annotated purple numbers $1, \ldots, 12$ synchronizing it with those in Figure 30. The terms in the green field are the begin term with overlapping 'redexes', and the end term $\mathtt{rsrtsr}$ of the two divergent reductions.

of generators. The associativity rule does hold for all terms that match the equation. If we orient these rules, say from left to right, we can determine the reduction graph of all reducts starting from the initial term $a(b(c(de)))$, just as for the associahedron on 5 letters. This reduction graph is the permutoassociahedron, consisting of 120 points. On 3 letters a, b, c it is the dodekagon as in Figure 39.

It was some 40 years ago analyzed by M. Kapranov [44]. Its importance resides in the foundations of category theory, extending the theorem of Mac Lane that we mentioned in Figure 15. The permutoassociahedron is also figuring in recent theories in theoretical physics as

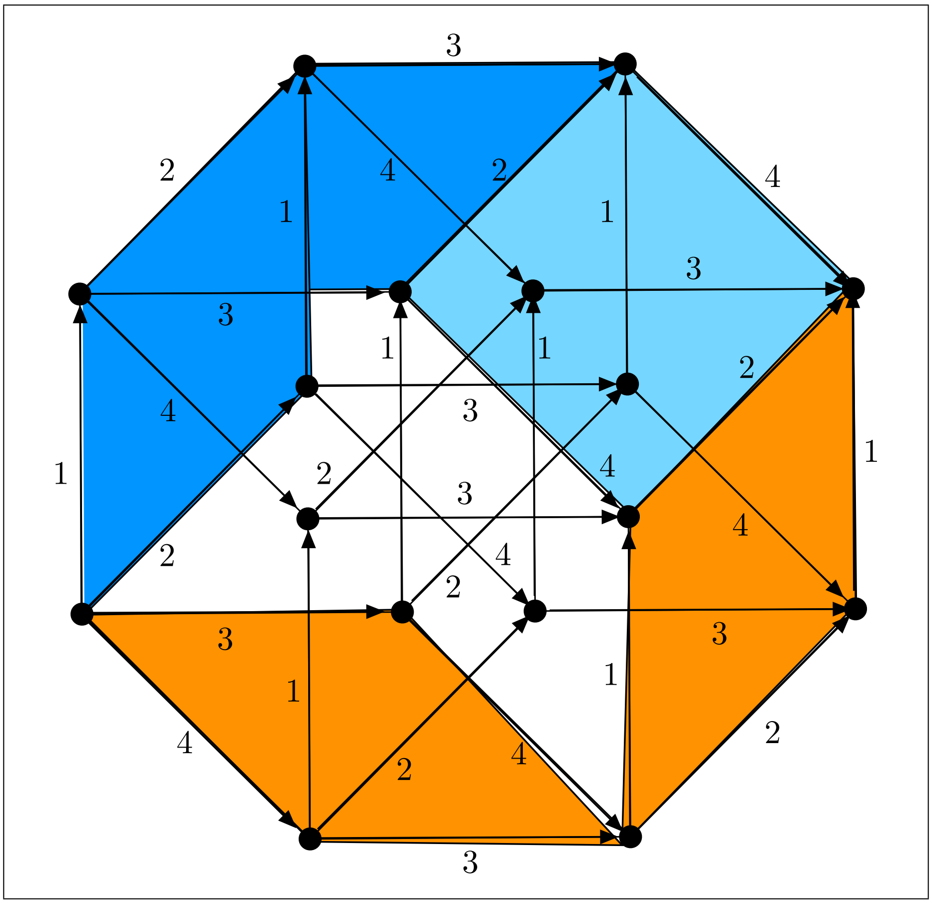


Figure 32: The 4D cube, the tessaract. The skeleton, with edges labeled $1, 2, 3, 4$ can be seen as an abstract reduction system, which is confluent (CR) and terminating (SN). It also displays the conversion of the word 1234 into the word 4321 in successive swaps. The swaps are elementary reduction diagrams, each of which is a transposition of two symbols. The partial order of the process of filling up these 'cells' yields the permutohedron P_4 of order 4. Displayed as colored cells are the transpositions $\alpha_{12}, \alpha_{23}, \alpha_{24}, \alpha_{43}, \alpha_{21}$, present around the 'North pole' of P_4. *Picture by author; it should be redone in tikz!*

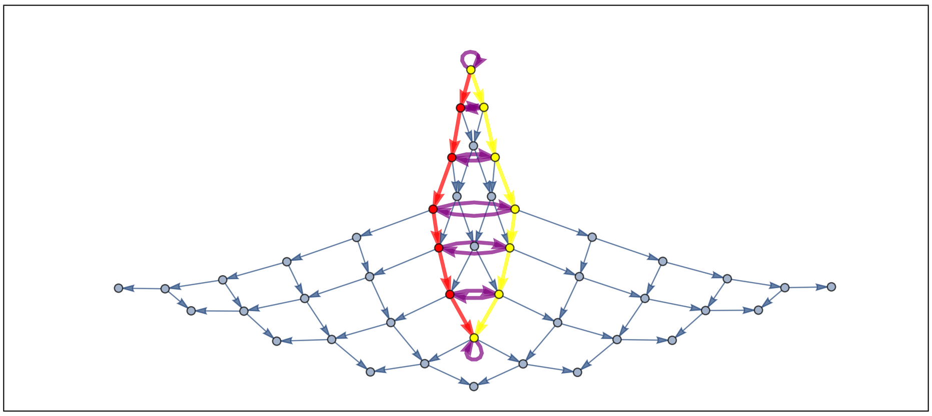


Figure 33: A visual impression of the *homotopy equivalence relation*, purple, in the grey-blue gridlike underlying space. How can the red and yellow path in this space be transformed into one another by nudges via intermediate cells in this space? *(This Figure is from Xerxes Arsiwalla et al., [2] [3], with permission.)*

in quantum gravity theory. For a link to that area of investigation see the paper [64] of M. Sheppeard, a paper that contains the Figure 42 included here. The author Marni Sheppeard deceased only a few years ago in tragic circumstances in the mountains of New Zealand. Her memory is honored by a website devoted to her life and work, maintained by some of her former coworkers.

The permutohedron figures in the homotopical completion of the cube. Is the associahedron also figuring in the homotopical completion of some structure? A degenerate cube maybe? And the permutoassociahedron?

The permutoassociahedron is also interesting for term rewriters. Presumably the permutohedron is embeddable in this structure, by collapsing the steps due to commutativity of the generators $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$. Then the associahedron is embeddable in it too, by transitivity. Furthermore it is likely that the elementary diagrams embodying this structure are *decreasing diagrams*, thus ensuring that tiling with these e.d.'s always will converge in finite reduction diagrams. Presumably this structure is also a Garside monoid?

We conjecture that the Cube Equation holds, so that the theory of *abstract residual systems* applies, delivering a host of syntactical results

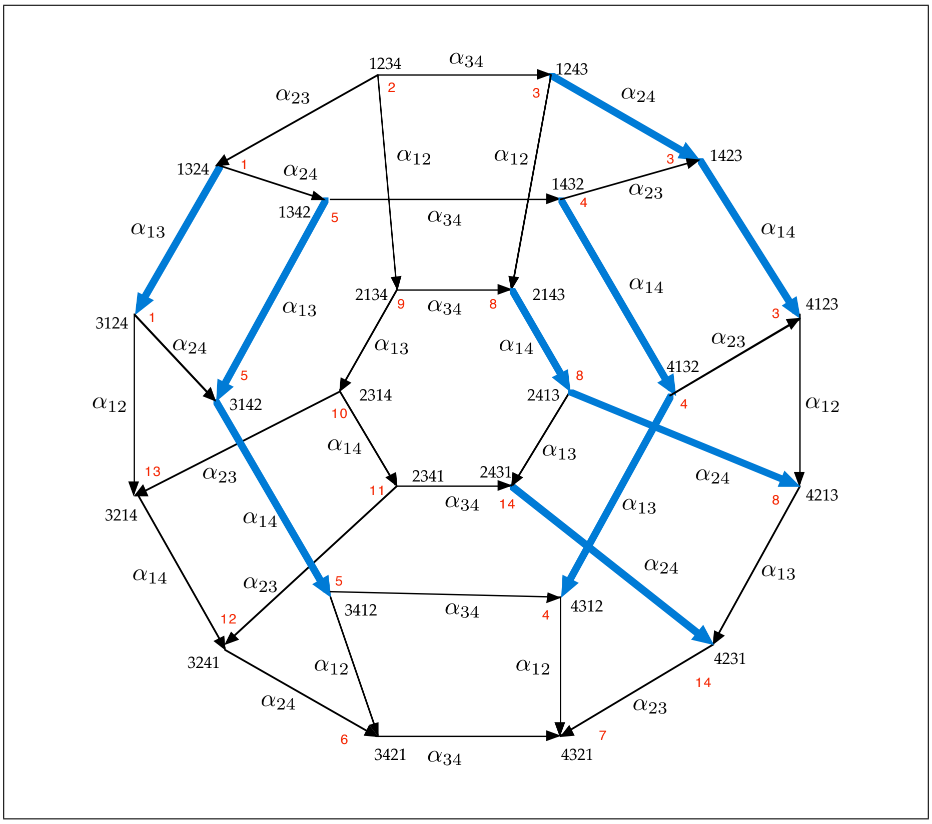


Figure 34: The permutohedron P_4 as seen from above, with the ten edges to be collapsed for the embedding of A_5 into P_4. The red numbers $1, \ldots 14$ indicate the vertices of A_5 that are mapped to that place in P_4. *Picture by J. Endrullis and author, inspired by Figure* 10*, p.*201 *in Stefan Forcey [34]; there with another ordering than the* weak order *used here on permutations.*

and lattice properties. Residual systems originated in the λ-calculus, in particular from Theorem 1.1 stated and proved on pages 120, 121 of Curry-Feys Vol.I.

Another interesting question is on a meta-level: the presentations of permutohedron and associahedron are so to say abstractions of those for the permutoassociahedron. Do we get a lattice homomorphism and a lattice congruence as for the case of the embedding of associahedron into permutohedron? What are the equations analogous as for permu-

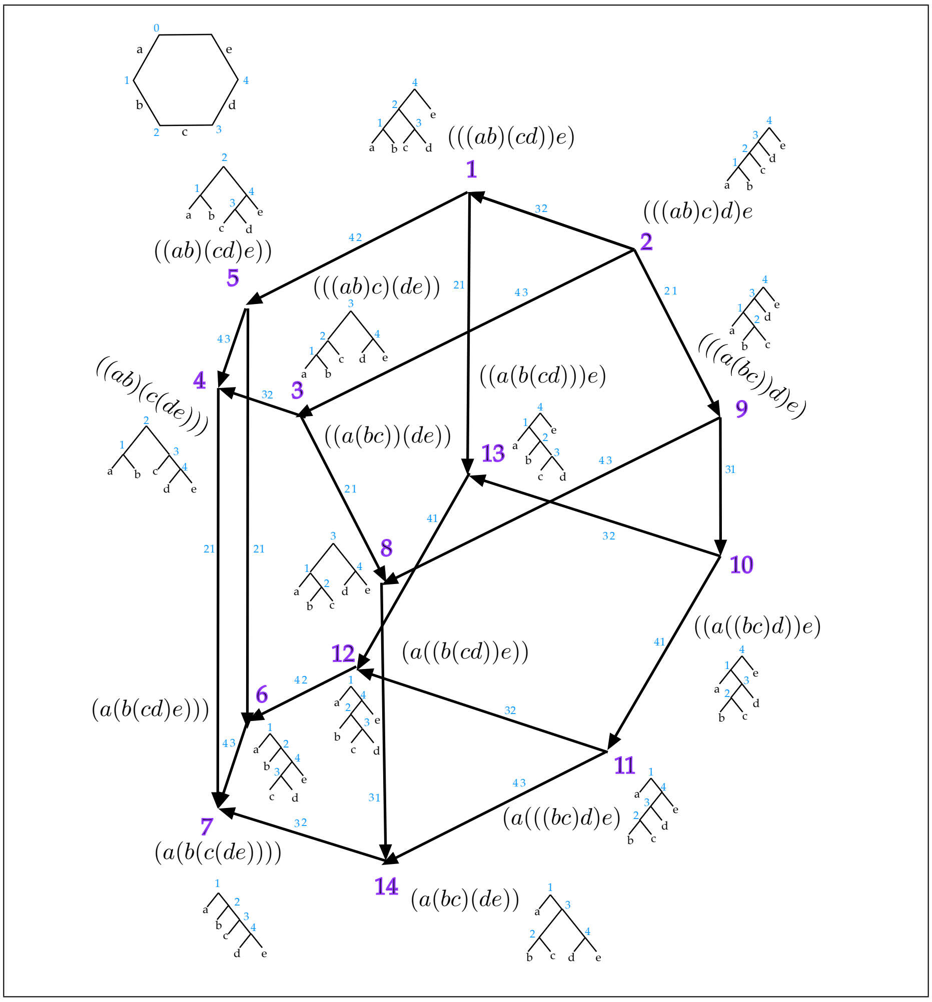


Figure 35: The associahedron A_5 with parenthesized terms and corresponding binary trees. The hexagon upper left with the blue numbers 1, 2, 3, 4 interleaved with the letters a, b, c, d, e is responsible for the numbering of nodes in the binary trees; we call it the canonical node numbering. The double digits 32, 21, 43 stand for the generators $\alpha_{32}, \alpha_{21}, \alpha_{43}$ and indicate the tree rotation, i.e. the edge that is flipped, between the vertices 1, . . . , 14. *Picture by author.*

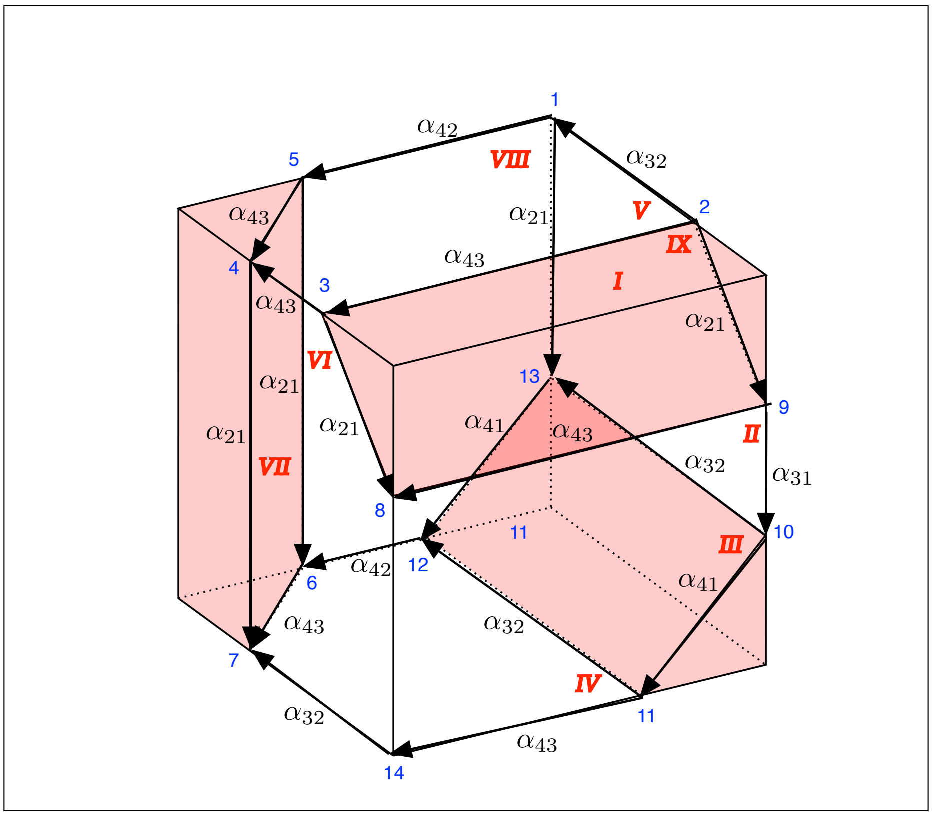


Figure 36: As is well-known, the associahedron A_5 can also be rendered as a truncated cube, by removing the three prisms as indicated. The capital Roman numbers I, ... IX enumerate the nine faces of this enneahedron. *Picture by J. Endrullis and author.*

tohedron and associahedron?

See Endnote [3]

10 An equational realization of the permutoassociahedron

The permutoassociahedron was discovered and realized thirty years ago, by Kapranov answering a question of MacLane and Stasheff. This beautiful jewel is also in recent years receiving much scrutiny. Its realization

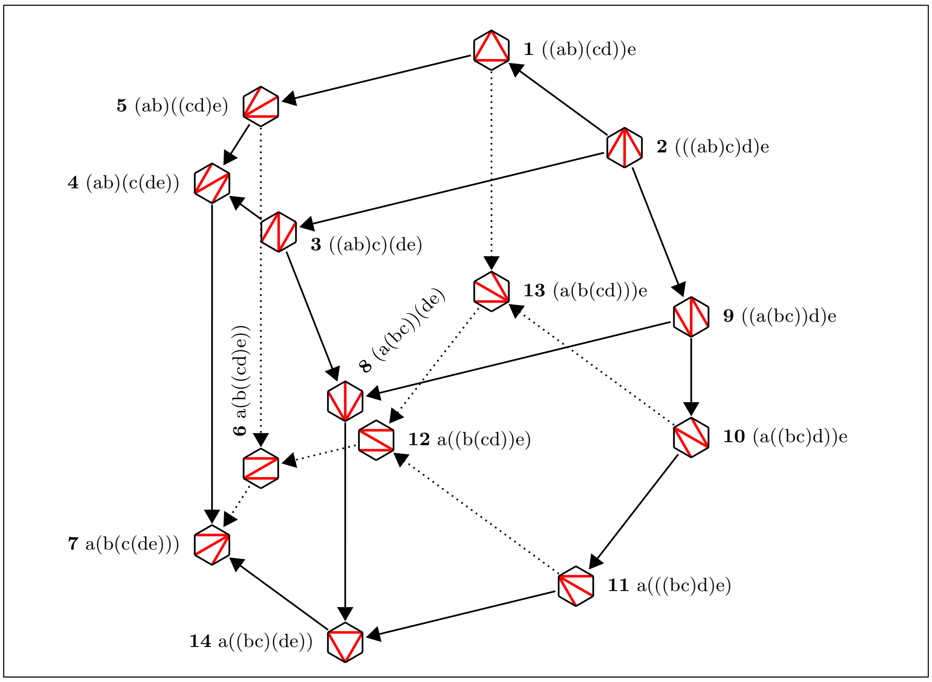


Figure 37: The associahedron A_5 with the parenthesized terms over a, b, c, d, e and the corresponding hexagon triangulations. *Picture by J. Endrullis.*

was achieved from various directions, according to Ceballos-Ziegler [10], among them the realization as convex closure of a set of points in R^n and as solution domain of a set of linear inequalities. Three other methods are mentioned by Ceballos-Ziegler [10], p.$120, 121$.

In this paper we have for the permutohedron and the associahedron endeavoured to find a realization from another direction, namelijk a purely *equational* one, where each face is an elementary diagram as in [46], and Terese 2003, [7], aka cell in homotopy rewriting. Thus we have discussed the YBE for the permutohedron, and a so-called degenerate form of YBE, known as the pentagon equation PE. A useful intuition here is to have a change of view (in Dutch blikwisseling) and view these bodies described by these equations as *processes*, performing actions such as transpositions or tree rotations. We can then abstract away some of the generators into the silent move τ of Milner's CCS and ACP, and

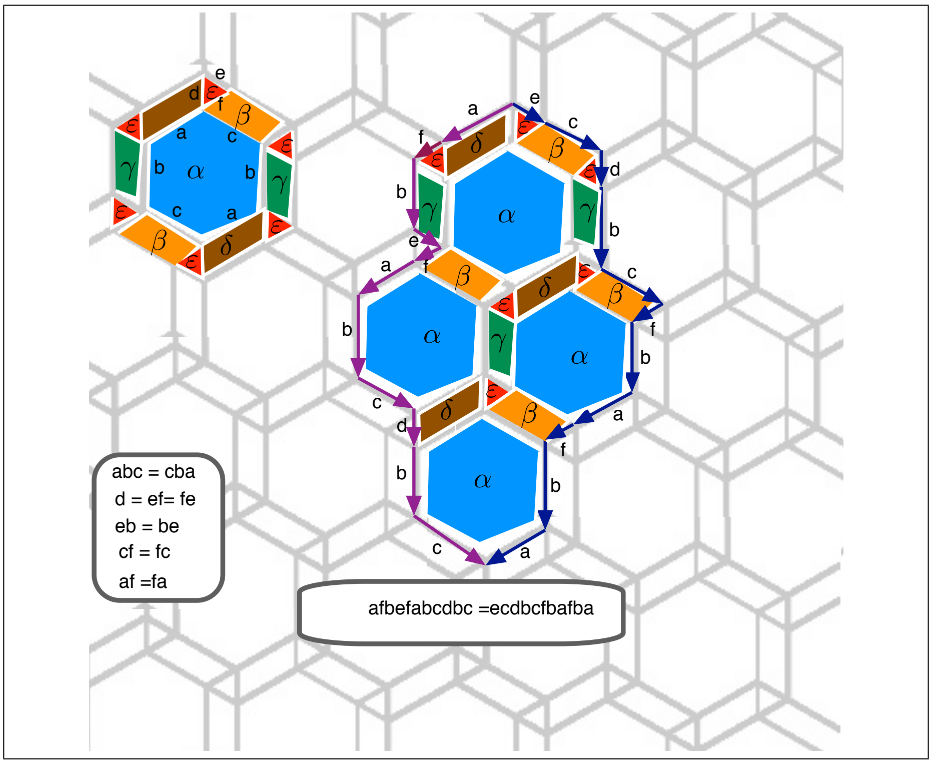
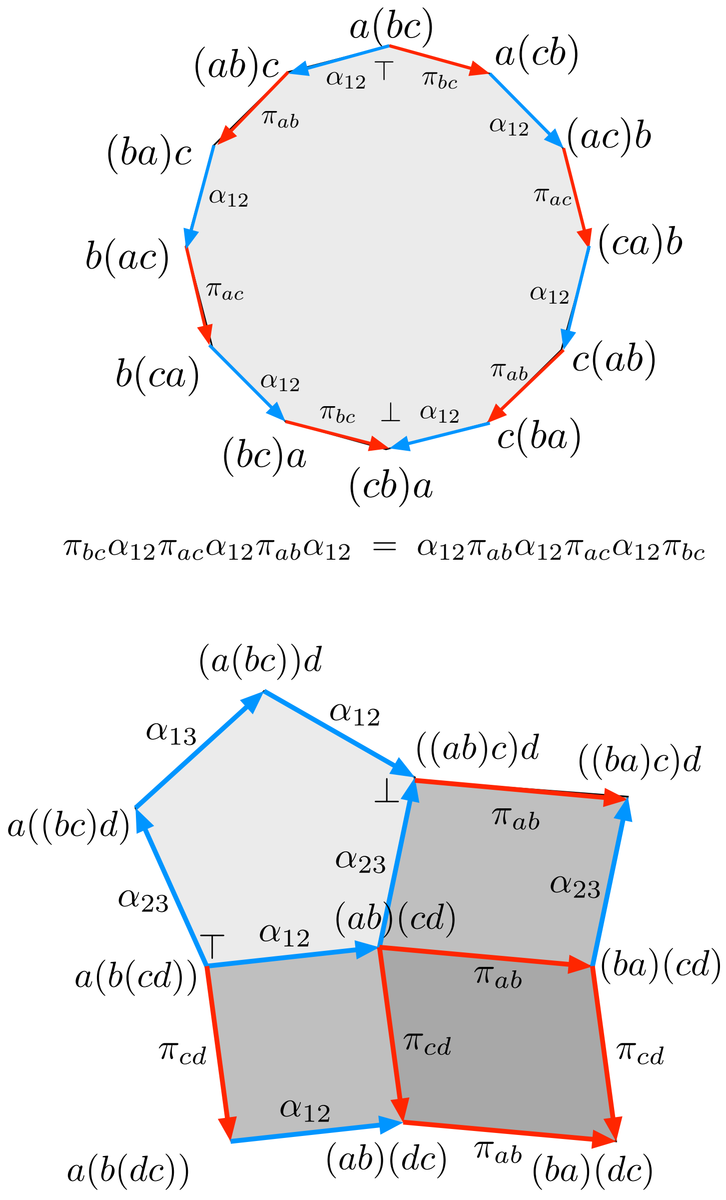


Figure 38: An example of the homotopical equivalence relation in a simple setting. All arrows are pointing downwards, and satisfy in the underlying chainmail-like space the five equations in the cadre. (Chainmail in Dutch = maliënkolder.) The two displayed paths are homotopy equivalent because they can be nudged towards each other by flipping the five colored cells $\alpha, \beta, \gamma, \delta, \varepsilon$. This example only gives a taste of the homotopical equivalence relation; in actual homotopy rewriting completion the underlying Squier space, aka Serre space, is more complicated, having words as points in the space and steps by a monoid presentation, viewed as rewrite relation. *Picture by author.*

invoke notions of *process simulation*, whereas current lattice theory uses the notion of embedding.

Another link with computer science, in particular with λ-calculus and the ensuing residual systems, originated by Curry, Lévy, Huet, Melliès, van Oostrom, is that these equations lend themselves for a check whether the cube equation CE holds; also here we have a direct connection with Garside theory created by Dehornoy and coworkers. A crucial technical



$$\pi_{bc}\alpha_{12}\pi_{ac}\alpha_{12}\pi_{ab}\alpha_{12} \;=\; \alpha_{12}\pi_{ab}\alpha_{12}\pi_{ac}\alpha_{12}\pi_{bc}$$

$$\alpha_{23}\alpha_{13}\alpha_{12} \;=\; \alpha_{12}\alpha_{23}$$

Figure 39: The three types of cells (faces) of the permutoassociahedron KP_4: dodekagons (12-gons), pentagons, squares. Note that the 12-gon yields a hexagon after suppressing the blue steps of the associahedron, i.e. the associativity steps. This large cell has a palindrome equation. Top and bottom are marked, also in the pentagon, that is subject to the pentagon equation PE. There are squares in two flavours, the ones with only red permutation steps, and the red-blue mixed ones. We expect that the cells (e.d.'s) are decreasing diagrams, as they are (semi)-palindromes, and there seems not to be a cyclic dependence for all the α- and π-generators, as can be seen by a suitable ordering from top to bottom of the whole permutoassociahedron. Note that the big cell, the dodekagon, is again arising by Knuth-Bendix completion. *Picture by author, after similar uncolored and unlabeled pictures of this 12-gon and the pentagon fragment on the web.*

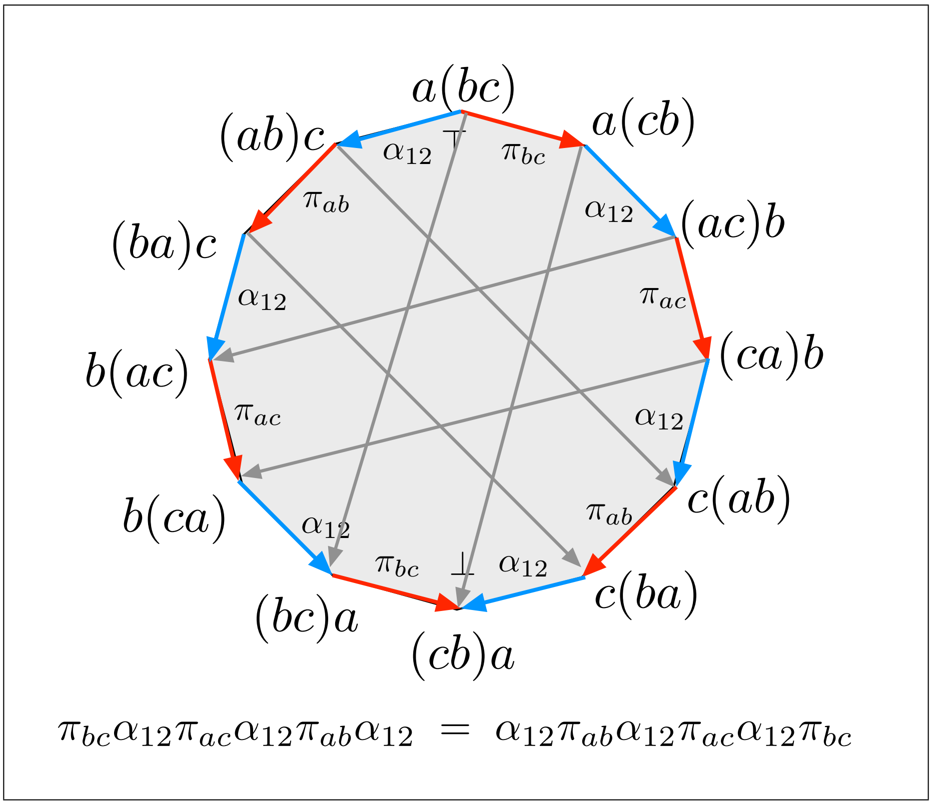


Figure 40: The commutativity equations imposed on the permutoassociahedron pertain only to applications of the generators, and not on all terms. For the general commutativity equation $xy = yx$ the reduction graph, the skeleton of the permutoassociahedron, becomes much more complicated; for the dodekagon cell we then have 6 more reduction steps, drawn in grey. This extension to general commutativity of the application operator may be 'spurious complexity', or it might be an interesting exercise in term rewriting to determine various properties. Actually, these more general commutativity steps are used by Mac Lane [53], page 40, in showing the coherence for the face diagrams of the permutoassociahedron; see his diagram in our Figure 43. *Picture by author.*

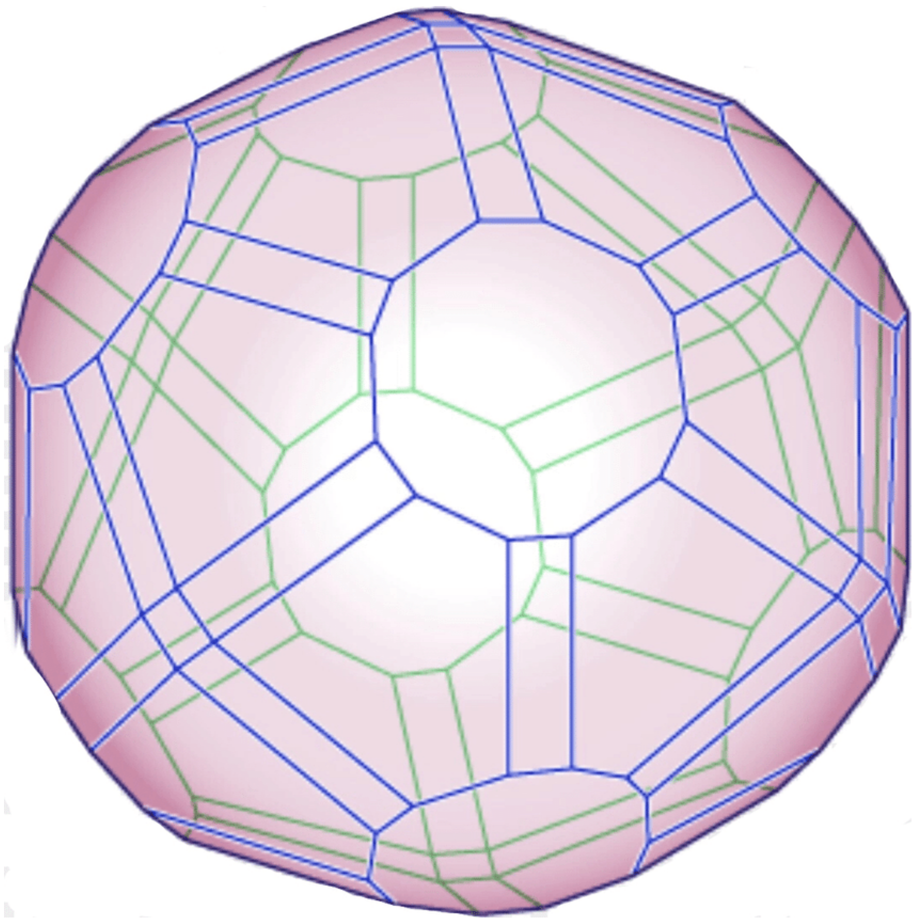


Figure 41: This picture shows the permutoassociahedron KP_4, whose 120 vertices are the parenthesations of the 24 permutations of 1, 2, 3, 4 subject to associativity as in the associahedron and transpositions as in the permutohedron. *This beautiful picture is copied from Marni Sheppeard [64], page 9. Due to her untimely death two years ago, I could not ask for consent for including this picture. It was found by googling 'permutoassociahedron', images.*

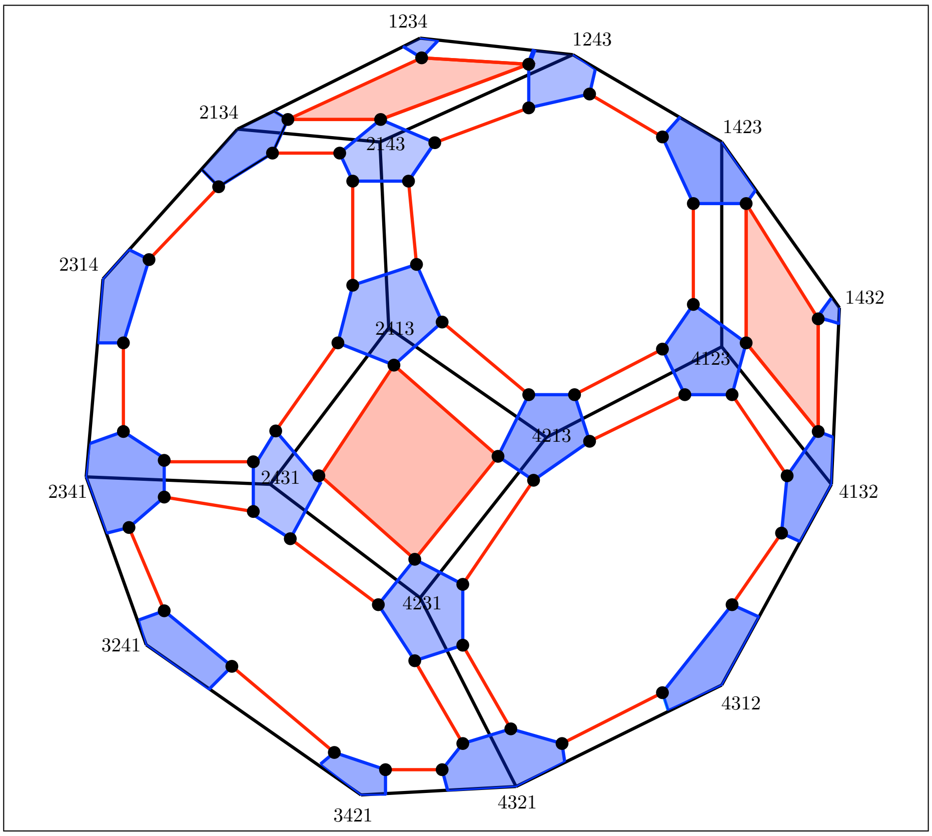


Figure 42: Following the construction recipe in Kapranov 1993 [44], p.127, see Endnotes, we obtain the permutoassociahedron as in this still somewhat rough drawing. At first sight, it seems that by some slight changes of the three types of cells one could make these faces regular, just as is possible for the permutohedron. But this is impossible, see Remark 11.3. Also the associahedron does not have a regular tiling of squares and pentagons; it is a so-called near-miss Johnson solid. *Imperfect picture by author.*

tool there is the method of *word reversing*, using exactly the elementary diagrams and reduction diagrams introduced in Klop 1980, see [46], and Terese [7].

In short, in our opinion there is a fourth way of realizing and scrutinizing polyhedra such as the three in this paper, namely via an equational approach. We then get the important notion from λ-calculus and

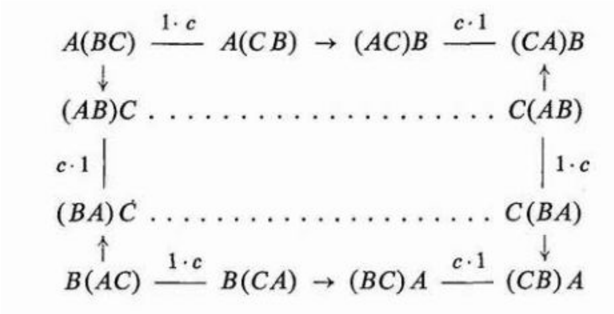


Figure 43: This diagram in Mac Lane [53] page 40 is according to Kapranov [44] the first occurrence of the dodekagon (12-gon) as a commuting/coherent diagram giving Mac Lane's coherence theorem for monoidal categories with commuting letters. Note the two dotted lines for general commuting steps, the same as the grey extra arrows in our Figure 40. See also Huet's approach to that coherence theorem as an application of Knuth-Bendix completion, page 7 of his 1987 lecture in this book.

orthogonal rewriting of Lévy-equivalence, which is akin to the homotopical perspective that nowadays is blooming. In Terese [7] it is clearly demonstrated by Van Oostrom, on the basis of Melliès axiomatic residual theory, how after establishing the CE, this key unlocks a host of syntactical information concerning Lévy equivalence, projection-equivalence, lattice properties. Another influx in this area is the notion of *decreasing diagrams*, which are e.d.'s or cells equipped with a well-founded labeling satisfying a beautiful invariant originally discovered by De Bruijn and fine-tuned by Van Oostrom. This technique and its relevance for homotopical rewriting is used and explained by Yudin [76].

The method of renaming certain labels, action labels or generators into τ, seems to have potential for the permutoassociahedron. We conjecture that given the homotopical completion of the permutohedron, in the form of the Zamolodchikov-cycle, we can collapse the ten steps as identified in the permutohedron collapse to the associahedron, and thus get the Zamolodchikov-cycle, being the homotopical completion of the associahedron. And similar for the permutoassociahedron, once we have distilled its equational realization as outlined above.

Remark 10.1. 1. Continuing the caption of Figure 46, another observation of Hans Zantema, March 2021 in personal communication,

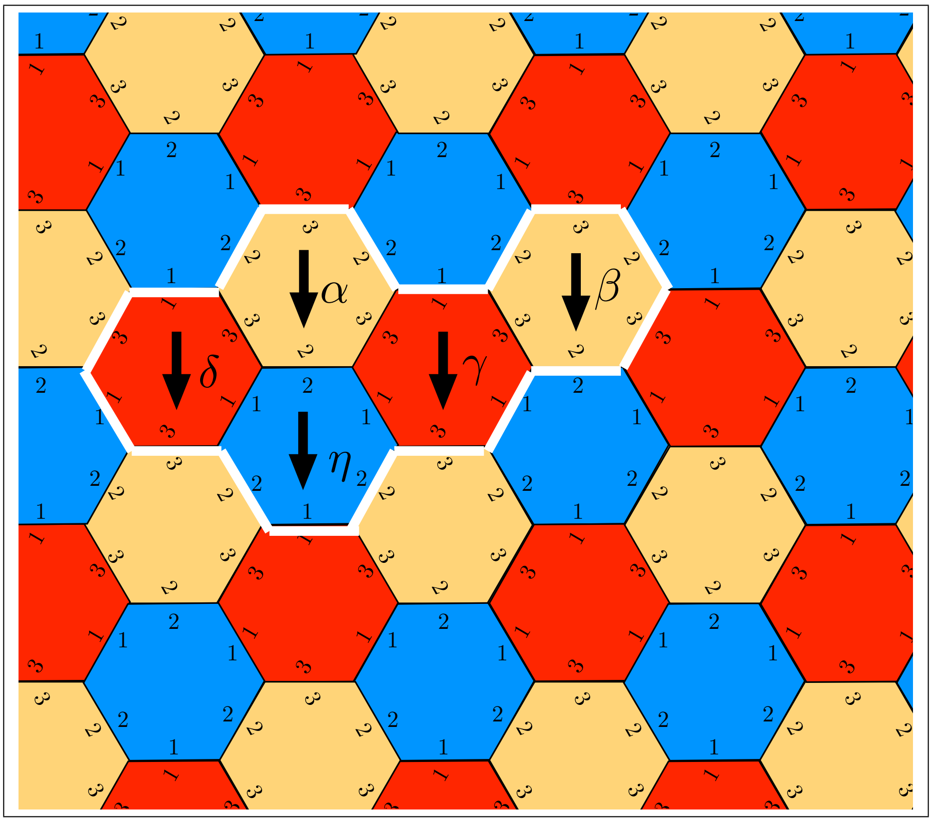


Figure 44: An illustration of two homotopically equivalent paths, in white, 31 232 1 232 respectively 131 212 31 23. The underlying space is very interesting, it is governed by the three equations $121 = 212; 323 = 232; 131 = 313$ of the Artin-Tits monoid. Curiously, this monoid is not tiling confluent as shown in the Escher-Klee figure 45. *Picture by author.*

is very worthwhile. Namely: consider the Tietze equivalent presentation

$$\langle \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d} \mid \mathsf{c} = \mathsf{dbc}, \mathsf{bd} = \mathsf{db}, \mathsf{a} = \mathsf{dbba} \rangle$$

The e.d.'s (elementary diagrams) or cells of this Tietze variant are decreasing diagrams, in contrast to the ones of the original Zantema monoid. Establishing that is a trivial exercise. This has

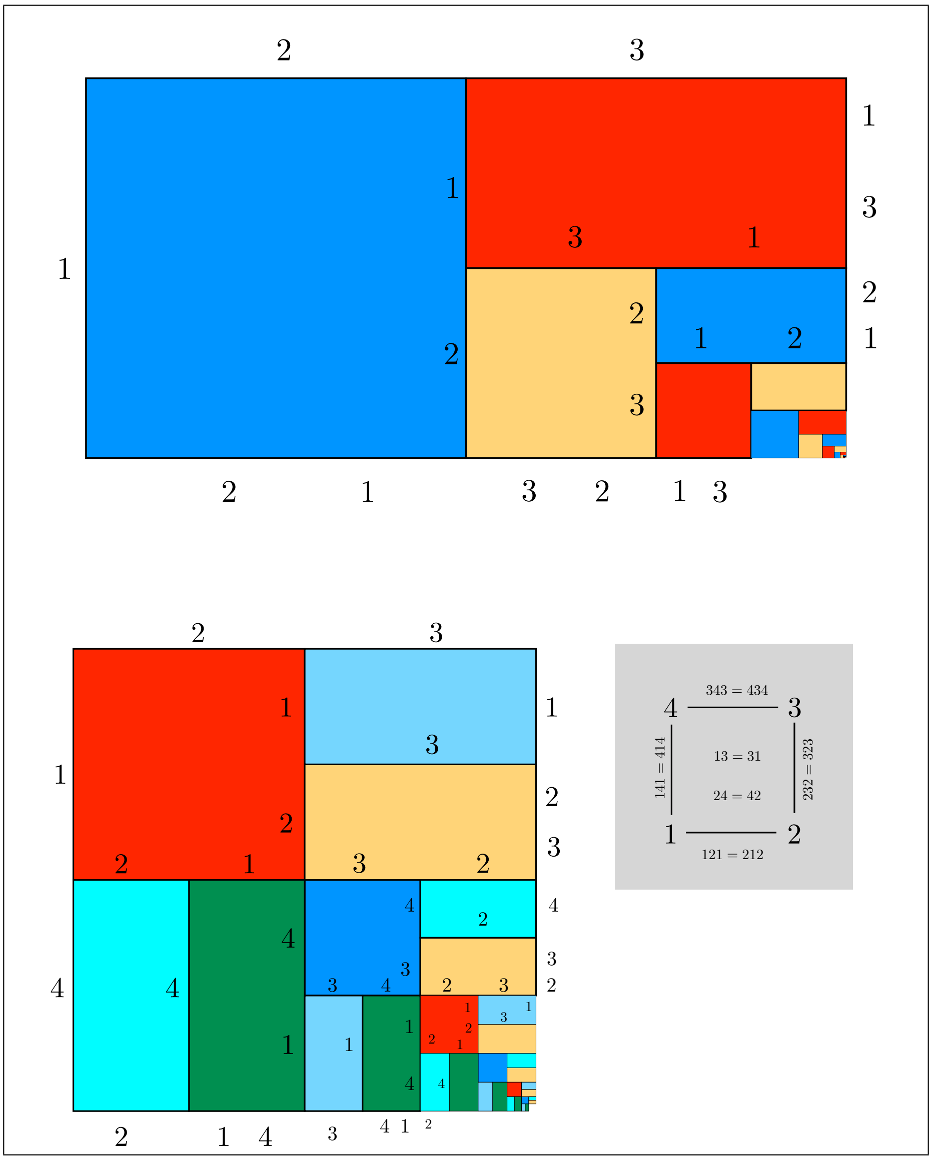


Figure 45: Upper figure: The diverging reduction diagram arising for the Artin-Tits monoid $\langle 1, 2, 3 \mid 121 = 212, 131 = 313, 232 = 323 \rangle$. It determines the three elementary diagrams or cells in blue, red, yellow. These tiles (cells), do not lead to 'tiling confluence', but generate a never-ending reduction diagram that we like to call the Escher-Mondriaan reduction diagram. In the context of word reversing for Garside monoids, where similar elementary diagrams are composed to reduction diagrams in order to find least common multiples of words, the divergence for this monoid presentation was also observed in Dehornoy et al. [20], page 73, Example 4.28, second part about the Artin-Tits monoid. Lower figure: Diverging reduction diagram for a generalized braid monoid, aka Artin-Tits monoid, subject to 'braid-like' equations as in the underlying connectivity graph in the grey square. Again the diverging reduction diagram is cyclic, which raises the question whether that is the case for every diverging reduction diagram for monoid equations. Zantema gave a remarkable example in Figure 46 to the contrary, there are non-cyclical diverging diagrams. *Picture by author.*

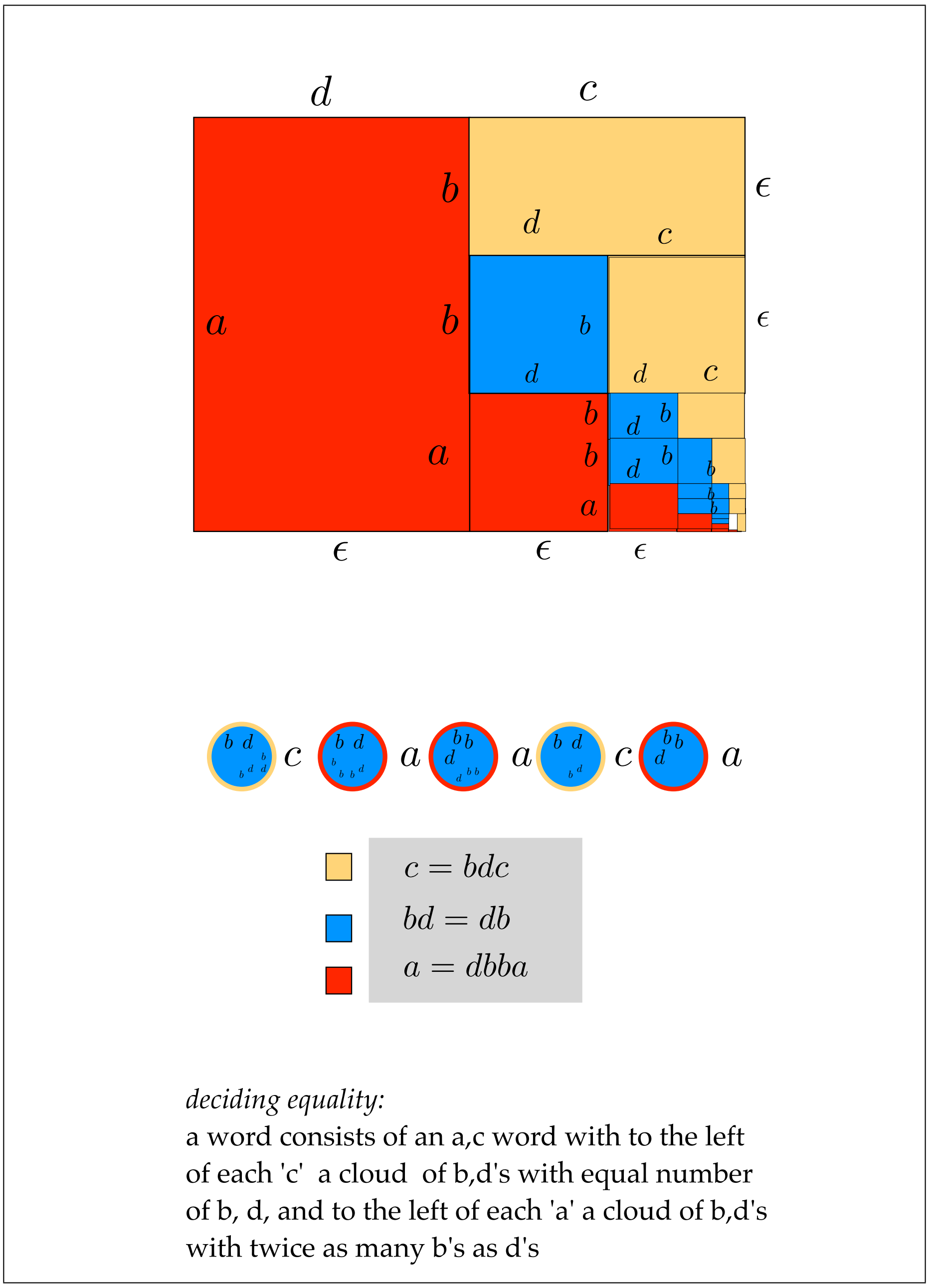


Figure 46: A very remarkable non-homogeneous monoid $\langle \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d} \mid \mathtt{c} = \mathtt{bdc}, \mathtt{bd} = \mathtt{db}, \mathtt{a} = \mathtt{dbba} \rangle$ found by Hans Zantema, personal communication March 2021, answering the question of the author whether all infinite reduction diagrams are periodical, such as the one of the Artin-Tits monoid in Figure 45, or the well-known Escher diagram of the Baumslag-Solitar monoid, which is the standard counterexample in abstract reduction systems. This one is aperiodic, as ever longer towers of blue squares arise. The monoid computes as it were the natural numbers. This example is even more remarkable, as a Tietze equivalent variation does have always converging reduction diagrams, see Remark 10.1. *Picture by author.*

two important consequences: Call a property of monoids *absolute*, if it is invariant under Tietze moves. So decidability of the equality, the word problem, is absolute. Famously, the Squier property FDT, finite derivation type, is absolute. So the property of decreasing diagrams is *not* absolute. Nor is the property of *confluence by tiling*, stating that all reduction diagrams must converge to a finite completed one, called CR^+ in Klop [80], see [46].

2. A more interesting but easy exercise is that Zantema's monoid is decidable as to its equality, conversion relation. Figure 46 gives the reason.

3. Some history: Reduction diagrams built from elementary diagrams (e.d.'s), often called 'cells', in the framework of homotopical/homological rewriting, were arising from the early λ-calculus literature, and made explicit in Klop [46], Klop, Oostrom, de Vrijer [48], Terese [2003]. Such diagrams figured also prominently in the introduction of decreasing diagrams, by De Bruijn and Van Oostrom, the 'master theorem' about confluence in abstract rewriting, having numerous corollaries there, such as Huet's strong confluence lemma and Newman's lemma and Staples' request lemma's. Independently, the useful method of reduction diagrams and e.d.'s was also discovered in the development of Garside theory, with the paradigm notion of the Braids monoid. There the method is called *word reversal* or *word reversing.* There is one difference, in the treatment of empty steps, leading to improper e.d.'s or cells, even one with all 4 edges empty. In the literature of Garside theory these empty steps are suppressed by a nice abbreviation mark in the diagrams, namely a round little arc connecting two identical diverging steps. Also, in the framework of Garside theory the method of word reversing has been fruitfully extended to deal with left and right word reversing. But the historical point is that this method found its origin in the λ-calculus. And on a meta-level, witnesses a nice confluence of ideas or methods.

4. Continuing with historical origins: also the Cube Equation arises directly from λ-calculus. And just as the method of word reversing, the notion of the CE is a key notion in Garside theory, and often

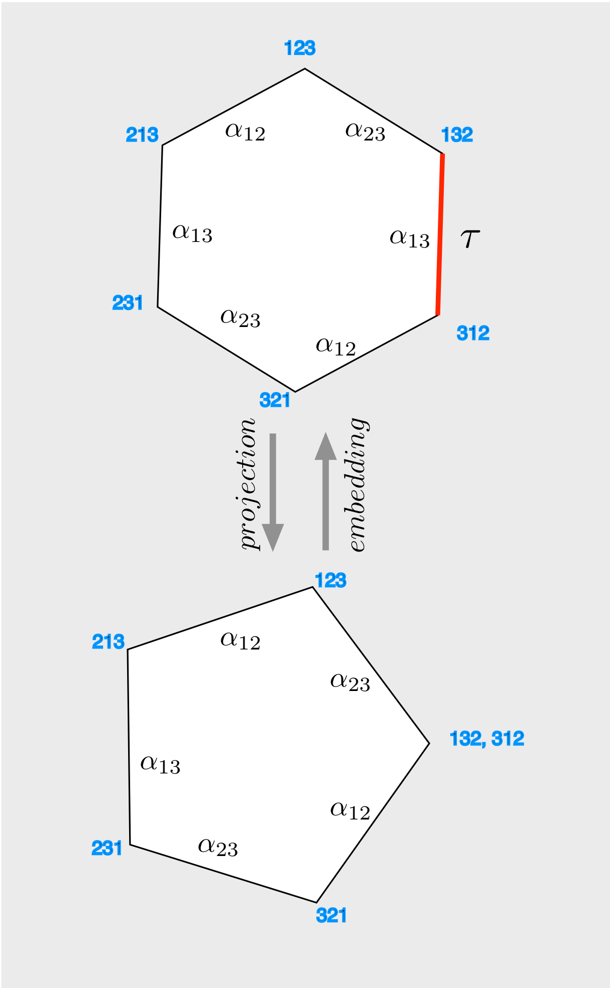


Figure 47: The permutohedron P_3 versus the associahedron A_4. The permutation vertex 312 above is peculiar, it is not 312 avoiding, and therefore part of the collapse (red edge) from the hexagon to the pentagon. This 312-avoidance phenomenon manifests itself also in the collapse of P_4 to A_5 in Figure 19. It holds in general for the collapse of P_n to A_{n+1}, according to Reading [60]. *Picture by author.*

used there. See for several pointers to occurrences of such usage Endrullis-Klop [28].

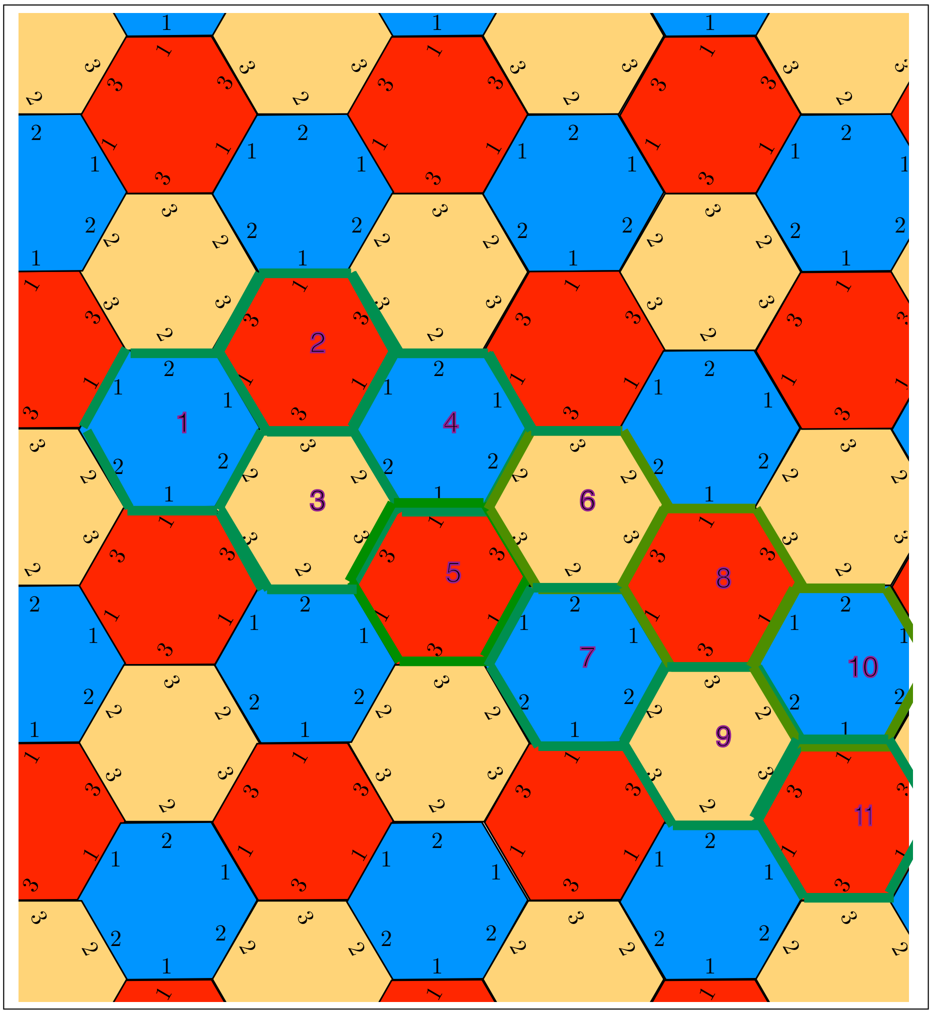


Figure 48: The hexagonal space arising from the Artin-Tits monoid contains a homeomorphic copy of the diverging Escher-Mondriaan diagram in Figure 45. It is the diagonal strip of e.d.'s numbered $1, 2, 3, \ldots$ that is periodic wih period 6. The direction at infinity corresponds with the accumulation point (condensation point) in the Escher-Mondriaan diagram. *Picture by author.*

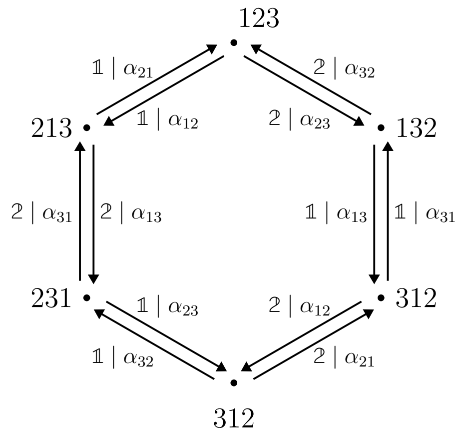


Figure 49: Permutohedron P_3 as the finite state transducer (FST) translating Artin's notation of braids with 3 strands into braid codes using the α_{ij}. A braid word in Artin's notation is entered at the top 123 and translated following the arrows, registering at each step the translation instruction $a|b$. Also the reverse translation, after flipping each $a|b$ into $b|a$. P_4 is a similar FST, for the case of 4 strands. Note that this translation FST bridges a state-independent presentation with a state-dependent presentation. *Picture by J. Endrullis, in Endrullis-Klop [28].*

Remark 10.2. Note that the permutohedron is also a finite state transducer (FST), translating words in Artin braid notation in colored braid notation and vice versa.

A metaphor for the levels of homotopical/homological rewriting is presenting itself as follows. The whole structure of these levels can be likened to an infinite skyscraper building, arising from ground level 0 to infinity ∞, via intermediate levels numbered by $n \in \mathbb{N}$. On level 0 we have ordinary rewriting of objects or words, or terms. On the first floor we rewrite sequences of reductions ($\rightarrow$) encountered at the ground floor, the rewrite arrow is now $\Longrightarrow$. On the second floor we are rewriting rewritings as on the first floor, using triple arrows $\Rrightarrow$. On the third floor we have quadruple arrows as in Figure 55 of Malbos and Guiraud.

Conjecture 10.1. Both the permutohedron and the associahedron can be 'cubified', as shown in Figures 52 and 55, and presumably also the permutoassociahedron, although in that case we shy away from that complicated pictorial attempt. Anyway, we conjecture the following.

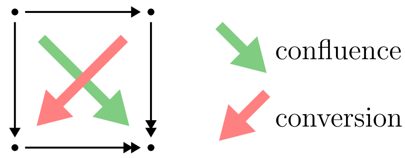


Figure 50: A dual view of an elementary reduction diagram (e.d.) or *cell* as it is called by Paul-André Melliès in [56]. The green arrow is the classical concern with confluence; the red arrow is the new point of view of homotopic rewriting. The dual view was exploited in Klop [46] for a new proof of the standardisation theorem in λ-calculus, and vastly extended in scope by Melliès in his paper as cited and other papers of his series of fundamental papers establishing axiomatic residual theory, further elaborated by Van Oostrom in Terese [7], section 8.7.1, page 430ff. *Picture by J.Endrullis.*

If we have a finite abstract reduction system with labeled arrows $a, b, c, \ldots$*, convergent, complete (i.e.* SN & CR*) in de notation of Terese [7], and this ARS fits on the sphere as a triangulation, then the associated monoid presentation obtained by reading the 'cells', or e.d,'s in our lingo, as equations, then this monoid presentation satisfies the Cube Equation* CE*, which is called in Dehornoy [17] p.*59*, Figure* 2.1 *the 'coherence property', and in Dehornoy [20] p.*67 *Definition 4.14 the (sharp)* θ*-cube condition. All such monoid presentations are then residual systems in the sense of Van Oostrom [7], Chapter* 8.7.1*, p.*430ff*. The cells must be atomic, i.e. not composed from other cells; they have to be 'prime'.*

10.1 Palindrome and semi-palindrome equations

In the following simple observation we assume familiarity with the notion of decreasing diagrams; for the precise definition see e.g. Van Oostrom [71], [58], Terese [7], Endrullis-Klop [28],[31], [27], Yudin [76], Klop, Oostrom and De Vrijer, Endrullis, Klop, Endrullis, Klop, Overbeek [30], Endrullis, Klop [29].

Definition 10.1. 1. A *palindrome equation* in a monoid presentation is one of the form

$$a_1 a_2 a_3 \ldots a_{n-1} a_n = a_n a_{n-1} \ldots a_3 a_2 a_1$$

for letters (generators) $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ from the monoid alphabet. We require that the 'outer' symbols a_1, a_n are different from the 'inner' symbols

$$a_2, a_3, \ldots, a_{n-1}$$

2. A *semi-palindrome equation* is a palindrome equation where some of the symbols of the inner symbols in the *rhs*, i.e. with indices between 1 and n, may be crossed out, deleted.

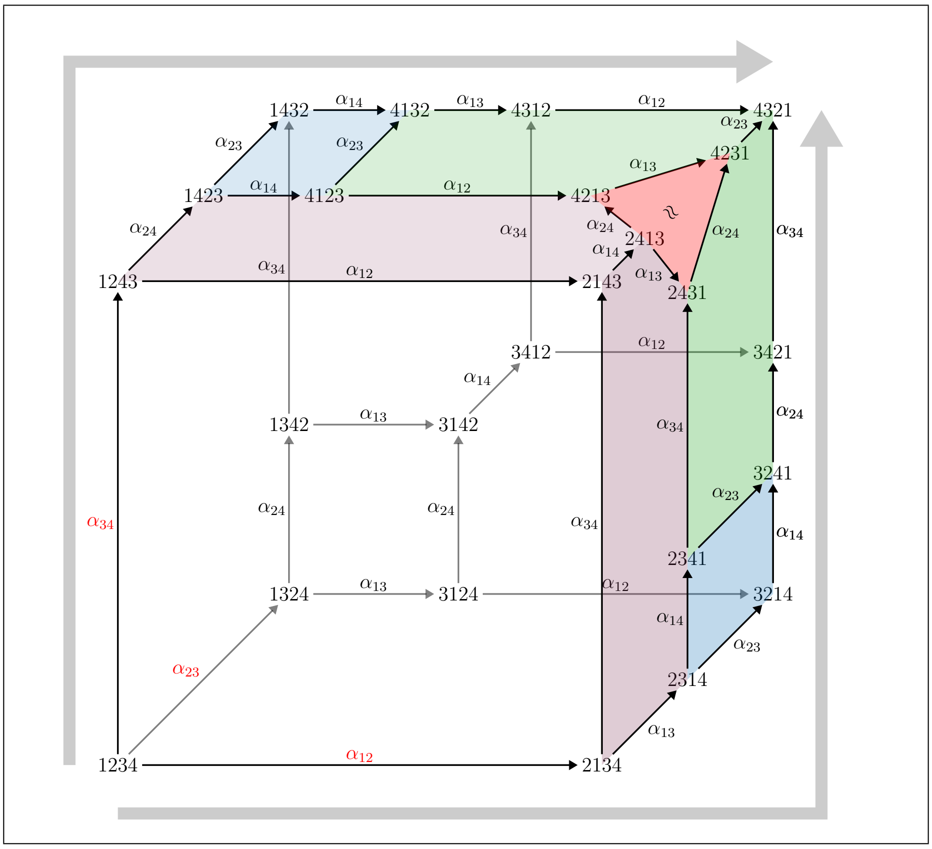


Figure 51: The 'cubification' of the permutohedron, obtained by repeated projection of the three initial steps in red leaving the 'state' 1234 over each other until the process closes. In the end we have thereby also established the Cube Equation CE for this presentation. The CE says that the two ways of computing by projection the the upper right edge of the big cube, yield the same; that is, the same modulo the equations of the presentation. A similar much easier cubification can be made for the associahedron which has only 9 faces; then the cube closed exactly, with literal equality, not modulo the equations. See Figure 55. For the permutoassociahedron we do not have a clue, if only because that seems to start with 4 initial steps? *Picture by J.Endrullis, in [28].*

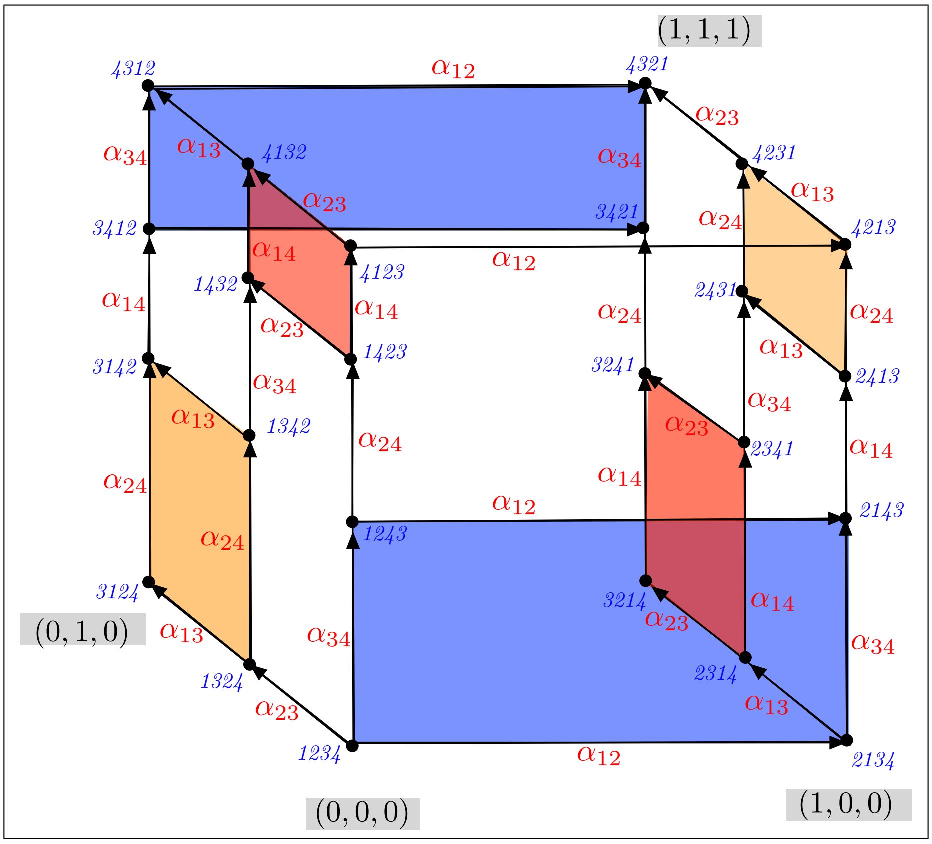


Figure 52: Another cubification of the permutohdron P_4, this time an 'exact' covering of the cube C_3, without the contorted square of the cubification in Figure 52. Actually, the basis of this figure without the edge and node labels and the colors is precisely given in Figure $2(\mathfrak{a})$ in [63]. *Picture by author.*

3. a *palindrome presentation* consists of *disjoint* (qua alphabet) palindrome equations and semi-palindrome equations, filled up to a full set of e.d.'s for the pairs (a,b) without corresponding e.d. the commutation equation $\mathfrak{ab} = \mathfrak{ba}$, also a palindrome.

Theorem 10.2. *A palindrome presentation consists of decreasing diagrams, and hence is confluent by tiling, convergent in the terminology of Dehornoy [17] p.*$53, 54$.

Example 10.3. 1. The braid monoids B_n^+ have in the α_{ij}-notation a palindrome presentation, with the YBE.

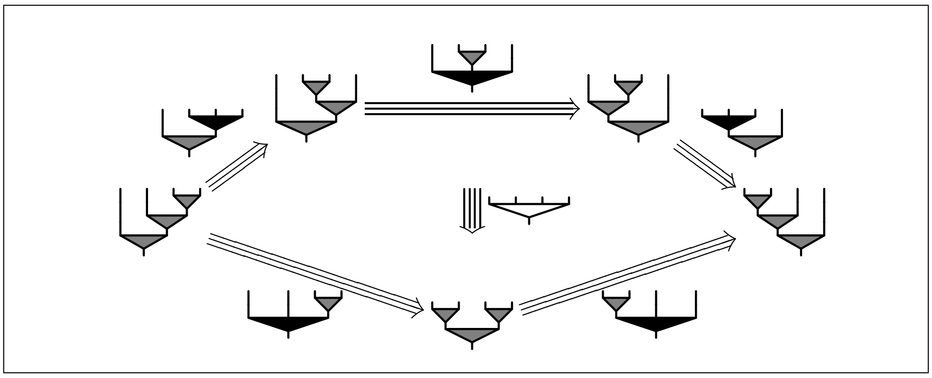


Figure 53: Balancing wineglasses! The pentagon relation also emerges in advanced developments in homotopical rewriting: An example of a quadruple rewriting from Guiraud-Malbos [38], p. 38, manifesting itself in their homotopical analysis of the monoid $\langle \mathsf{a} \mid \mathsf{aa} = \mathsf{a} \rangle$ related to the famous Thompson groups T and F, of which F is discussed extensively in its relation with the associahedron in Dehornoy [18]. See also [14] about Thompson group F, Ch. 16. See also [37] about diagram groups, p.108. Note that this picture is a pentagon of triple steps, with an obvious resemblance to the pentagon of binary tree rotations in Figure 22. There is an impressive homotopical/homological machinery behind this quadruple pentagonal rewrite step! In our metaphor this object lives on the third floor of the infinite homotopy skyscraper building; our pictures in this paper inhabit only lower floors, $0, 1, 2$. *Picture from Guiraud and Malbos in [38], p. 38, with consent of the authors.*

2. The associahedron has a semi-palindrome presentation, namely with the pentagon equation PE.
3. *Conjecture:* The permutoassociahedron KP_n has a semi-palindrome equation presentation, probably a sort of interweaving conjunction of YBE for the permutohedron part and PE for the tree rotation (associahedron) part.
4. The permutohedron equation, (conjectured) aka Zamolodchikov equation in Figure 58, is a (near) palindrome presentation.

10.2 Tietze triangulation and orefication

Remark 10.4. Explanation of Figure 62: To prove confluence by tiling, aka CR^+ in Klop [80] or convergence in Dehornoy's book [17], there are several methods that one can try. An important method is that of *decreasing diagrams*, created by de Bruijn and Van Oostrom [58], [27], [48]. An interesting method that occurs in Van Oostrom [72] p. $7 - 11$, and also is mentioned in Dehornoy's book at several places, is the method of *projection closure*: if one can find a finite set of words containing the constants (generators), and closed under projection, then we have convergence,

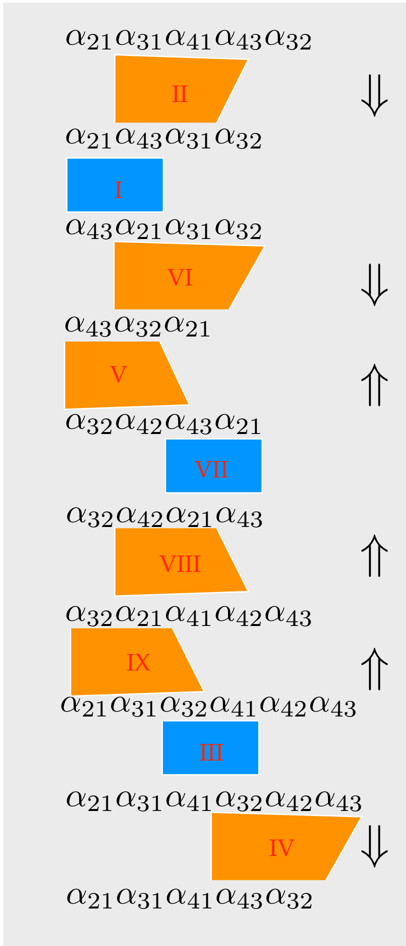


Figure 54: Walking around the associahedron A_5. The 3 blue squares and 6 yellow (oriented) trapezia indicate how faces are flipped; they stand for up $\Uparrow$ and down $\Downarrow$ double arrows in the notation of homotopical rewriting. They also indicate with their upper and lower side the 'redex' and the 'contractum', in λ-calculus vernacular. The whole sequence is a cycle, that as we expect can be obtained from the Zamolodchikov cycle for the permutohedron P_4, by abstraction, that is, replacing the ten collapsing edges identified in Figure 19 by the silent move τ. The red capital Roman numbers $I, \ldots, IX$ indicate the 9 faces of A_5, in the same notation as in the other figures of A_5. *Picture by author.*

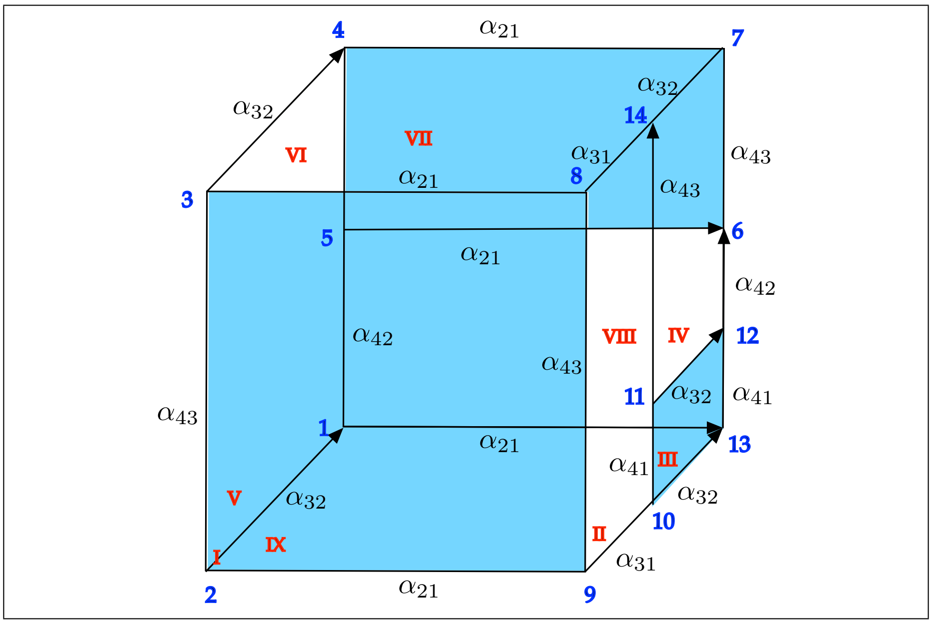


Figure 55: Cubification of the associahedron A_5, showing that the presentation satisfies the Cube Equation CE, and thus is an *abstract residual system* in the sense of Van Oostrom, see Terese [7], section. 8.7.1, pages 430−474. All arrows are directed away from vertex 2 to vertex 7. The three squares of A_5 are in blue. The other 6 faces are the pentagons. The red capital Roman numbers I, . . . , IX indicate the 9 faces of A_5, in the same notation as in the other figures of A_5. *Picture by author.*

all reduction diagrams will be succesfully completed, without diverging. We can also call this method *confluence by completion*. There is an equivalent method as follows. Call an equation in a monoid presentation *short* if it is of the form $ax = by$ or $cx = d$. This condition is called in ring theory the *Ore-condition*. For such short equations confluence by tiling is trivial, because the e.d.'s or cells are non-splitting. So we can attempt a completion process that we can call *orefication*, that is, by means of Tietze moves strive towards a presentation of only short equations:

To do that we introduce new constants abbreviating two-or-more-letter words, putting the new letter steps against each other, see whether the projection generates new more-letter words, and abbreviating these again, until hopefully, closure of the process happens. That is indeed the case for all the braids monoids, but notoriously not for the Artin-Tits monoid, or the Baumslag-Solitar monoid or many generalized braid monoids, as defined in Maccammond [54], also defined in Dehornoy's books at several places. Now performing such an orefication completion process, is in some cases facilitated by what we might call a Tietze triangulation, of which Figure 61 gives an example. There the orefication is succesful, and the result can be recorded in

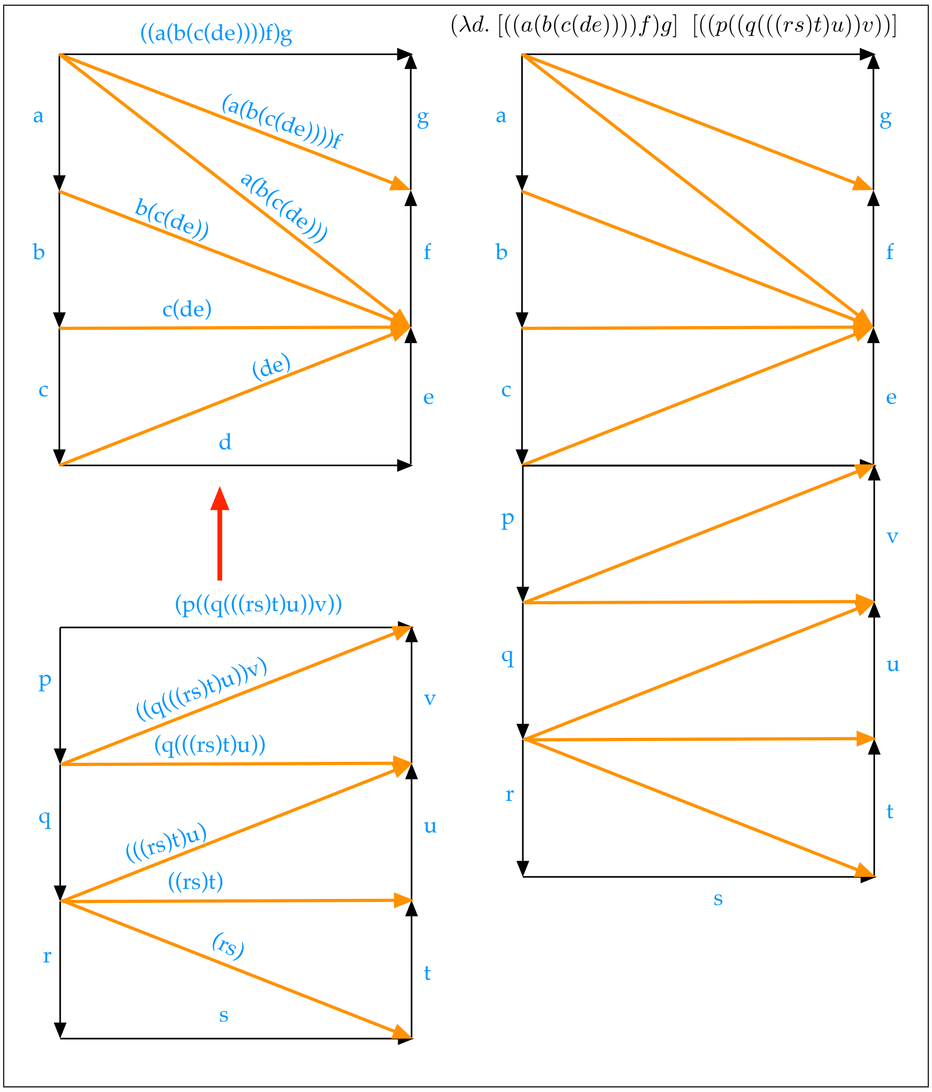


Figure 56: Composing triangulations amounts to substitution: the edge d in the upper figure is refined (red arrow) to pqrstuv in the lower figure, triangulated as in the figure and thus bracketed as (p((q(((rs)t)u))v)). Substituting this bracketed word for d in the upper figure for ((a(b(c(de))))f)g yields the resulting bracketing ((a(b(c(((a(b(c(de))))f)ge))))f)g. So, composing triangulations along an edge is tantamount to a β-reduction step! Hence the λ-term, a β-redex, in the 'result edge' on top of the composed triangulation to the right. *Picture by author.*

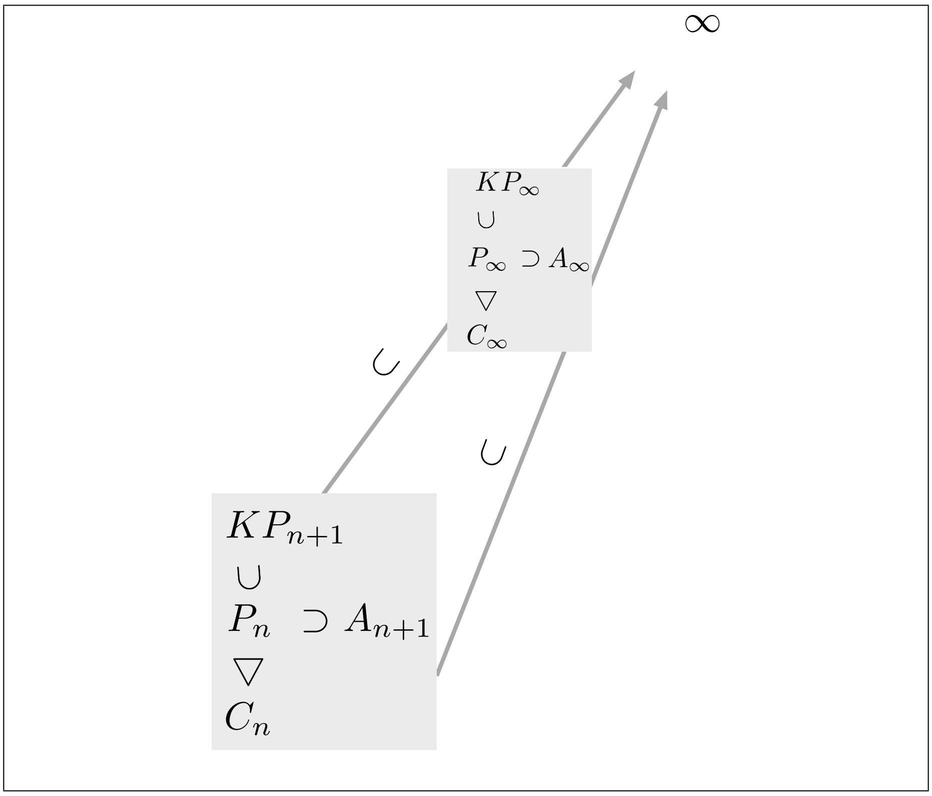
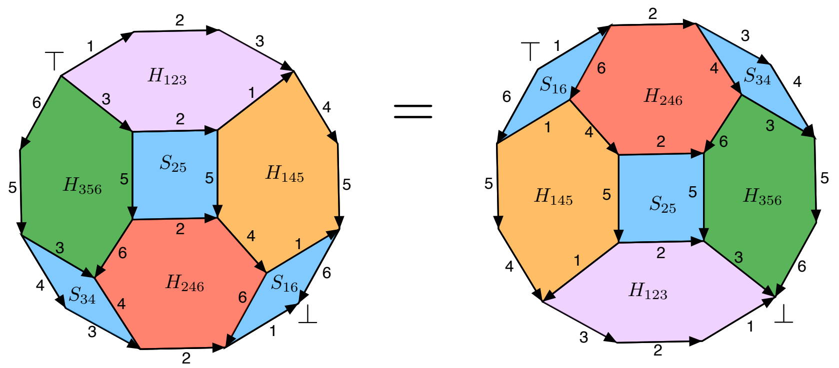


Figure 57: A grand perspective, in part conjectured. The situation at the foreground is clear for $n = 2, 3, 4$; for $n = 4$ we have seen that the tessaract C_4 gives rise by a homotopical 'lifting' to the next floor in which the single rewrite arrow $\rightarrow$ becomes double arrow $\Longrightarrow$, designated by the upward Δ; the permutohedron P_4 embeds the associahedron A_5, which itself is embeddable via P_4 into the permutoassociahedron KP_5. Each of the four structures is embeddable in the next one of his own sort by incrementing the n, for instance the cube is embedded in the tessaract, and the hexagon in the permutohedron, etc. These chains converge no doubt to infinite limit versions as in the background, but the relations between these limits should be considered more carefully than we could. The limit A_∞ is considered by Dehornoy [18], p. 213, there called $\mathcal{T}^\infty$, obtained as a direct limit; it is related to Thompson's group F. See also Sunic [69]. The infinite associahedron is also analyzed in [35]; in that paper the binary trees figuring as elements of the infinite associahedron are represented by constellations of arcs connecting points on a circle, somewhat reminiscent to the arcs displayed in our Figure 18 as being tantamount to the diagonals in a polygon triangulation. The perspective extends even beyond these structures; associahedron and permutoassociahedron are crucial in establishing coherence theorems in category theory, a theme adressed in Gérard Huet's contribution in this book. For an explanation of coherence in that context, which nowadays goes all the way to the fascinating mountains of ∞-categories, see the marvellous recent books by Eugenia Cheng [13] and Emily Riehl [62]. *Picture by author.*



$$H_{123}H_{145}S_{25}S_{16}H_{246}H_{356}S_{34} = S_{34}H_{356}H_{246}S_{25}S_{16}H_{145}H_{123}$$

Figure 58: This interesting picture occurs without colors, edge directions (in general direction from top ⊤ to bottom ⊥) and in other notation for S, Square and H, Hexagon, in the book by Carter and Saito [9], full of marvellous visualisations of so-called movies and stills; see p.210, Figure 5 there for the present figure. Typical for the homotopical outlook it displays a *graphic equation*, between words where the letters are cells. The cited reference gives the linear equation too, and calls it the *permutohedron equation*, remarking that it has aspects of both the YBE and of the Zamolodchikov equation. The order of the letters in the words is the order of construction of the cells. The circumference in both sides has numbers 123456, respectively 654321(note their direction) and are intended to be used to glue the two figures together at the same numbered side. Then we have the actual spherical permutohedron. So the two pictures are front and back side of P_4. What is remarkable is that this permutohedron equation, which seemingly coincides with the Zamolodchikov cycle in Figures 30 and 31, is just as the YBE a *palindrome*. Well, almost: it is so if the order of $S_{25}S_{16}$ is reversed in one of the two sides of the equation.That has the important consequence of facilitating the method of *decreasing diagrams*, as we will make explicit in Theorem 10.2. So after suitably orienting this permutohedron equation, we have a confluent reduction (if other equations are also suitable). Note that the equations of the cells themselves are also palindromes, e.g. $123 = 321, 25 = 52, \ldots$. So we have here a palindrome equation of palindrome equations! *Picture by author.*

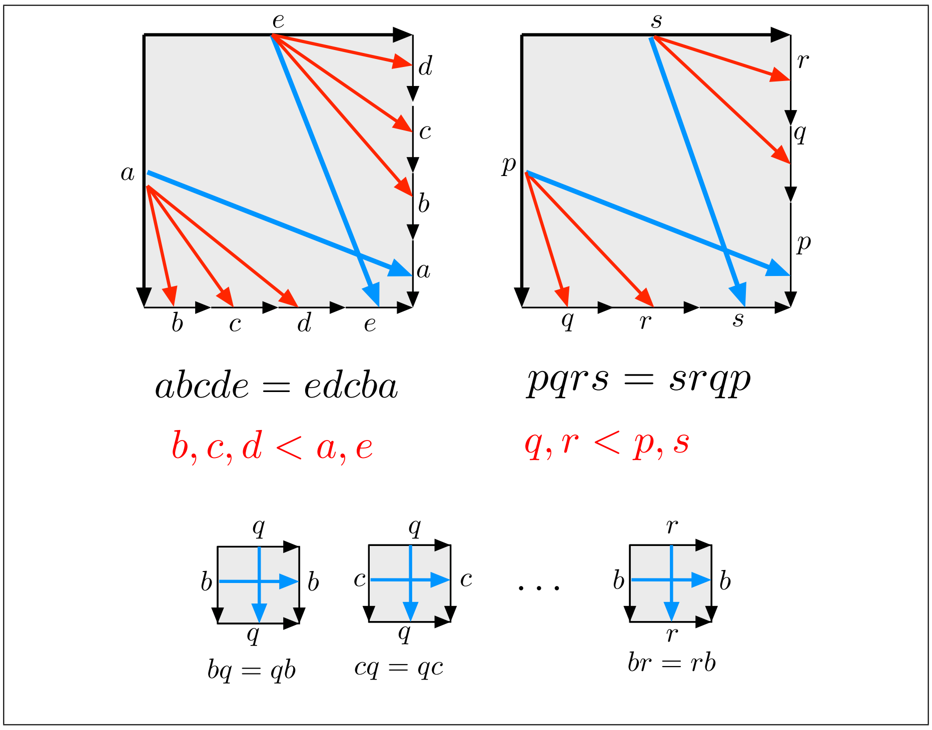


Figure 59: Two disjoint palindrome equations with their elementary diagrams (cells), which are decreasing diagrams under the red inequality conditions. The blue edges are 'for free'. To make a full set of e.d.'s commuting equations $\mathtt{bq} = \mathtt{qb}, \mathtt{br} = \mathtt{rb}, \ldots$ have been added. the result is a full set of decreasing diagrams that hence yields confluence by tiling; every reduction diagram (called 'grid' in Dehornoy [19], p. 71 section 3.2.1, Reversing grids), is converging, i.e. will terminate in a closed diagram, thereby delivering least common multiples (lcm's) for the initial horizontal and vertical words along the divergent sides. *Picture by author.*

a list of equations, the short presentation, but also neatly in a sort of multiplication tabel, or better, *projection table.*

For P_3, i.e. B_3^+, this projection table or closely similar ones, is also included in Dehornoy's Garside Theory book [20]. We would like to determine the analogous table also for $\mathfrak{n} = 4$. In fact that can be read off from providing the permutohedron with the blue and red extra arrows as in Figure 61, in all hexagon and square cells of the permutohedron. Here it is convenient to use the back and front plane rendering as in Figure 58 by Carter and Saito [9].

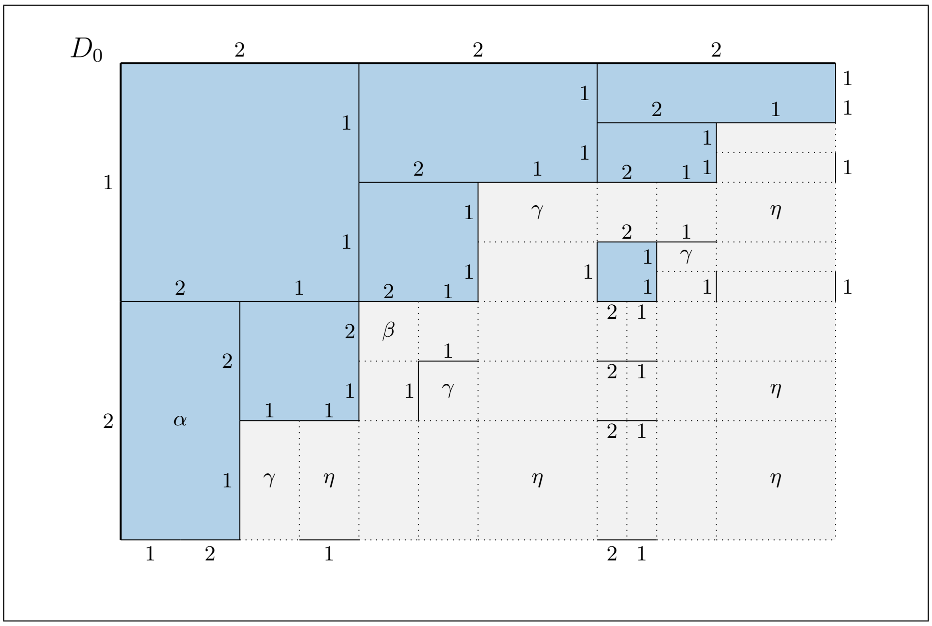
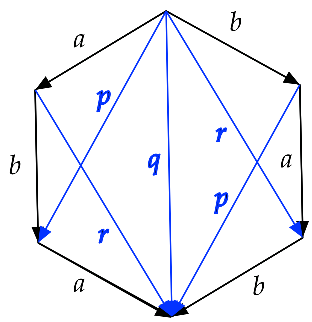


Figure 60: A completed reduction diagram, nicely called 'grid' in Dehornoy [19], of the Chinese Monoid. The equations can be read off from the cells; remarkably, they are decreasing diagrams, so we know in advance that the diagram will close. (See [29]). The proper cells or e.d.'s are blue, the improper ones are grey. Note the *absorption cells* γ, where identical steps cancel out each other. There are even totally empty cells. In Dehornoy [17] and [20] another, more 'economic', way of dealing with absorption steps is adopted, but in the more recent very appetizing book Dehornoy [19] (see in particular the reward for a marvellous open problem in group theory) the full set of improper e.d.'s is adopted, as introduced in Klop 80 for λ-calculus and Combinatory Logic. The reason for our preference of absorption steps as in this Figure 60, is that it is convenient to have the information which steps in the process of constructing a reduction diagram, cancel out each other. *Picture by J.Endrullis and author in [29].*

11 Reassociations and Retransitivizations

In some papers about diagonal flips of $n-\mathsf{gon}$ diagonalisations the moves that we are considering, tree rotations or diagonal flips, are called *reassociations*. See [51], and also [23], [45], for a fascinating link between associahedra and tree rotations and the four color theorem 4CT. Well, associativity is such a basic notion that it occurs in every field of logic and mathematics.

There is a related notion, equally fundamental, that arises more in the mind of term rewriters, including λ-calculus aficionados, and that thus far in this story was



/	ε	a	b	p	q	r
ε	ε	ε	ε	ε	ε	ε
a	a	ε	p	ε	ε	b
b	b	r	ε	a	ε	ε
p	p	b	p	ε	ε	b
q	q	r	p	a	ε	b
r	r	r	a	a	ε	ε

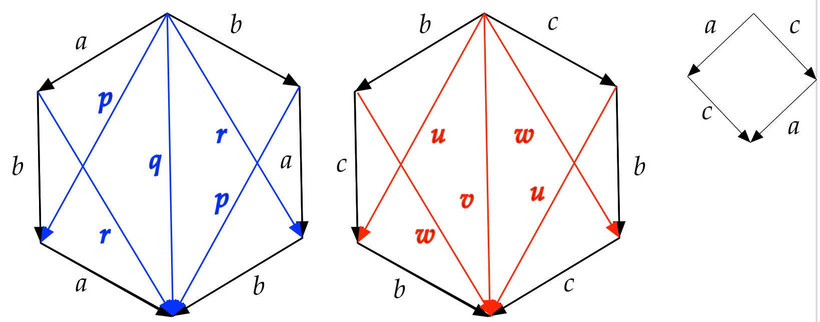
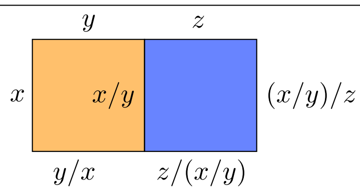


Figure 61: Tietze triangulation of the three cells of the permutohedron yielding for the triangulated left cell `aba` = `bab` which is P_3 a 6x6 projection table. Exercise: it is an abstract residual system a la Van Oostrom; the Cube Equation holds. Note the column with all ε's, corresponding to the fundamental or Garside word. For the whole P_4 we obtain a 24x24 projection table, expectedly, not included, but certainly forthcoming. *Picture by author.*



$$x.\varepsilon = x$$

$$\varepsilon.x = x$$

$$x/\varepsilon = x$$

$$\varepsilon/x = \varepsilon$$

$$x/x = \varepsilon$$

$$x/yz = (x/y)/z$$

$$(yz)/x = (y/x).(z/(x/y)))$$

Figure 62: These equations are originally due to Jean-Jacques Lévy, first appearing in his Ph.D-thesis of 1978, and frequently used in residual theory, by Paul-André Melliès, Vincent van Oostrom, also in a fundamental paper by Eugene Stark on the interface of term rewriting and process algebra, see [68], analyzing 'true' concurrency versus interleaving concurrency. The Cube Equation CE occurs in that analysis too. The Lévy-equations are also used often in the books of Dehornoy and Garside theory. With a look at the yellow-blue diagram, the equations are almost self-explaining. Orienting the equations left to right as rewrite rules, we meet an interesting critical pair, which is convergent; the reduction is also SN and hence by a well-known theorem of Huet the TRS is complete, i.e. confluent and terminating. Thus it presents an algebraic method for computing reduction diagrams. It is nice to do that for the Artin-Tits monoid, then divergence, a loop, arises. *Picture by author.*

not yet mentioned once. That is the property of *transitivity* of a binary relation. On purpose, we have drawn our hexagons and other n-gons as directed arrows, as we do in term rewriting; also the diagonals are directed. Now constructing a diagonalisation, we were actually making the initial given graph, which was not transitive, more and more transitive. The final diagonalisation is a transitive relation.

From this idea that we are actualy dealing with transitivisations, and the corresponding *re-transitivisations*, to use a for the moment an ugly word, we arrive easily at the idea of starting with the infinite line of natural numbers, as directed arrows, not yet transitive, and making this ARS (abstract reduction system) more and more transitive. We then get an infinite tree-like figure, well-founded, but for some examples without root, that may vanish in upward infinity, into the sky, as in Figure 63. But the nodes are canonically numbered as in the example of Figure 18. And so the swaps, or tree rotations α_{ij}, make sense, just as before, and we get, as we conjecture with some confidence, a neat presentation of the infinite monoid A_∞. Indeed, exactly by the Pentagon Equation PE as we discussed at length. Related to the presentation of Dehornoy in the Tamari-volume, but without his quasi-commutation equations.

OK, if we can do this with the ARS consisting of the infinite line of steps, labeled with generators a_i and interlaced with natural numbers as in the figure, why not start with a general ARS, initially possibly non-transitive, and considering its final transitivisations and in particular the moves between such, the re-transitivisations α_{ij}?

This is a fun question for rainy Sunday afternoons and sleepless nights instead of counting sheeps. Now we are in fact considering again the infinite line of steps, but with identifications of some of its points i, j. That gives identifications between the moves α_{ij}. And so we guess that we get associahedra, finite or infinite, that are also subject to identifications of some edges, in other words, subject to collapses just as we saw for the embedding of A_5 into P_4 by contracting ten edges.

I wonder whether for such collapsed associahedra the famous Cube Equation still holds! And as said before that would be a key unlocking a wealth of syntactic information in terms of Lévy-equivalence, congruence properties etc. Maybe this is a nice question for a Master Thesis (In Holland 4 ECT study points, a month work)! Or a full Ph.D-thesis if you extend such questions with homotopical completion questions! But such a mountain expedition is alas beyond me, I would stay in base camp. But I would love to see the photos taken of such an expedition. Photos? Yes: the etymology of the word 'theorem' is 'that what one sees', related to 'theos' and 'theater'. Photos also have to do with seeing. The Flemish priest and poet Guido Gezelle (1830-1899, Brugge, Belgium) called a photo, in Flemish or Dutch, a *lichtdrukmaal*, light-print-matter or light-print-item. Great obsolete Dutch word.

Remark 11.1. (*Infinitary rewriting.*) For term rewriters and lambda calculus researchers, mainly. We started this story with a central theorem about finite λ-calculus, that has repercussions beyond λ-calculus, the Cube Equation. Nowadays there is also a satisfactory theory about infinitary λ-calculus and infinitary term rewriting. Even transfinite rewriting, for all countable ordinals.

Why infinitary λ-calculus? Because it is just as natural as being able to compute π in infinitely many decimals. And because it yields more directly computations that used to require the powerful Scott's Induction Rule. The present point in this story with consideration of infinitary versions of all three jewels encountered above, permutohedron,

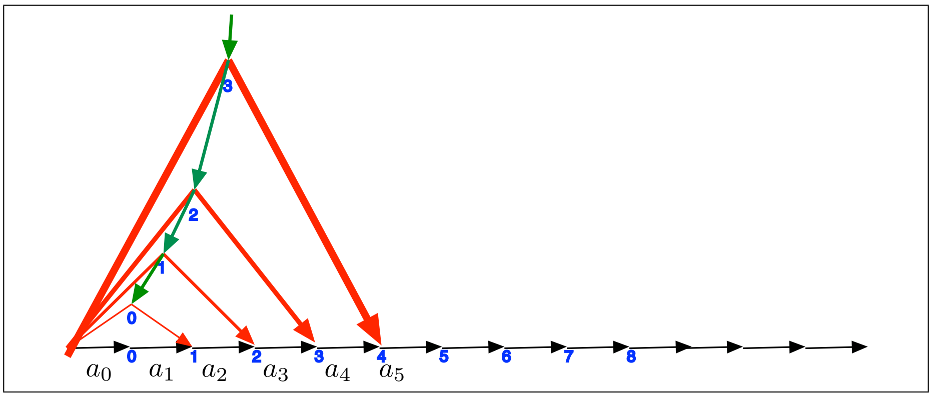


Figure 63: Infinite binary tree, green, with one upward branch with nodes the stream $0, 1, 2, \ldots$. Every infinite permutation of the stream of natural numbers gives rise to such a green tree. Every green tree amounts to a bracketing of the infinite word $a_0a_1a_2\ldots$. In the figure the bracketing with prefix $(((((a_0a_1)a_2)a_3)a_4)a_5)$. *Picture by author.*

associahedron and permutoassociahedron, invites a look at some issues here from this infinitary/transfinite term rewriting point of view.

For the infinite permutohedron P_∞ we consider infinite streams of natural numbers, subject to transpositions of adjacent entries, where we can impose the direction to swap two entries to say decreasing order, like in P_4 above. Now the nontrivial question is whether we have the property of infinitary confluence CR^∞. Because of the critical pairs this is a challenge.

The same problem emerges for the associahedron with the associativity rule. Insight here of CR^∞ could be interesting in studying A_∞. And finally, the same questions pertain for the infinitary version of the permutoassociahedronhedron KP_∞, with associativity and transpositions combined. What makes these questions exciting is that in the background we have a perfectly nice infinitary confluence of first order orthogonal rewriting; but a less nice situation for infinitary λ-calculus, there CR^∞ fails but UN^∞ (the property of unique normal forms with respect to infinitary reductions) holds; and for infinitary λβη-calculus CR^∞ breaks down dramatically, even UN^∞ then fails. So infinitary confluence CR^∞ for streams with transpositions and/or associativity is a matter to approach with some care. In general, infinitary rewriting in the presense of critical pairs is a challenge, up to now not addressed, except in Endrullis-Klop [24].

A conjecture whether CR^∞ holds for these stream rewriting questions would therefore be premature for the present writer (rewriter). *Hic sunt leones!* But we like leones, not?

For infinitary lambda calculus and its properties, and in general transfinite rewriting, see [5], [7], [26], [32], [65].

Acknowledgments

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Many thanks to Jörg Endrullis for providing the basis of several of the pictures in tikz. Subsequent inaccuracies in some figures are due to the author, who is not proficient with sophisticated tools like tikz... The picture of the permutohedron occurs in our joint work [28]. Thanks to Jörg and Henk Barendregt for assistance with the final deliverance in the right style files. And to Jane Spurr (with fond memories of out earlier cooperation in 1992 concerning the Handbook of Logic in Computer Science [47]) further in the process of routing the papers!

Also many thanks to Vincent van Oostrom for discussions over several years concerning braids and his theory of residual systems in Terese [7].

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Finally, thanks to grand-daughter Charlène (age 8 years) for showing and explaining the fidget ball in Figure 28 to me. And to my wife Marianne (age 80, like Jonathan) for endurance in living with and unfailing support of the present author of this story.

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[33] *Todo: for the embedding of ass into perm*
See endnote [4]

Question 11.2. We have without good reason the fractal cube drawn in such a way that the infinite limit border stream of the upper plan fractal coincides with the border stream of the right fractal plane, as can be seem by the coloring. Here is something to check in due time. We can observe that the halves of the cube, the upper half, the right half, the lower half are fully defined. They have not yet reached the 'white hole of undefinedness' that centers the cube's corner point G. It could be that for these half cubes an infinitary cube equation holds, if we admit as projections infinite words. In general it is a nice research question to study the infinite words, aka streams, that arise in infinite diagrams. Are they ω-regular languages?

Endnote [5]

Remark 11.3. 1. Concerning the rendering of the permutoassociahedron according to the recipe of Kapranov, it is seductive (at least it was for the present author) to try to manipulate this tessellation into a *regular* one, by nudging the faces to become regular squares, pentagons and dodekagons. However this is impossible: there is a

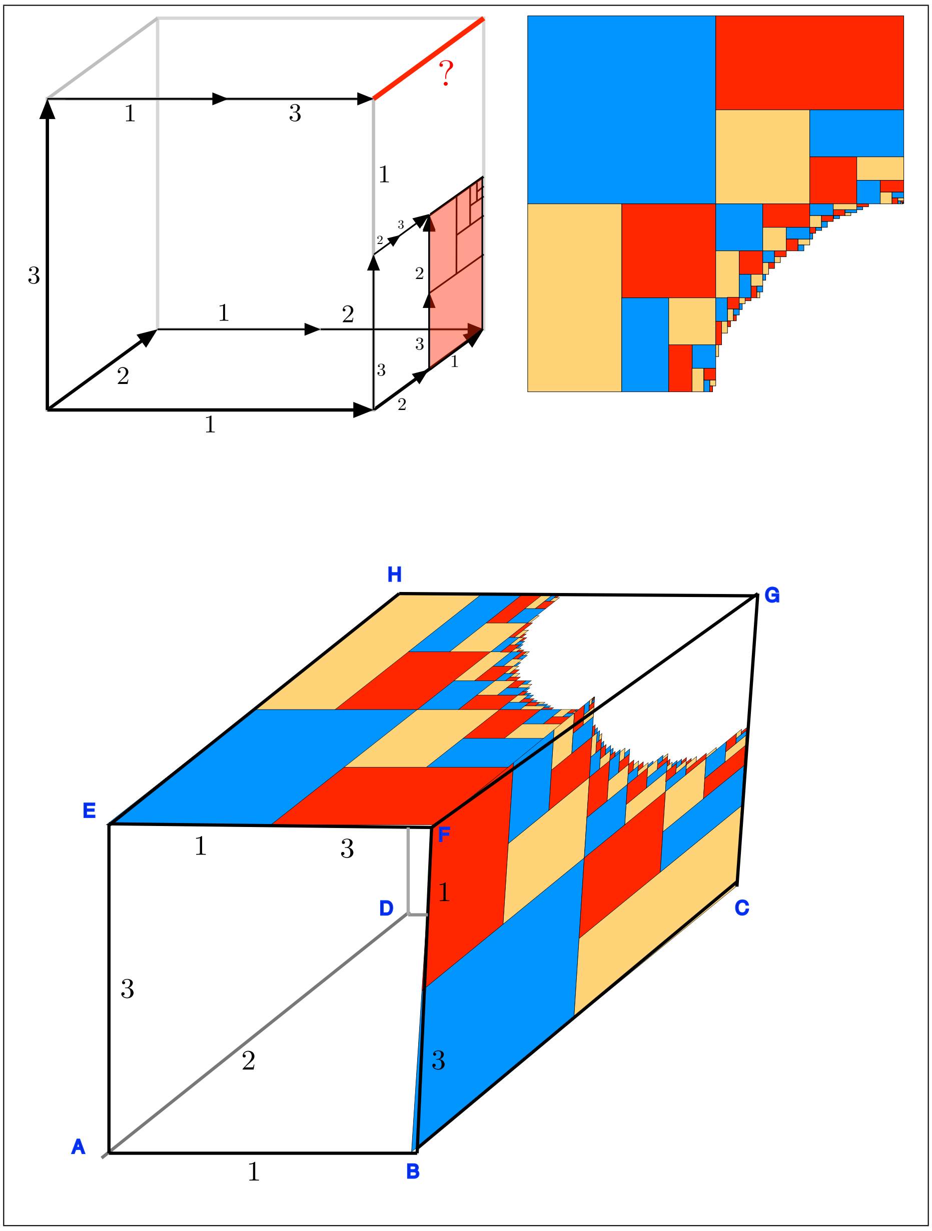

Figure 64: For the Artin-Tits monoid the Cube Equation fails, both sides of the CE are undefined. (We do not use Kleene-equality, see endnote.) If we persist in the computation, i.e. keep going on adding 'cells' as given by the presentation, we get a nice fractal. The figure gives a sketch (to be improved in due time by due author) of the fractal planes covering the cube in the three faces converging at point G, right, upper, behind. We can view the two-dimensional and three dimensional graphs as abstract reduction systems; they are non-terminating and also non-confluent. *Picture by author.*

list of 92 polyhedra called Johnson solids, a generalization of Platonic solids and of Archimedean solids, that are defined as sphere tessellations using only regular polygons. For Archimedean polyhedra there is an extra requirement about the way vertices are similar to each other. For Platonic solids the tesselating regular polygons must be the same polygon. The Johnson solids can only consist of regular n-gons where $n = 3, 4, 5, 6, 10$. There is no such polyhedron containing a dodekagon.

2. A pitfall (for the present author) is that there is a well-known rendering of the associahedron with 6 pentagons and 3 squares that *prima facie* looks very much to be a Johnson solid. But in fact it is a so-called *near-miss Johnson solid*! The faces are almost but not quite regular and cannot be made so. For plenty of studies and information for this class of polyhedra see Wikipedia. I wonder how close to a Johnson solid the permutoassociahedron can be made; there are measures for such closeness, see Amir-Sequin [1].

Remark 11.4. *(Confluence versus coherence.)* It's important to note that coherence is a much stronger property than confluence. This was already remarked in Huet's lecture in Boulder Colorado, 1987, included in this book. See also Melliès [55] for a clear explanation of the extras in coherence, formulated in terms of homotopy equivalence and push-outs in the categories that are involved. For the classical notion of residuals in λ-calculus and orthogonal term rewriting systems the theory was established by Huet and Lévy. That theory pertained primarily to the 'conflict-free' case where there are no critical pairs, *aka* orthogonal rewriting. After that foundation of residual theory, attention was directed to the non-orthogonal case where critical pairs are present in the presentations, such as braids and Garside monoids and the associativity rule in connection with monoidal categories, leading to Mac Lane's coherence theorems and Garside theory. Melliès [55] endeavoured to extend the classical orthogonal residual theory to cover these cases too, by extending the notion of residual to *treks*.

12 Endnotes

Notes

[1]The text of this endnote is from Dennis Sullivan, in his math forum explanation: A chemist showed this way to factor a permutation into transpositions. Draw n dots on each of two parallel lines in the plane. Connect the dots on one line with those on the other line by straight segments according to the given permutation. There will be one transposition for each intersection point of two segments. To see this slide one line over to the other allowing the dots to follow along their segments. The order of the dots is changed by a transposition when an intersection point is crossed. This provides a fast mechanical way to compute the sign of a permutation, which was the motivation from theoretical chemistry.

[2]The underlying chainmail-like space in Figure 38, drawn faintly grey, was obtained as *turtle graphics* by viewing the Fibonnacci stream as a program with graphical instructions corresponding to the 0's and 1's of the infinite stream. The program converting a stream in a fractal or infinite plane covering, was devised by Jörg Endrullis around 2010, in joint work with the present author on finite state transducer degrees of infinite streams, see [25]. The Fibonacci stream reads as follows:

$$0100101001001\ldots$$

This is a *morphic stream*, generated from start word 0 by repeated simultaneous substitutions $0 \to 01, 1 \to 0$.

[3]Citation from Kapranov 1993 [44], p.127: The 'polytope' KP_4, contains 120 vertices and is difficult to draw in detail. Here are simple directions how to do this. Start from the permutohedron P_4. Its faces are squares and hexagons. Inside each square mark one point near each of the four vertices. Inside each hexagon mark two points near each vertex. Each vertex of P_4 is contained in three faces: one square and two hexagons. Thus there will be five marked points near each vertex. Join them by edges to form a pentagon and then join these pentagons to each other as shown in Fig. 5. In this way each hexagon will be replaced by a dodecagon, each square by a smaller square, each vertex will be blown up to a pentagon and each edge will be doubled to give a rectangle.

[4]Here is a slight pitfall: We mentioned that the Artin-Tits monoid does not satisfy the Cube Equation CE. Figure 64 witnesses that, the computation is divergent. Yet, in Dehornoy et al. [20] p.114 it is stated that 'one easily checks that the Cube condition is satisfied for $\{\sigma_1, \sigma_2, \sigma_3\}$'. These are the three generators, we named them $\{1, 2, 3\}$ in our Figure 64. The explanation is that in the Definition (4.15) on p.67 of the cited book one uses so-called Kleene-equality, which has the curious effect that the equation is also satisfied if both *lhs* and *rhs* are undefined, which is indeed the case here; the computation along the other path in the cube also diverges. We have implicitly adopted the to our taste more sensible reading that the equation holds if both sides are defined and equal. As to the intended equality, that can be literal equality or convertibility; this distinction is called in the cited book, the sharp θ-

cube condition respectively the θ-cube condition. The permutohedron satisfies the ordinary cube condition, the associahedron the stronger sharp one.

[5]We have encountered in this story several times the Cube Equation, a property called in the beautiful books by Dehornoy and coworkers the coherence property, or the cube condition. Another property that we encountered a number of times is the property of confluence by tiling, or CR^+ in Klop 80, in the books of Dehornoy called convergent. We have seen that convergence alone is not an absolute property, i.e. not invariant under Tietze moves. That was demonstrated by Zantema's monoid, see Figure 46. Now a remarkable theorem of Dehornoy and Paris mentioned in page 95 of Dehornoy [17] states:

Theorem 12.1. *(Dehornoy-Paris* 1999, *[21])*
The property coherent & convergent is absolute.

13 Appendix: Confluence by Completion

There is a method of obtaining confluence that at first sight is quite miraculous. It is a completion procedure, somewhat in the spirit of the well-known critical pair completion procedure of Knuth & Bendix. The miracle in our case of braid words is the termination of the completion algorithm. But later on, with the acquaintance of the Permutohedron, the termination can be fully understood.

13.1 Braids with 3 strands

Consider the monoid for positive 3-strand braids $M = \langle a, b \mid aba = bab \rangle$. The confluence problem for this monoid means that we have to show termination of tiling with (copies of) the elementary diagram (e.d.) with divergent steps a, b and converging sides ab and ba, as we have seen. *Todo: we have not seen that yet, remedy the order*We solved this problem with a non-trivial termination argument of certain string rewrite rules.

Now we will consider a simpler solution, that uses an interesting 'completion' method described in the notes Van Oostrom [**?**], based on an idea of Hans Zantema (personal communication). We will give a somewhat different (but equivalent) procedure. We will call this method *confluence by completion*.

The problem with proving confluence, or equivalently, termination of diagram tiling, is the splitting in the two converging sides; they comprise each two steps. If these were just single steps, confluence and diagram termination was trivial. Thus we attempt to make the converging two step sides into single steps, simply by adopting new constants c, d and abbreviations $c = ba$ and $d = ab$. Doing so is nothing else than applying a *Tietze move* om the monoid presentation; we will come back to Tietze moves later. What we have obtained now is the e.d. squ(b,a,c,d) which is described in the two 'projection' equations $[a/b = d, b/a = c]$.

However, we now have to consider also the case in a diagram tiling construction divergent pairs of steps (a,c) and (a,d) These are treated by the mini-diagram tiling doublesqu(c, squ(b,a,d,e), squ(b,a,c)), yielding squ(d,a,b,e). If one prefers , we can avoid these mini-diagrams in favour of equations; we strive for equations of the form ac = bd, because they they are tantamount to squ(b,a,c,d). So we proceed by equational computation:

1. Case $(a, c) : ac = bd; d = ab; ac = bab = cb$.
2. Case $(a, d) : d = ab; de = ab$.
3. Case $(c, d) : d = ab; ac = cb; c = ba; da = aba = ac = cb; da = cb$.
4. Case $(b, c) : c = ba; ce = ba$.
5. Case $(b, d) : d = ab; bd = ac; c = ba; bd = aba = da$.
6. Cases $(z, a), (z, b), (z, c), (z, d) : z = da = bd = ac = cb$.

We have now achieved a transformation of the original presentation into one with constants a, b, c, d and equations

$$\begin{aligned} ac &= bd \\ c &= ba \\ d &= ab \\ ac &= cb \\ d &= cb \\ bd &= da \end{aligned}$$

This set of equations is *full*: every pair of constants appears as the head symbols in tthe lhs and rhs of an equation. [9] (Note that for equations like $c = ba$ we use to this end the equivalent equation $ce = ba$.) Therefore, we also have a full set of e.d.'s; for every divergent pair of steps in a tiling diagram construction there is an e.d. that matches the divergence.

And this is sufficient to have terminating diagrams, and thereby confluence. Also, the completed diagrams solve the word problem as we have seen, and finally, convertibility is equivalent to diagram equivalence, meaning that two braids absorb each other.

[9] Such equations of the form $ac = bd$ are known in ring theory as *Ore conditions*.

13.2 Braids with 4 strands

For the case of 4 strands, we will now attempt the same confluence completion procedure. Note that it is not at all clear yet that this completion procecure will indeed terminate with success. Indeed, for some monoid presentations it fails.

Example 13.1. Consider the monoid

$$\langle \mathsf{a}, \mathsf{b} \mid \mathsf{aab} = \mathsf{bab} \rangle$$

Abbreviate $\mathsf{c} = \mathsf{ab}$, obtaining $\mathsf{ac} = \mathsf{bc}$. Now the divergent pair of steps (a,c) leads to a failure by cyclicity, as the figure shows. For this monoid it is not possible to transform to a full set of equations, or e.d.s. The method of confluence completion fails here. Not surprisingly, since diagram tiling with the original e.d. as in Figure xx also fails in a cyclic construction.

A confluence completion as for 3 strands is again possible, but laborious. Instead, we will reproduce a computation in Van Oostrom xx, where one works with longer words. This done in several rounds. We consider the presentation

$$\langle 1, 2, 3 \mid 121 = 212, 232 = 323, 13 = 31 \rangle$$

1. In Round 1 we have the three letters $1, 2, 3$. For the projections of these three words over each other we obtain composite converging sides
$$21, 12, 32, 23$$
These are our new 'letters', but other than above, we will now work with these words themselves, without introducing abbreviations by introducing new symbols. This is anyway equivalent to the procedure with abbreviations. The four words just obtained, we project again over each other, thereby obtaining words
$$2132, 123, 321, 132$$
2. In the next iteration we obtain
$$13, 12321, 2321, 1213$$

3. The fourth iteration leads to

$$213, 1321, 1232, 121, 232$$

4. The fifth iteration gives

$$12132, 21321$$

   and hereafter we have reached succesful completion; now all projections yield words already found. At this point is rather mysterious why the completion procedure did terminate, and what is special about the words obtained. The key to understanding resides in the permutohedron of order 4. That contains 24 nodes, and what we have found are exactly the 23 non-bottom points, or better, words leading to these points, as can be seen in Figure xx. Thus all the words found are in fact simple positive braids, actually, representants from their convertibility equivalence classes. It is best to add also the bottom point, the node 4321, this is also a simple braid. These 24 simple 4-strand braids are closed under projection. In other words, the permutohedron is an *abstract residual system* in the sense of van Oostrom.

Exercise 13.2. Prove that the 24 simple positive 4-strand braids are closed under projection.

Proof sketch: Consider the simple braids in their absolute notation with the α-crossings α_{ij}. A simple braid has no repetitions of α's, by definition. Now project one simple braid x over another y. Use here the fact that the diagram construction of $D(x, y)$ terminates, as proved earlier by other means. Now observe (see Figure [?]), that the α-s propagate through the diagram unchanged. This means that the converging sides of the diagram $D(x, y)$, the projections x/y and y/x , are also repetitionfree, i.e. simple.

Todo: correction: not a repetition, but occurrence of two 'opposite' alphas α_{ij} and α_{ji}

13.3 Braids with 5 or more strands

Also for general n permutohedra exist, with similar properties as for the case $n = 4$. We conclude that the confluence by completion procedure works for general n.

Todo: This Appendix is an old version, to be edited. The reference for this confluence-by-completion by repeated projection of the words, starting with the generators, as in the syllabus on Braids of Vincent, is not clear: Vincent when asked referred to Hans, but Hans does not recall when I asked him in 2022, that it originated from him. Fact is, that the method is mentioned a couple of times in the three books of Dehornoy c.s.; precise references to pages and remarks there are to be included. The precise originator thus remains unclear.

Exercise 13.3. Consider the monoid

$$\langle a, b \mid aab = bab \rangle$$

with generators a, b and the relation $aab = bab$. Show that the confluence by completion procedure fails.

14 Subject Index

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