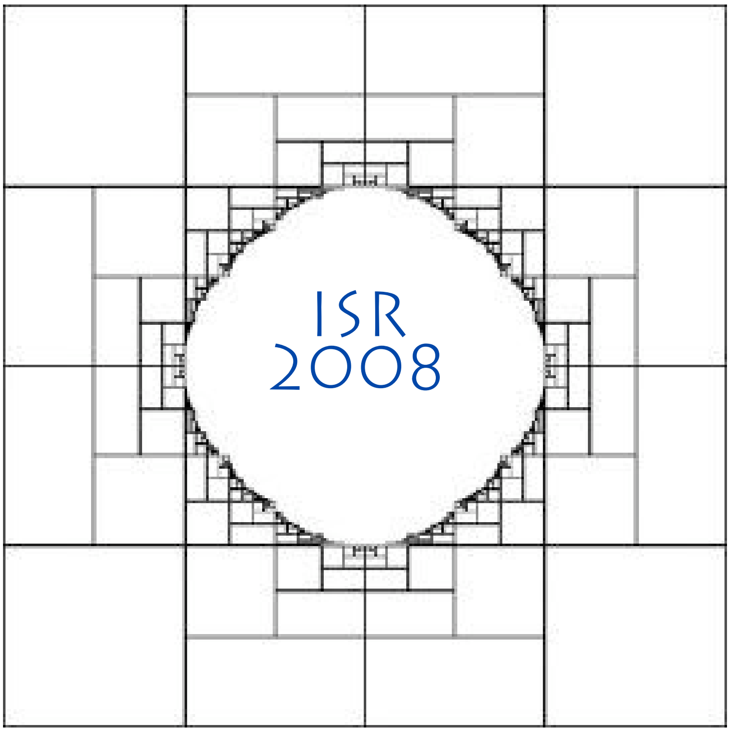
A Course in

Infinitary Rewriting



Monday, July 14, 2008

Contents

Introduction.
This syllabus starts with an introduction in infinitary rewriting for the first order case, based on a prolonged cooperation with Richard Kennaway, Ronan Sleep and Ferjan de Vries, from 1986 to ca. 2000 which has been recorded in a chapter in Terese 2003, and in a series of papers mentioned there.

The second chapter is based on a paper in preparation together with Henk Barendregt, and adopts a λ-calculus perspective.

The third chapter tells about the first part of an investigation of the last two years of the 'productivity' (≈ infinitary normalisation in a constructor discipline) of definitions in a certain simple format of infinite streams of data. This work was carried out in the framework of NWO project Infinity, a cooperation project of VU University Amsterdam, CWI Amsterdam, and Utrecht University, in a cooperation involving in particular Dimitri Hendriks, Jörg Endrullis, Ariya Isihara, Clemens Grabmayer, Roel de Vrijer, Vincent van Oostrom and the present author.

The fourth chapter relates about tree ordinals in infinitary first order term rewriting. Whereas the second chapter dealt with streams of just data, here we consider streams of streams of streams, ..., in a well-founded manner. This chapter is based on a study performed by Ariya Isihara, and Marek Kwiatkowski, who investigated this subject up to ε_0, in his master's thesis. Ariya Isihara showed how to proceed to Γ_0, among other contributions which are forthcoming in his PhD thesis.

The final chapter contains some juicy facts about celebrated streams such as the Thue-Morse stream and the Toeplitz stream, as well as some questions about a comparison of streams.

This syllabus is still under construction, and is intended to be in a process of continued evolution, to be found on the author's homepage
http://web.mac.com/janwillemklop/Site/Home.html

Let us now elaborate more about the conceptual contents of these notes. All of this is a study of infinite computation in the canonical regime of orthogonal rewriting, as it is known for the settings of first order rewriting, and higher order rewriting in the form of λ-calculus and Combinatory Reduction Systems (CRSs). The latter framework (CRSs) will not explicitly considered except for a few simple extensions of λ-calculus.

The properties analyzed are unique normalisation, which can be seen as consistency with respect to different computations. As a counterpart of succesful computations, we investigate at the same time the notion of undefined. How computation can fail, and the different forms of such undefined behaviour. This gives rise to computational anomalies, objects (terms) which we like to think of as 'black holes', capable of activity but not yielding any information (in contrast with black holes in the cosmos). So this is a study of computation in its most fundamental aspects, in a strongly constrained regime, which we feel is canonical. Of course we also try to stretch the boundaries of the constraining regime, and look e.g. at weakly orthogonal rule sets. Eighty years ago we saw the first steps in this setting of canonical computation, with the discovery of λ-calculus and Combinatory Logic (CL). In more recent years two extensions have been added:

- The introduction of patterns, leading to term rewriting systems and extensions of λ-calculus, in general to higher-order rewriting in various forms (HRSs, CRSs and other formats).
- The introduction of infinite terms, and infinite computations.

We note that the latter extension is in mathematics entirely natural: there we have also infinite terms such as power series, infinite summations and so on.

As a statement of personal belief, we remark that in our opinion orthogonal term rewriting constitutes the most attractive paradigm of computation, and captures the most essential aspects of computation. Of course there are several other paradigms of computation, Turing Machines being the most prominent. But λ-calculus, CL, OTRSs, HRSs, CRSs have an innate elegance and directness that TMs with their machine language are lacking. Moreover, the rewrite paradigm is very much intertwined with logic; there is an extensive model theory for λ-calculus and rewriting, and not for Turing Machines or other computational paradigms. To appreciate the directness of rewriting, construct a Turing Machine definition of the infinite Thue-Morse sequence! Or compare the elegance of Church's numerals and arithmetic on them in λ-calculus, with the 'equivalent' treatment in TMs! Cf. the chapter in Penrose's book, who not for nothing includes a chapter devoted to λ-calculus.

To continue this statement of belief, we describe an image, a 'Gestalt' that we find captivating. (The precise statement is intended in a next release to be included in Chapter 1, as the Consistency Theorem.) Imagine a finite or infinite term, λ-term or TRS term U_0. Conceive it as a mini-universe, that can develop non-deterministically in various directions, along possibly infinite evolution paths α, β, γ. It may do so in a transfinite time-scale, though this is not essential.

What can become of U_0? Is there an end result? Are there more final results that U_0 can evolve to? The rules of evolution are very minimalistic, which is a good deal of the charm (similarly present in S. Wolfram's book NKS with its central notion of one-dimensional cellular automaton; simple rule sets may produce complex objects).

The 'evolution rules' are very constrained: they are orthogonal to each other.

(In the case of λβ-calculus there is just one rule.) Now due to this orthogonality, a remarkable phenomenon emerges. There is already from the start, the moment zero of evolution, a predestined image as a blueprint of the completion of the evolution, and it is unique. On the way, a part of this faint image to be, has already evolved. In such an intermediate term, there is an initial part (prefix) that is fixed and immutable in whatever future development. However, the catch is that we cannot decide how far this stable part extends, what places are already stable and which are not. All we know is that it exists. Now the miracle is that a different evolution path β may lead to quite another partially completed situation, whose fixed part however is consistent with that of the other, obtained via the evolution path α! And secondly, the stable, fixed part of a term grows with the progression along an evolution path. In the end all partially completed images come together in the same completed term. From U_0 to U_Ω. Thirdly, the evolution of U_0 to U_Ω need not go smoothly along an evolution path α. It can stagnate, and there may appear anomalies, spots in the picture that are like black holes. They do not yield information, they just are singularities, failures, undefined entities. The evolution α registers them, and that's all. But they are there, and form an unavoidable companion to the regular, 'productive' normal parts. Also the position and emergence of these black spots is predetermined from the start. All this has a strong metaphoric appeal. Summing up, our statement of belief is that orthogonal term rewriting as a model of computation is a thing of beauty.

Chapter 1
First order infinitary rewriting

1.1.1. Ordinals

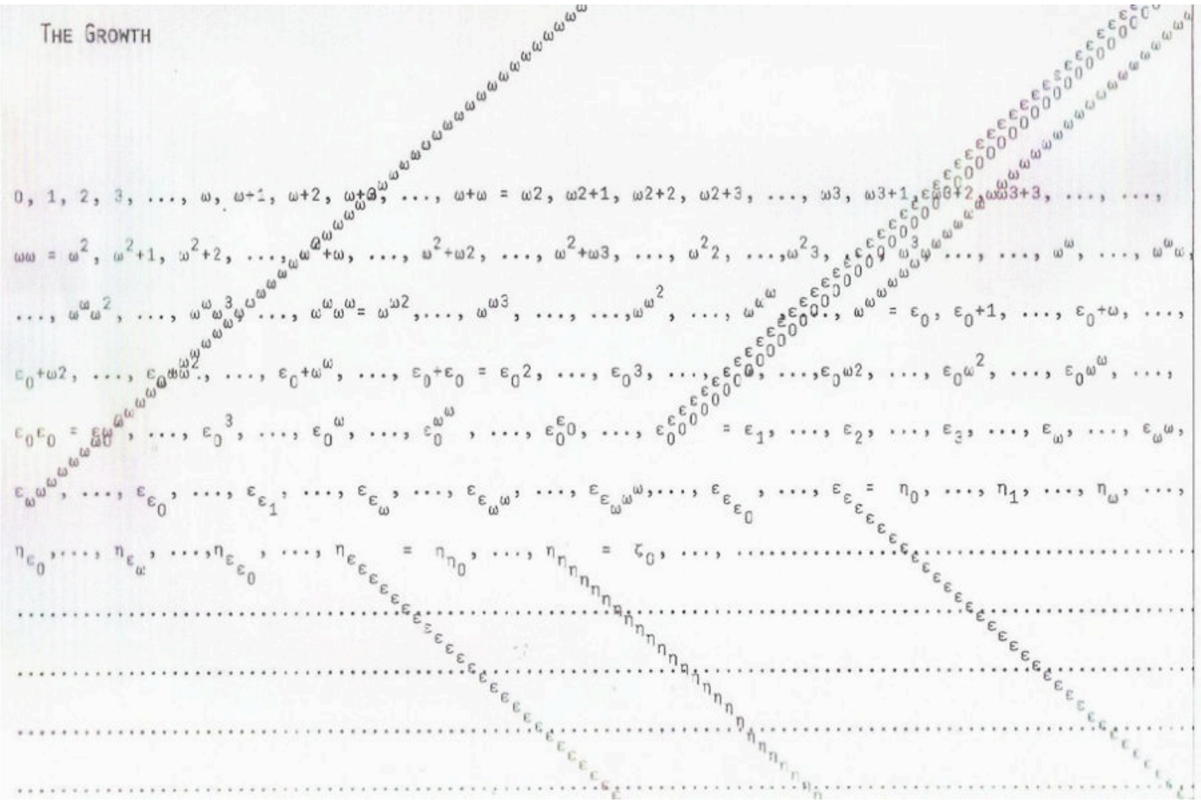


Figure 1.1. The Growth, by Henk Barendregt. In: Hans Koetsier, Advertisements 1969-1981, Staatsuitgeverij, page nr. 81, no structure left, published 25-1-1975 in dutch weekly journal Vrij Nederland.

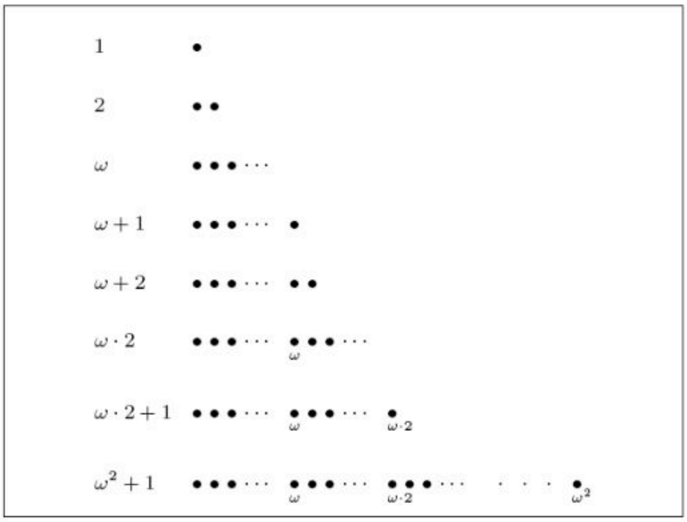


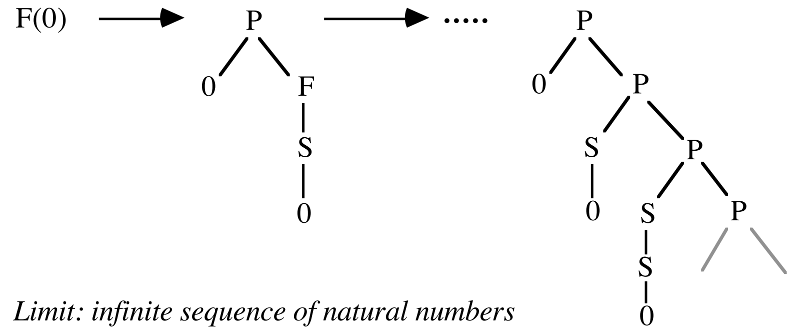
Fig. 3. Dot diagrams of the ordinals 1, 2, ω, $\omega + 1$, $\omega + 2$, $\omega \cdot 2$, $\omega \cdot 2 + 1$, and $\omega^2 + 1$.

Figure 1.2. From: Benedikt Löwe, Visualisation of ordinals.

1.1.2. Convergence

In this chapter we will consider infinite terms over a first order signature. Our starting point is an ordinary TRS (Σ, R). In fact, we will suppose throughout that our TRSs are *orthogonal*. Now it is obvious that the rules of the TRS (Σ, R) just as well apply to infinite terms as to the usual finite ones. First, let us explain the notion of infinite term that we have in mind. As before, let Ter(Σ) be the set of finite Σ-terms. Then Ter(Σ) can be equipped with a distance function d such that for t, s ∈ Ter(Σ), we have $d(t, s) = 2^{-n}$ if the n-th level of the terms s, t (viewed as labeled trees) is the first level where a difference appears, in case s and t are not identical; furthermore, d(t, t) = 0. It is well-known that this construction yields (Ter(Σ), d) as a metric space. Now infinite terms are obtained by taking the completion of this metric space, and they are represented by infinite trees. We will refer to the complete metric space arising in this way as ($Ter^{\infty}(\Sigma)$, d), where $Ter^{\infty}(\Sigma)$ is the set of finite and infinite terms over Σ.

A natural consequence of this construction is the emergence of the notion of *Cauchy convergence*: we say that $t_0 \to t_1 \to t_2 \to \ldots$ is an infinite reduction sequence with limit t, if t is the limit of the sequence t_0, t_1, ... in the usual sense of Cauchy convergence. See Figure 1.1 for an example, based on a rewrite rule F(x) → P(x, F(S(x)) in the presence of a constant 0.



Limit: infinite sequence of natural numbers

Figure 1.1

In the sequel we will however adopt a stronger notion of converging reduction sequence which turns out to have better properties. First, let us argue that it makes sense to consider not only reduction sequences of length ω, but even reduction sequences of length α for arbitrary ordinals α. Given a notion of convergence, and limits, we may iterate reduction sequences beyond length ω and consider e.g.

$$t_0 \to t_1 \to t_2 \to \ldots \to t_n \to \ldots$$
$$s_0 \to s_1 \to s_2 \to s_3 \to \ldots \;\; r$$

where $\lim_{n\to\infty} t_n = s_0$ and $\lim_{n\to\infty} s_n = r$. See Figure 1.2 for such a reduction sequence of length ω + ω, which may arise by evaluating first the left part of the term at hand, and next the right part. Of course, in this example a 'fair' evaluation is possible in only ω many reduction steps, but we do not want to impose fairness requirements at the start—even though we may (and will) consider it to be a desirable feature that reductions of length α could be 'compressed' to reductions of length not exceeding ω steps, yielding the same 'result'.

We will give a formal definition now.

1.1. DEFINITION. Let (Σ, R) be a TRS. A *(Cauchy-) convergent* R-*reduction sequence of length* α (an ordinal) is a sequence $\langle t_\beta \mid \beta \leq \alpha \rangle$ of terms in $Ter^\infty(\Sigma)$, such that

(i) $t_\beta \to_R t_{\beta+1}$ for all $\beta < \alpha$,

(ii) $t_\lambda = \lim_{\beta<\lambda} t_\beta$ for every limit ordinal $\lambda \leq \alpha$).

Here (ii) means: $\forall n\, \exists \mu < \lambda\, \forall \nu\, (\mu \leq \nu \leq \lambda \;\Rightarrow\; d(t_\nu, t_\lambda) \leq 2^{-n})$.

Notation: If $\langle t_\beta \mid \beta \leq \alpha \rangle$ is a Cauchy-convergent reduction sequence we write

$t_0 \rightarrow_\alpha^c t_\alpha$ (‘c’ for ‘Cauchy’).

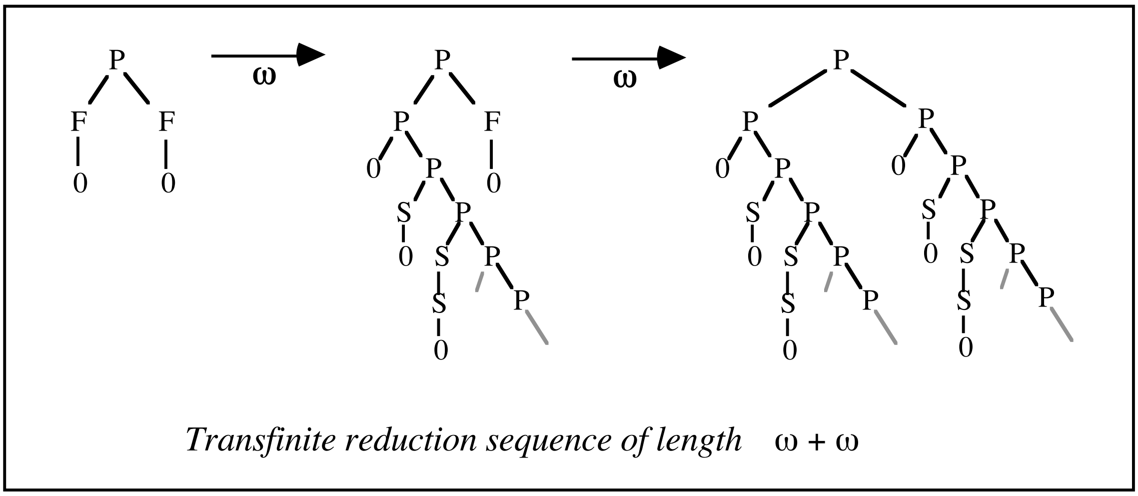


Figure 1.2

The notion of normal form as a final result has to be considered next. We simply generalize the old finitary notion of normal form to the present infinitary setting thus: a (possibly infinite) term is a normal form *when it contains no redexes*. The only difference with the finitary case is that here a redex may be itself an infinite term. But note that a redex is still so by virtue of a finite prefix, that was called the redex pattern—this is so because our rewrite rules are orthogonal and hence contain no repeated variables.

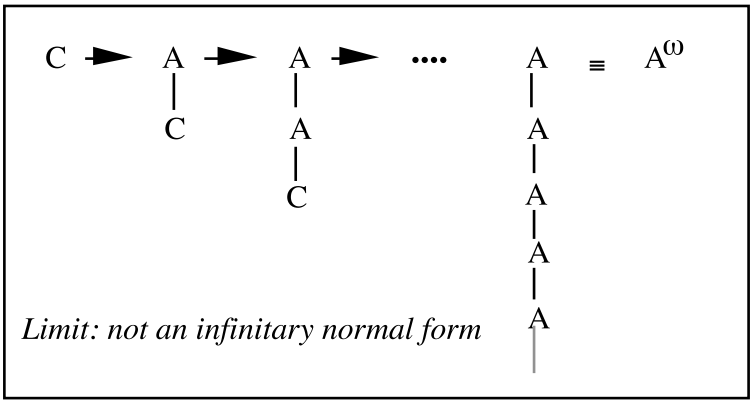


Figure 1.3

So, in Figure 1.3 we have, with as TRS {C → A(C), A(x) → x}, a (Cauchy-) converging reduction sequence with as limit the infinite term A(A(A(A..., abbreviated as A^ω; this limit is not a normal form: A^ω reduces to itself: $A^\omega \to A^\omega$, and only to itself. (Note that this step can be performed in infinitely many different ways, since every A in A^ω is the root of a redex.) Normal forms are shown in Figures 1.1, 1.2 as the rightmost terms (if no other reduction rules are present than the one mentioned above). Henceforth we will often drop the reference 'infinite' or 'infinitary'. Thus a term, or a normal form, may be finite or infinite. The notion of Cauchy converging reduction sequence that was considered so far, is not quite satisfactory. We would like to have the *compression property*:

$$t_0 \to_\alpha{}^c t_\alpha \;\Rightarrow\; t_0 \to_{\leq\omega}{}^c t_\alpha .$$

That is, given a reduction $t_0 \to_\alpha{}^c t_\alpha$, of length α, the result t_α can already be found in at most ω many steps. ('At most', since it may happen that a transfinite reduction sequence can be compressed to finite length, but not to length ω.) Unfortunately, $\to_\alpha{}^c$ lacks this property:

1.2. COUNTEREXAMPLE. Consider the orthogonal TRS with rules $\{A(x) \to A(B(x)), B(x) \to E(x)\}$. Then $A(x) \to_\omega A(B^\omega) \to A(E(B^\omega))$, so $A(x) \to_{\omega+1} A(E(B^\omega))$. However, we do not have $A(x) \to_{\leq\omega} A(E(B^\omega))$, as can easily be verified.

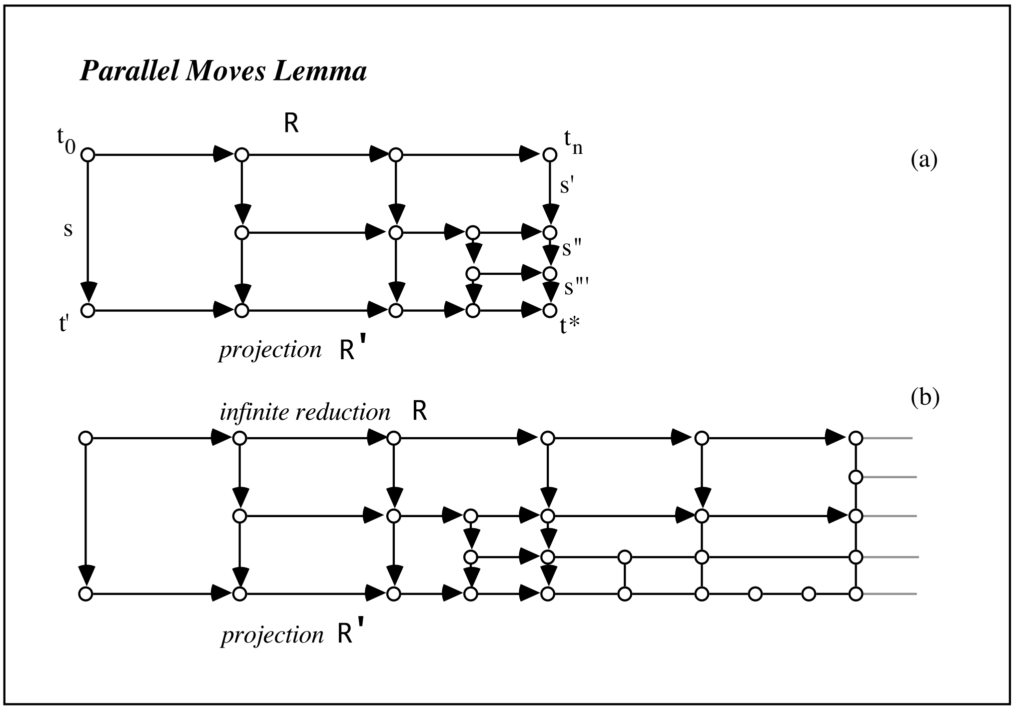


Figure 1.4

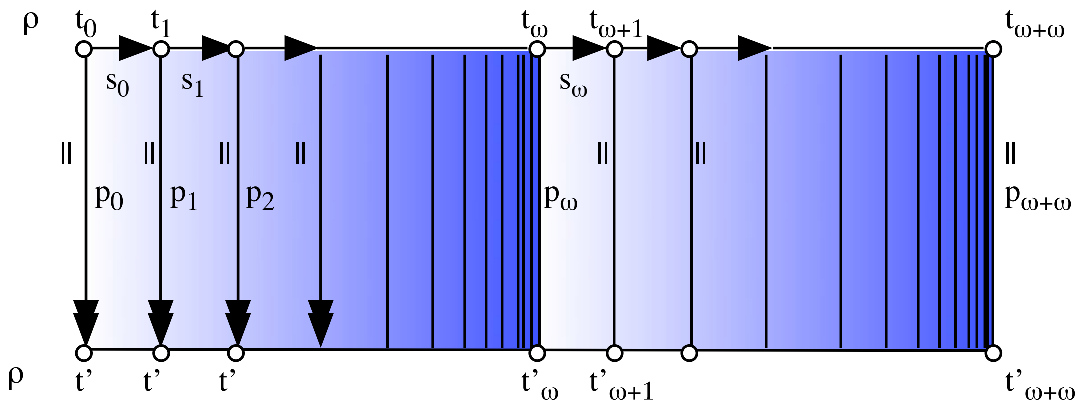


Figure 1.5.

Another obstacle for $\rightarrow_\alpha{}^c$ is that the well-known Parallel Moves Lemma resists a generalization to the present transfinite case. We recall the PML in Figure 1.4(a): setting out a finite reduction R: $t_0 \twoheadrightarrow t_n$ against a one step reduction $t_0 \rightarrow_s t'$ (where s is the contracted redex), one can complete the reduction diagram in a canonical way, thereby obtaining as the righthand side of the diagram a reduction $t_n \twoheadrightarrow t^*$ which consists entirely out of contractions of all the *descendants of* s *along* R. Furthermore, the reduction R': $t' \twoheadrightarrow t^*$ arising as the lower side of this reduction diagram, is called the *projection of* R *over the reduction step* $t_0 \rightarrow_s t'$. Notation: $R' = R/ (t_0 \rightarrow_s t')$.

We would like to have a generalization of PML where R is allowed to be infinite, and converging to a limit. In this way we would have a good stepping stone towards establishing infinitary confluence properties. However, it is not clear at all how such a generalization can be established. The problem is shown in Figure 1.5. First note that we can without problem generalize the notion of 'projection' to infinite reductions, as in Figure 1.4(b): there R' is the projection of the infinite R over the displayed reduction step. This merely requires an iteration of the finitary PML, no infinitary version is needed. Now consider the two rule TRS {A(x, y) → A(y, x), C → D}. Let R be the infinite reduction A(C, C) → A(C, C) → A(C, C) → ... , in fact a reduction cycle of length 1. Note that R is converging, with limit A(C, C). The projection R' of R over the step A(C, C) → A(D, C), however, is no longer converging. For, this is A(D, C) → A(C, D) → A(D, C) → ..., a 'two cycle'. So, the class of infinite converging reduction sequences is not closed under projection. This means that in order to get some decent properties of infinitary reduction in this sense, one has to impose further restrictions.

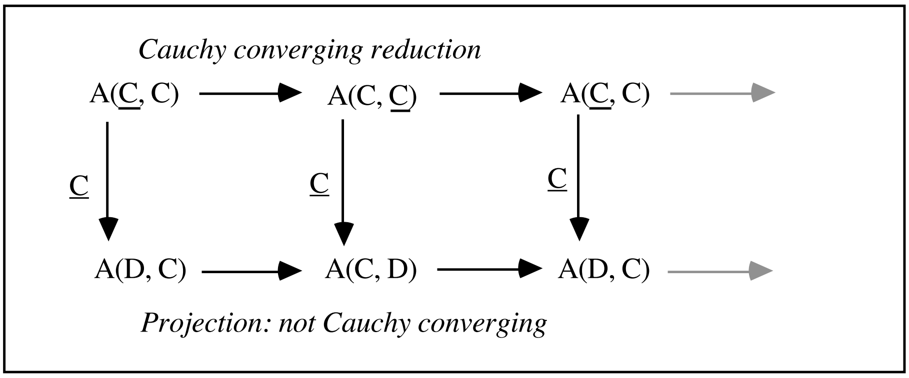


Figure 1.6

As the last example shows, there is a difficulty in that we loose the notion of descendants which is so clear and useful in finite reductions. Indeed, after the infinite reduction A(C, C) → A(C, C) → A(C, C) → ... , with Cauchy limit A(C, C), what is the descendant of the original underlined redex C in the limit A(C, C)? There is no likely candidate.
We will now describe the stronger notion of converging reduction sequence that does preserve the notion of descendants in limits. If we have a converging reduction sequence $t_0 \to_{s0} t_1 \to_{s1} \ldots t$, where s_i is the redex contracted in the step $t_i \to t_{i+1}$ and t is the limit, we now moreover require that

$$\lim_{i\to\infty} \text{depth}(s_i) = \infty. \qquad (*)$$

Here depth(s_i), the depth of redex s_i, is the distance of the root of t_i to the root of the subterm s_i. If the converging reduction sequence satisfies this additional requirement (*), it is called *strongly convergent*. The difference between the previous notion of (Cauchy) converging reduction sequence and the present one, is suggested by Figure 8.6. The circles in that figure indicate the root nodes of the contracted redexes; the shaded part is that prefix part of the term that does not change anymore in the sequel of the reduction. The point of the additional requirement (*) is that this growing non-changing prefix is required really to be non-changing, in the sense that no activity (redex contractions) in it may occur at all, even when this activity would by accident yield the same prefix.

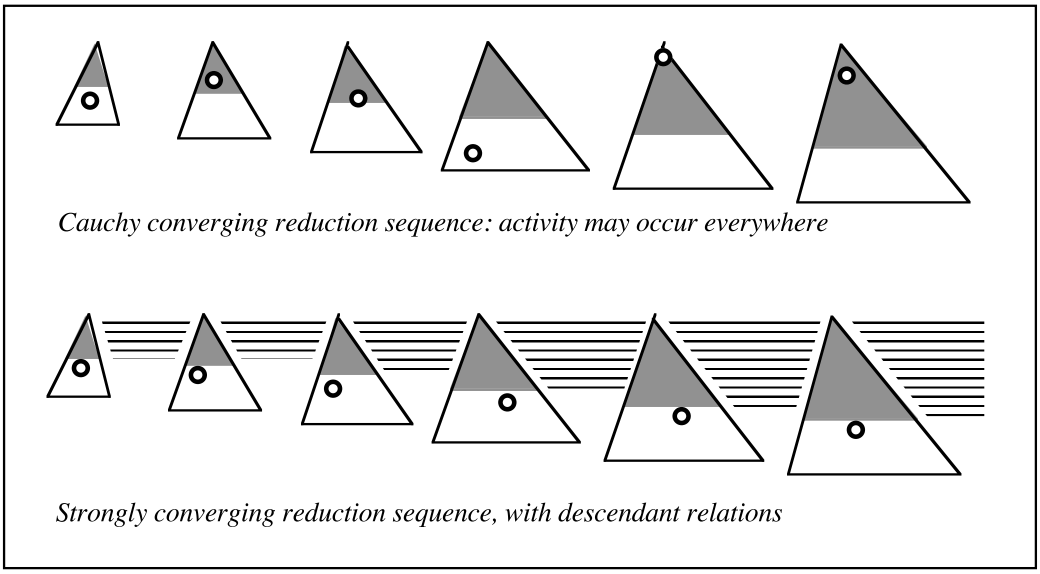


Figure 1.7

Note that there is now an obvious definition of descendants in the limit terms; the precise formulation is not hard to make explicit.

In fact, we define strongly converging reductions of length α for every ordinal α, by imposing the additional condition (*) whenever a limit ordinal $\lambda \le \alpha$ is encountered. (It will turn out however that only countable ordinals may occur.) More formally:

1.3. DEFINITION. Let (Σ, R) be a TRS. A *strongly convergent* R-*reduction sequence of length* α is a sequence $\langle t_\beta \mid \beta \le \alpha \rangle$ of terms in $Ter^\infty(\Sigma)$, such that

(i) $t_\beta \to_R t_{\beta+1}$ for all $\beta < \alpha$,

(ii) for every limit ordinal $\lambda \le \alpha$:

$$\forall n\ \exists \mu < \lambda\ \forall \nu\ (\mu \le \nu \le \lambda\ \Rightarrow\ d(t_\nu, t_\lambda) \le 2^{-n}\ \&\ depth(s_\nu) \ge n).$$

Here s_ν is the redex contracted in the step $t_\nu \to t_{\nu+1}$. (See Fig. 1.7.)

Notation: If $\langle t_\beta \mid \beta \le \alpha \rangle$ is a strongly convergent reduction sequence we write $t_0 \twoheadrightarrow_\alpha t_\alpha$.

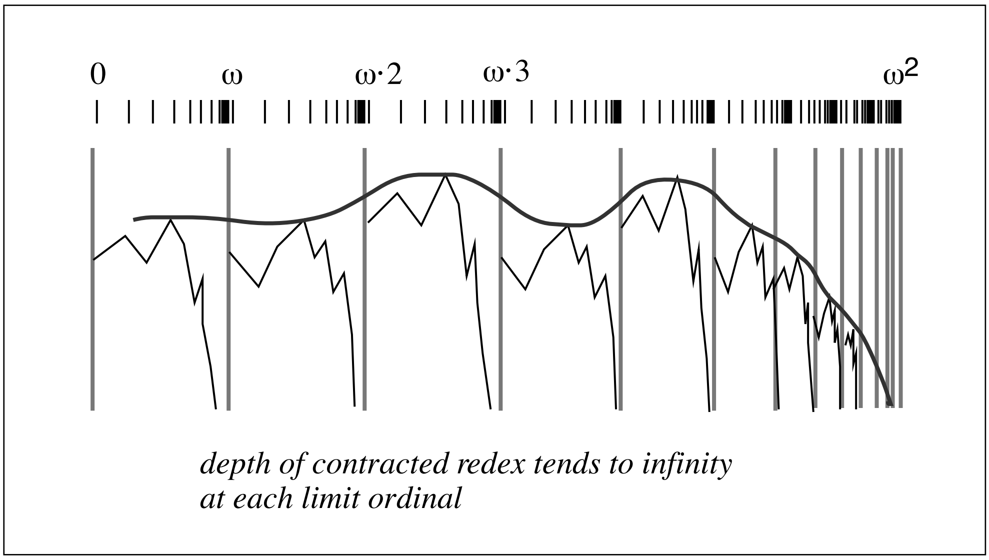


Figure 1.8

Henceforth strong convergence will be the default notion. Now we can state the benefits of this notion.

1.1.3. Compression

1.4. COMPRESSION LEMMA. *In every orthogonal TRS:*

$$t \to_{\alpha} t' \quad \Rightarrow \quad t \to_{\leq \omega} t'.$$

(Note that the counterexample 1.2 to compression for Cauchy converging reductions was not strongly converging.)

1.1.4. Infinitary properties

	Ordinary rewriting (finitary)	**Infinitary or transfinite rewriting**
1	*finite reduction*	*strongly convergent reduction*
2	*infinite reduction*	*divergent reduction (‘stagnating’)*
3	*normal form*	*possibly infinite normal form*
4	CR: *two coinitial finite reductions can be prolonged to a common term*	CR^{∞}: *two coinitial strongly convergent reductions can be prolonged by strongly convergent reductions to a common term*

	Ordinary rewriting (finitary)	**Infinitary or transfinite rewriting**
5	UN: *two coinitial reductions ending in normals forms, end in the same normal form*	UN^∞: t*wo coinitial strongly convergent reductions ending in (possibly infinite) normal forms, end in the same normal form*
6	SN: *all reductions lead eventually to a normal form*	SN^∞: *all reductions lead eventually to a possibly infinite normal form, equivalently: there is no divergent reduction*
7	WN: *there is a finite reduction to a normal form*	WN^∞: *there is a strongly convergent reduction to a possibly infinite normal form*

Table 1.1.

1.1.5. Zero times infinity

Let us discuss all the concepts introduced so far by means of the example given in Figure 1.9, where the reduction rules for Addition and Multiplication due to Dedekind are stated, in combination with a reduction rule defining the constant ∞ for 'infinity'. The constant 0 and the binary S for successor generate the finite natural numbers. These rules compute some familar identities for ∞, such as

$$A(S^n(0), \infty) = A(\infty, S^n(0)) = A(\infty , \infty) = \infty,$$

in the sense that these terms reduce to the same infinite normal form, namely S^ω = S(S(S(.... . There are also some plausible identities involving multiplication and ∞, to wit

$$M(S^{n+1}(0), \infty) = M(\infty, S^{n+1}(0)) = M(\infty , \infty) = \infty$$

How about zero times infinity? The equation $M(\infty, 0) = \infty$ is immediate, but the term $M(0, \infty)$ is interesting, since it it turns out to be undefined, in a formal sense. The whole reduction graph including all finite and infinite reducts of $M(0, \infty)$ is displayed in Figure xx. It turns out to be full of cycles, the shortest one displayed in red. All terms in the graph are hypercollapsing; the term below right, a regular tree that we render in abbreviation as $\mu x.\ A(x, 0)$ is reducible only to itself, even in infinitely many different one step reductions. None of the terms in the graph has a normal form, i.e. they are not WN^∞. There is no longest strongly convergent reduction, in fact there are strongly convergent reduction of any countable ordinal length. The same holds for divergent reductions. The diagonal steps are all collapsing steps; they seem to emanate from the term $\mu x.\ A(x, 0)$; but note that this term is not a starting point of any of these collapsing steps.

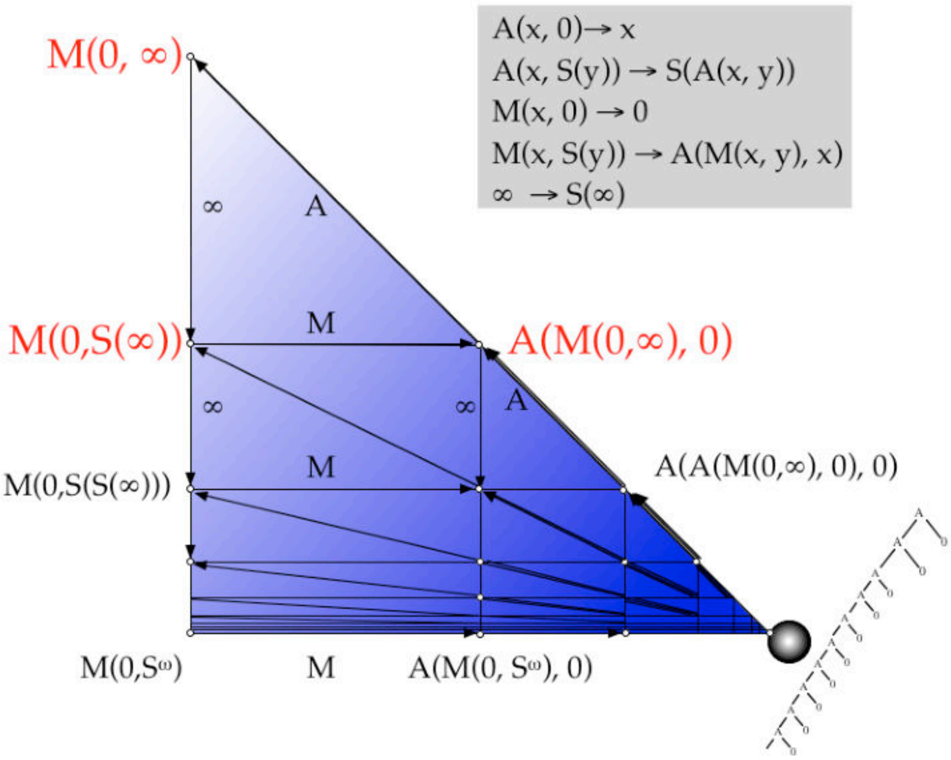
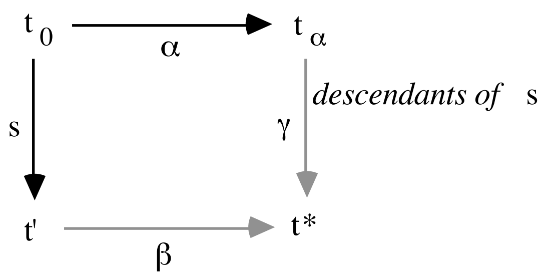


Figure 1.9: *Zero times infinity*

1.5. INFINITARY PARALLEL MOVES LEMMA (PML$^\infty$). *In every orthogonal TRS:*



That is, whenever $t_0 \to_\alpha t_\alpha$ *and* $t_0 \to_s t'$, *where* s *is the contracted redex (occurrence), there are infinitary reductions* $t' \to_\beta t^*$ *and* $t_\alpha \to_\gamma t^*$. *The latter reduction consists of contractions of all descendants of* s *along the reduction* $t_0 \to_\alpha t_\alpha$.

Actually, by the Compression Lemma we can find $\beta, \gamma \leq \omega$.

As a side-remark, let us mention that in every TRS (even with uncountably many symbols and rules), all transfinite reductions have countable length. All countable ordinals can indeed occur as length of a strongly convergent reduction. (For ordinary Cauchy convergent reductions this is not so: the rewrite rule C → C yields arbitrarily long convergent reductions $C \to_\alpha^c C$. However, these are not strongly convergent.)

The infinitary PML^∞ is "half of the infinitary confluence property". The question arises whether full infinitary confluence (CR^∞) holds. That is, given $t_0 \to_\alpha t_1$, $t_0 \to_\beta t_2$, is there a t_3 such that $t_1 \to_\gamma t_3$, $t_2 \to_\delta t_3$ for some γ, δ? Using the Compression Lemma and the finitary PML all that remains to prove is: given $t_0 \to_\omega t_1$, $t_0 \to_\omega t_2$, is there a t_3 such that $t_1 \to_{\leq\omega} t_3$, $t_2 \to_{\leq\omega} t_3$? Surprisingly, the answer is negative: *infinitary confluence for orthogonal rewriting does not hold*. The counterexample is in Figure 1.8, consisting of an orthogonal TRS with three rules, two of which are 'collapsing rules'. (A rule $t \to s$ is collapsing if s is a variable.) Indeed, in Figure 1.8(a) we have $C \to_\omega A^\omega$, $C \to_\omega B^\omega$ but A^ω, B^ω have no common reduct as they only reduce to themselves. Note that these reductions are indeed strongly convergent. (Figure 1.8(b) contains a rearrangement of these reductions that we need later on.)

However, the good news is that in spite of the failure of CR^∞ we do have unicity of (possibly infinite) normal forms (UN^∞).

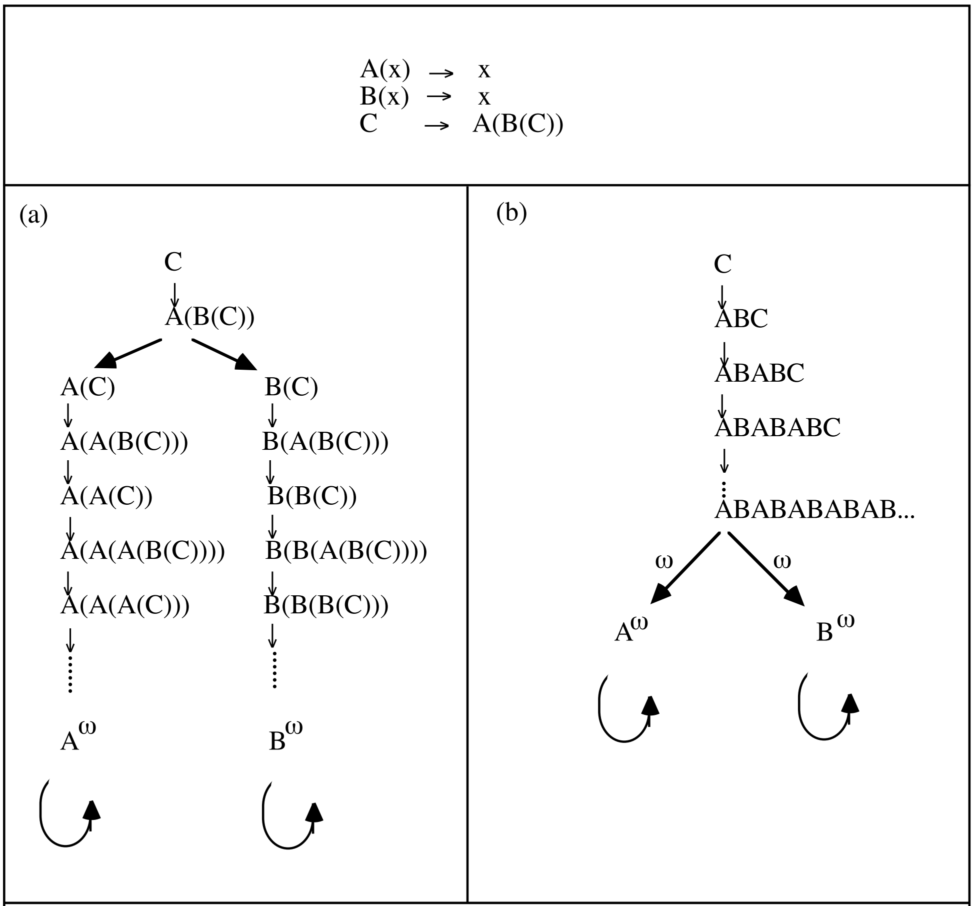
1.6. THEOREM. *For all orthogonal TRSs: Let* $t \to_\alpha t'$, $t \to_\beta t''$ *where* t', t'' *are (possibly infinite) normal forms. Then* $t' \equiv t''$.

Here $\equiv$ denotes syntactical equality. Note that in the ABC counterexample in Figure 1.8 the terms A^ω and B^ω are not normal forms.

This Unique Normal Form property, by the way, also holds for Cauchy converging reductions, that is, with $\to_\alpha$ replaced by $\to_\alpha^c$ and likewise for β. The reason is that we have:

$$t \to_\alpha^c t' \ \&\ t' \textit{ is a normal form} \quad \Rightarrow \quad t \to_{\leq\omega} t'.$$

(For $\alpha = \omega$ this is easy to prove; in fact a converging reduction of length ω to a normal form is already strongly convergent. For general α, the proof of the statement requires some work.)



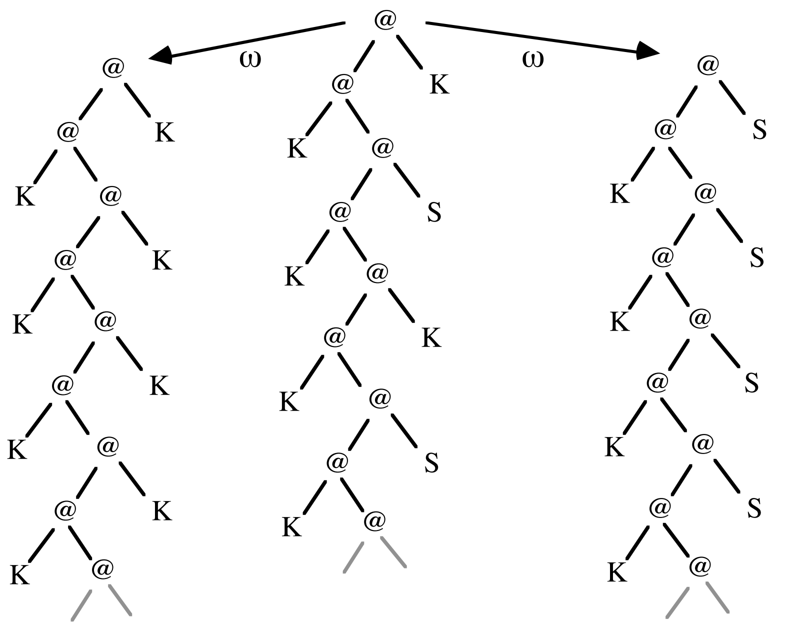
Failure of infinitary confluence

Figure 1.9

The ABC example (Figure 1.9) is not merely a pathological example; the same phenomenon (and therefore failure of infinitary confluence) occurs in Combinatory Logic, as in Figure 1.10, where an infinite tower built from the two different collapsing contexts K □ K and K □ S is able to collapse in two different ways. (Note that analogous to the situation in Figure 1.9, the middle term, built alternatingly from K □ K and K □ S, can be obtained after ω steps from a finite term which can easily be found by a fixed point construction.) Also for λ-calculus one can now easily construct a counterexample to infinitary confluence.

Sxyz $\rightarrow$	xz(yz)	@(@(@(S, x), y), z)	$\rightarrow$ @(@(x, z), @(y, z))
Kxy $\rightarrow$	x	@(@(K, x), y)	$\rightarrow$ x

collapsing contexts



Failure of infinitary confluence for Combinatory Logic

Figure 1.10

We will now investigate the extent to which infinitary orthogonal rewriting lacks full confluence. It will turn out that non-confluence is only marginal, and that terms which display the bad behaviour are included in a very restricted class. The following definition is inspired by a classical notion in λ-calculus; see Barendregt [84].

1.7. DEFINITION. (i) The term t is in *head normal form* (hnf) if $t \equiv C[t_1,\ldots,t_n]$ where $C[t_1,\ldots,t_n]$ is a non-empty context (prefix) such that no reduction of t can affect the prefix C[,...,]. More precisely, if $t \twoheadrightarrow s$ then $s \equiv C[s_1,\ldots,s_n]$ for some s_i (i =1,...,n), and every redex of s is included in one of the s_i (i =1,...,n).
(ii) t *has a hnf* if $t \twoheadrightarrow s$ and s is in hnf.

Actually, this definition is equivalent to one of DKP[89]; there a term t is called 'top-terminating' if there is no infinite reduction $t \to t' \to t'' \to \ldots$ in which infinitely many times a redex contraction *at the root* takes place. So:
t is top-terminating ⇔ t has a hnf. We need one more definition before formulating the next theorem.

1.8. DEFINITION. If t is a term of the TRS R, then the *family* of t is the set of subterms of reducts of t, i.e. $\{s \mid t \twoheadrightarrow_R C[s]$ for some context $C[\,]\}$.

1.9. THEOREM. *For all orthogonal TRSs: Let* t *have no term without hnf in its family. Then* t *is infinitary confluent*.

Here we want to reconsider the last theorem. Actually, it can be much improved. Consider again the ABC example in Figure 1.9. Rearranging the reductions $C \to_\omega A^\omega$, $C \to_\omega B^\omega$ as in Figure 1.9(b) into reductions $C \to_\omega (AB)^\omega \to_\omega A^\omega$ and $C \to_\omega (AB)^\omega \to_\omega B^\omega$ makes it more perspicuous what is going on: $(AB)^\omega$ is an infinite 'tower' built from two different collapsing contexts A( ), B( ), and this infinite tower can be collapsed in different ways. Remarkably, it turns out that the collapsing phenomenon is the *only* cause of failure of infinitary confluence. (The full proof is in KKVS[95a].) Thus we have:

THEOREM. (i) *Let the orthogonal TRS* R *have no collapsing rewrite rules* $t(x_1,...,x_n) \to x_i$. *Then* R *is infinitary confluent*.
(ii) *If* R *is an orthogonal TRS with as only collapsing rule:* $I(x) \to x$, *then* R *is infinitary confluent*.

Call an infinite term $C_1[C_2[...C_n[...]...]]$, built from infinitely many non-empty collapsing contexts $C_i[\ ]$, a *hyper collapsing* (hc) term. (A context C[ ] is collapsing if C[ ] contains one hole □ and C[ ] ↠ □.) Also a term reducing to a hc term is called a hc term. E.g. C from the ABC example in Figure 1.9 is a hc term. Clearly, hc terms do not have a hnf.

THEOREM. *Let* t *be a term in an orthogonal TRS, which has not a hc term in its family. Then* t *is infinitary confluent.*

This theorem can be sharpened somewhat, as follows. Consider the rewrite rule:

$t \to_\Omega \Omega$ *if* t *is a hc term*.

Of course this rule is not 'constructive', i.e. the reduction relation $\to_\Omega$ may be undecidable (as it is in CL, Combinatory Logic). However, we now have that orthogonal reduction extended with $\to_\Omega$ is infinitary confluent.

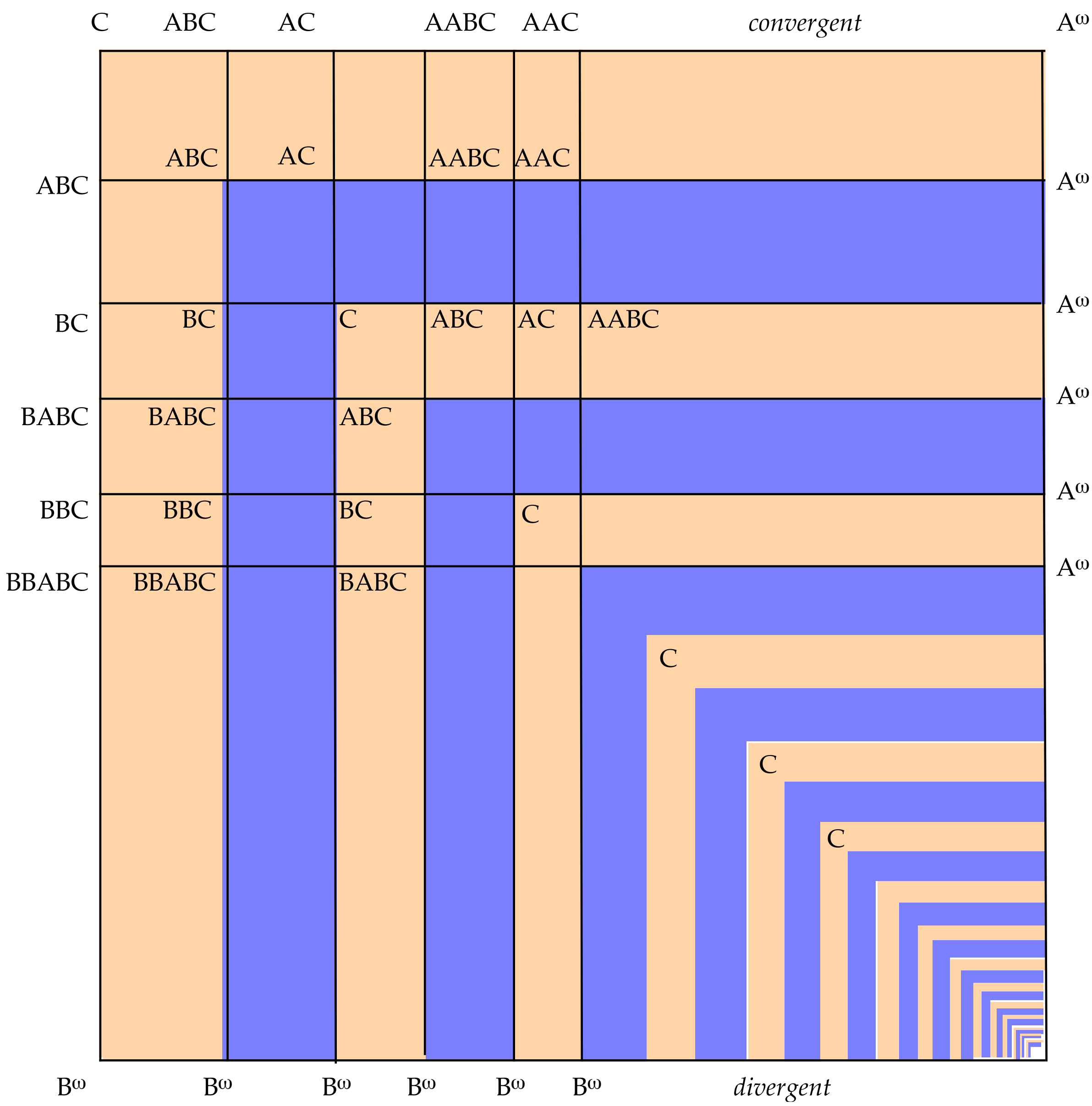


Figure 1.11. Reduction diagram of ABC counterexample

Steps in light-colored strips are trivial (empty) steps, in dark strips are proper steps. Note that the projection of the upper horizontal strongly convergent reduction over the left vertical strongly convergent reduction, yields the divergent reduction

$$B^\omega \equiv B^\omega \to B^\omega \equiv B^\omega \to B^\omega \equiv B^\omega \to B^\omega \equiv B^\omega \to \ldots$$

where in each proper step the the top B-redex is contracted.

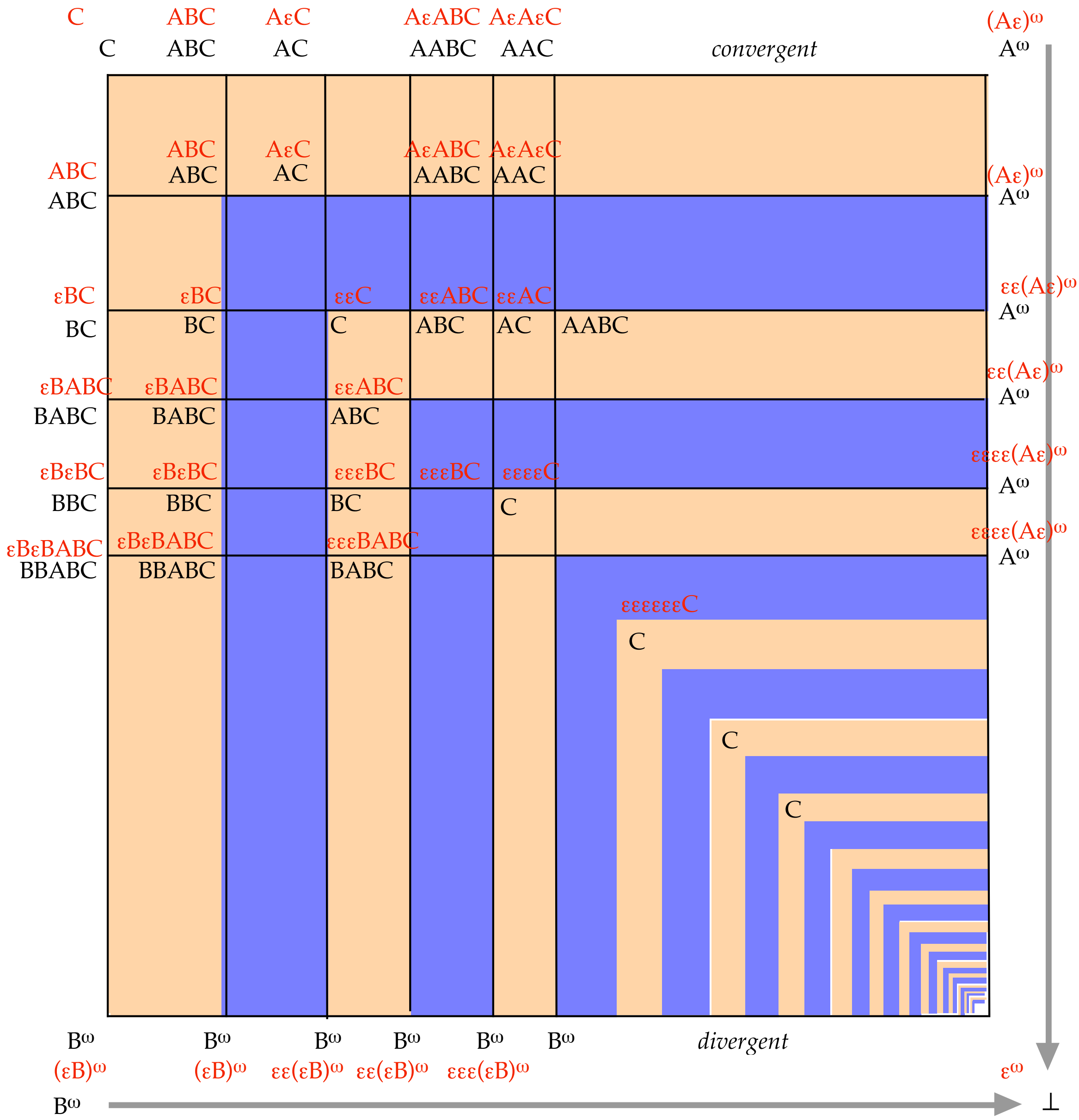


Figure 1.12. ε-lifting of ABC diagram

All terms in the lifted diagram are SN^{∞} (Exercise), and all infinite reductions are strongly convergent. The lifted terms and lifted reductions all project back (by deleting ε's) to the original terms and reductions–except for the lower-right corner point, the infinite normal form ε^ω. This term is replaced by bottom. We now look at the lifted reductions that yield ε^ω, and read off that the original terms B^ω and A^ω are hypercollapsing. These are both reduced to botom and confluence is restored.

1.3. Infinitary normalisation

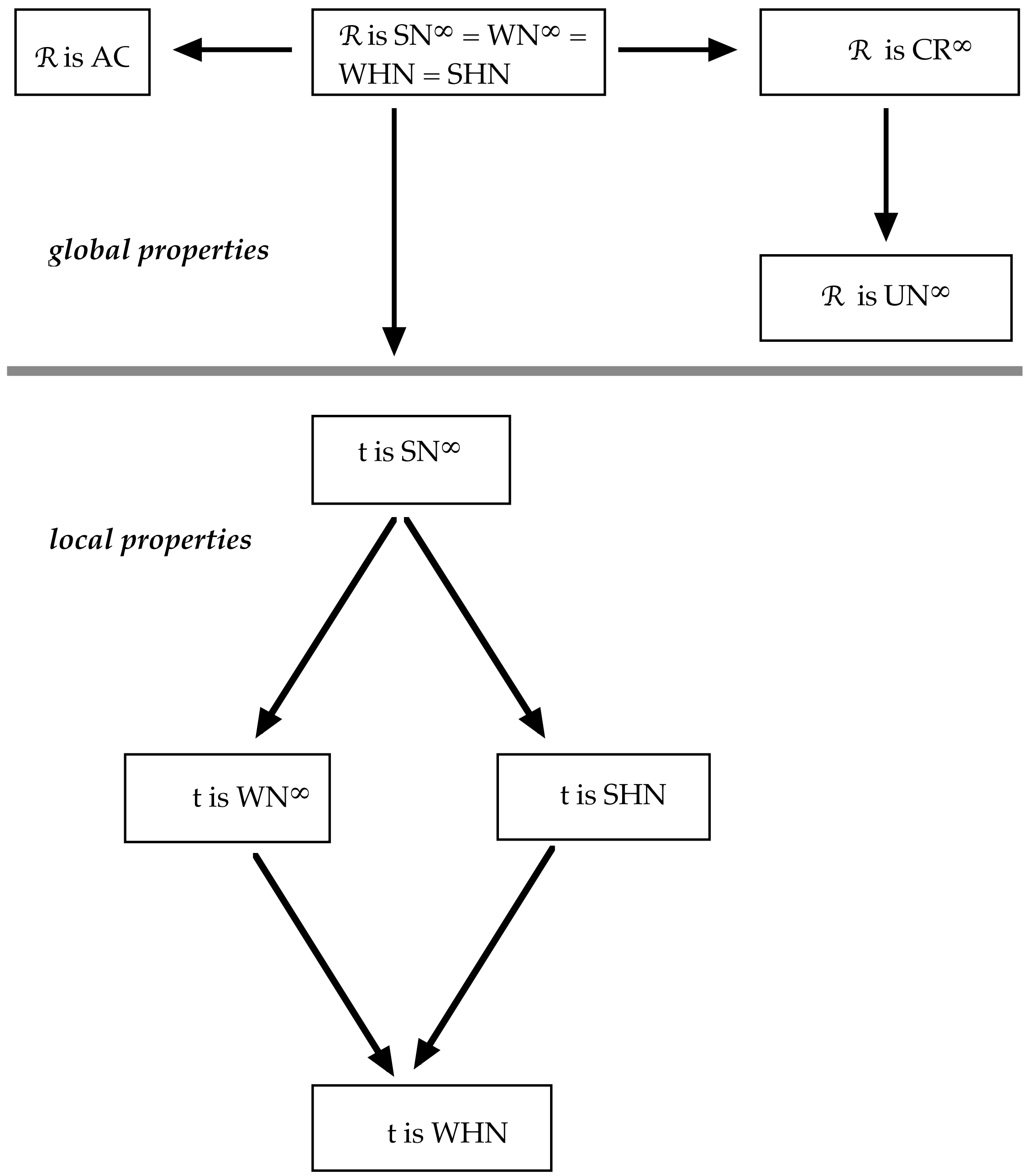


Figure 1.13

	PML	**CR**	**UN**	**PML$^\infty$**	**CR$^\infty$**	**UN$^\infty$**
OTRS	*yes*	*yes*	*yes*	*yes*	*no*	*yes*
WOTRS	*yes*	*yes*	*yes*	*yes*	*no*	*yes*
$\lambda\beta$	*yes*	*yes*	*yes*	*no*	*no*	*yes*
OCRS	*yes*	*yes*	*yes*	*no*	*no*	*yes*

Table 1.2

REMARK. The notions of finitary SN and infinitary SN$^\infty$ are independent, somewhat surprisingly.

(1) SN does not imply SN$^\infty$. Consider the rules for addition. This TRS is SN. But SN$^\infty$ does not hold, see the infinite term μx. A(x, 0). Another counterexample is given by the term I$^\omega$ and the TRS with rule I(x) → x.

(2) SN$^\infty$ does not imply SN.Take the fragment of Combinatory Logic (CL) consisting of the finite S-terms, with the S reduction rule. According to Waldmann [2000] this TRS is top-terminating. But the term AAA with A ≡ SSS has an infinite reduction:

SSSAA
SA(SA)A
AA(SAA)
SSSA(SAA)
SA(SA)(SAA)
A(SAA)(SA(SAA))
SSS(SAA)(SA(SAA))
S(SAA)(S(SAA))(SA(SAA))
SAA(SA(SAA))(S(SAA)(SA(SAA)))
A(SA(SAA))(A(SA(SAA)))(S(SAA)(SA(SAA)))
SSS(SA(SAA))(A(SA(SAA)))(S(SAA)(SA(SAA)))
S(SA(SAA))(S(SA(SAA)))(A(SA(SAA)))(S(SAA)(SA(SAA)))
SA(SAA)(A(SA(SAA)))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))
A(A(SA(SAA)))(SAA(A(SA(SAA))))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))
SSS(A(SA(SAA)))(SAA(A(SA(SAA))))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))
S(A(SA(SAA)))(S(A(SA(SAA))))(SAA(A(SA(SAA))))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))

1.4. Exercises and notes

EXERCISE 1.4.1. (i) Prove that an infinite reduction diagram must possess an infinite proper reduction (i.e. one without empty 'steps'. (Solution included.)
(ii) Obtain Newman's Lemma as a corollary of (i).

SOLUTION. Consider a construction of the infinite diagram in stages, by repeatedly adjoining an e.d. After each finite stage the diagram contains finitely many proper reductions starting from the initial term of the two diverging reductions that constitute stage 0 of the construction. Also, after each finite stage there must be eventually an adjunction of an e.d. with splitting converging sides; otherwise the construction would terminate. But such an adjunction will prolong, with at least one proper step, at least one of the finite proper reductions from the initial term that are present in the diagram at that stage. Now apply König's Lemma: in the limit an infinite proper reduction must arise.

EXERCISE 1.4.2. (A. Visser). As the dot diagrams of ordinals in Figure 1.2 suggest, ordinals can be mapped (embedded) into the segment of real numbers [0,1] in an order-preserving way. Prove that precisely the countable ordinals can be embedded in this way.

EXERCISE 1.4.3. (From Terese 2003, p.675.)
(i) Show that all finite reductions are strongly converging.
(ii) Show that all reductions in the 'binary tree' TRS given by the rule {C → B(C,C)} are strongly convergent and that they can be of any countable length.
(iii) Describe the normal form by strongly convergent transfinite reduction of F(A,B), given by the rule F(x,y) → G(x,F(y,H(x,y))).
(iv) Consider the TRS {J(x) → J(x)}. Let $\mathcal{R}$ be the strongly convergent reduction $J^\omega \to^\omega J^\omega$ in which the redex contracted at the n-th step is at depth n. Why is the ω^2 long sequence $\mathcal{R}^\omega = \mathcal{R};\mathcal{R};\mathcal{R};\ldots$, where ':' is concatenation of reduction sequences, not strongly converging?

EXERCISE 1.4.4. (*Failure of infinitary Newman's Lemma.*)
This Exercise establishes that we do *not* have the implication WCR & $SN^\infty \Rightarrow CR^\infty$. In other words, the infinitary version of Newman's Lemma WCR & SN ⇒ CR for abstract reduction systems (ARSs) fails for infinitary term rewriting. Consider the TRS with the three rules {C → A(C), A(C) → B(C), A(B(x)) → B(A(x))}. Note that we do not have A(x) → B(x)!

(i) Check that WCR holds, by looking at the critical pairs and applying Huet's Lemma stating that if all critical pairs are convergent, we have WCR.
(ii) We also have SN^∞; proving that is a nice exercise.
(iii) Observe that CR^∞ fails, as C reduces in ω steps to A^ω and B^ω, both infinite normal forms.
(iv) Some other counterexamples to the infinitary NL are as follows.

EXERCISE 1.4.5. Consider the collapsing rule I(x) → x as in CL, and consider the hypercollapsing term I^ω.
(i) Prove that has I^ω continuum many strongly convergent reductions.
(ii Prove that I^ω also has continuum many divergent reductions.

EXERCISE 1.4.6. Note that the proof of UN^∞ relied heavily on the properties of orthogonal rewriting. The question arises whether orthogonality is *necessary* for UN^∞ ; would mere confluence (finitary CR) not be enough? In fact we can answer this question $CR \Rightarrow UN^\infty$ negatively, with the following counterexample.
Consider the TRS R with the three rules:

$C \to A(C)$
$C \to B$
$A(B) \to B.$

So R is not orthogonal. We have the following reductions:

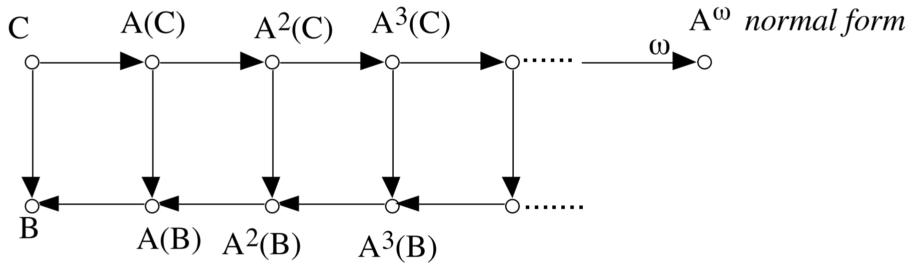


Figure 1.14. Counterexample to $CR \Rightarrow UN^\infty$

These are all the reducts of C. There are two normal forms, A^ω and B. Hence UN^∞ does not hold. All terms of Ter(R) are displayed; clearly, R is CR. R is not CR^∞, though.

EXERCISE 1.4,7. The property SN^∞ is called *top termination* in Waldmann [00]. It holds for the fragment of CL consisting of terms built from S's only, the S-terms. Note that the top or root redex in a CL term is not the same as the head redex. Indeed there are S-terms having an infinite head reduction. Give an example of such an S-term.

Solution.(H.P. Barendregt, private communication, January 2004.)
Define A = SSS and B = SAA.
Define $P > Q \Leftrightarrow P \twoheadrightarrow_h QR_1...R_n$, for some $R_1,...,R_n$.
Note that > is reflexive and transitive.
Define P^nQ as follows: $P^0Q = Q$; $P^{n+1}Q = P(P^nQ)$.

Lemma 1. $A^nxy > xy$, for $n > 0$.
Proof. Induction on n.
Case n = 0: trivial.
Case n+1: $A^{n+1}xy = A(A^nx) = SSS(A^nx)y > S(A^nx)(S(A^nx))y > A^nxy > xy$, by the IH.

Lemma 2. $A^nB(A^nB) > A^{n+1}B$.

Proof. By induction on n.

Case n = 0:

BB = SAAB > AB(AB)

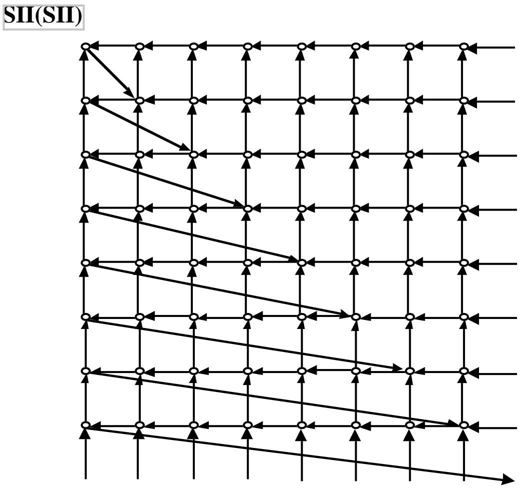
Case n+1:

$A^{n+1}BA^{n+1}B = A(A^nB)(A^{n+1}B = SSS(A^nB)(A^{n+1}B) > S(A^nB)(S(A^nB))(A^{n+1}B) > A^nB(A^{n+1}B) >$
$B(A^{n+1}B)$ (by Lemma 1), $= SAA(A^{n+1}B) > A^{n+2}B(A^{n+2}B)$.

Conclusion: BB has no weak-head nf.

Proof. BB > AB(AB) > $A^2B(A^2B)$ > ... . This is a proper infinite head reduction. (To be checked!)

EXERCISE 1.4.8.



The CL-term SII(SII) has the infinite reduction graph displayed to the left. Abbreviating ω ≡ SII, the terms at the nodes of this graph are $I^n\omega(I^m\omega)$ for n,m ≥0.

(i) Show that all these terms are root active, but not hypercollapsing.

(ii) Prove that all continuum many infinite reductions contained in this reduction graph, are divergent; in particular, they are root active.

Figure 1.15

SII(SII)
I(SII)(I(SII))
SII(I(SII))
I(I(SII))(I(I(SII)))
I(SII)(I(I(SII)))
SII(I(I(SII)))
I(I(I(SII)))(I(I(I(SII))))
I(I(SII))(I(I(I(SII))))
I(SII)(I(I(I(SII))))
SII(I(I(I(SII))))
I(I(I(I(SII))))(I(I(I(I(SII)))))
I(I(I(SII)))(I(I(I(I(SII)))))
I(I(SII))(I(I(I(I(SII)))))
I(SII)(I(I(I(I(SII)))))
SII(I(I(I(I(SII)))))
I(I(I(I(I(SII)))))(I(I(I(I(I(SII))))))
I(I(I(I(SII))))(I(I(I(I(I(SII))))))
I(I(I(SII)))(I(I(I(I(I(SII))))))
I(I(SII))(I(I(I(I(I(SII))))))
I(SII)(I(I(I(I(I(SII))))))
SII(I(I(I(I(I(SII))))))

I(I(I(I(I(SII))))))(I(I(I(I(I(SII)))))))
I(I(I(I(SII)))))(I(I(I(I(I(SII)))))))
I(I(I(SII))))(I(I(I(I(I(SII)))))))
I(I(SII)))(I(I(I(I(I(SII)))))))
I(I(SII))(I(I(I(I(I(SII)))))))
I(SII)(I(I(I(I(I(SII)))))))
SII(I(I(I(I(I(SII)))))))
I(I(I(I(I(I(SII)))))))(I(I(I(I(I(I(SII))))))))

Figure 1.16

EXERCISE 1.4.9. (*An alternative definition of strong convergence.*)
Definition.
(i) CC, *Cauchy convergence*, defined as before.
(ii) SC, *strong convergence*, defined as before.
(iii) CCC, *Cauchy convergence with colors*. Given is the first order signature Σ. We extend the signature by adding a colored activity marker α, a unary symbol with the reduction tule α(x) → x.The old reduction rules are changed in such a way that the rhs is prefixed with α. For CL this gives the rule for S:
Sxyz → α(xz(yx)).
Given an old reduction sequence, we can lift it to the colored version by applying the rules as modified, introducing the markers α. These are removed the next step using the α-rule.
So the infinite reduction SII(SII) →I(SII)(I(SII) → SII(I(SII)) → SII(SII → ... which is CC but not SC, is lifted to SII(SII) → α(I(SII)(I(SII)) → (I(SII)(I(SII) → α(SII)(I(SII)) → SII(I(SII)) → SII(α(SII)) → SII(SII) → ... which is not CC.

Proposition. SC ⇔ CCC.

So we can remove the depth requirement in favour of a signature extension and the old concept CC. We could view CCC as 'the' definition for SC, and then derive the depth requirement.

EXERCISE 1.4.10. In this Exercise we prove the Head Normalization Theorem. In fact, the theorem is proved in various places, among them Terese [03]. The relevant theorem there is xx, stating that outermost fair reductions are (head)-normalizing. That theorem as obtained in Terese [03] is concerned with a more general situation than the first-order framework that we adopt in this Exercise, and therefore uses some more advanced notions such as external redex, outermost fair reductions, and the SR-measure.

Here we will give a proof just for first-order TRSs, employing only the basic notion of elementary diagram for orthogonal TRSs, diagram construction by tiling with these elementary diagrams, and the notion of projection of a reduction over a step.

A1. DEFINITION. (i) A *head step* is a reduction step that takes place at the head or the root. This means that the redex contracted coincides with the whole term t. Notation: $t \to_h t'$.
(ii) A step which is not a head step is called an *internal* step; notation $t \to_i t'$.

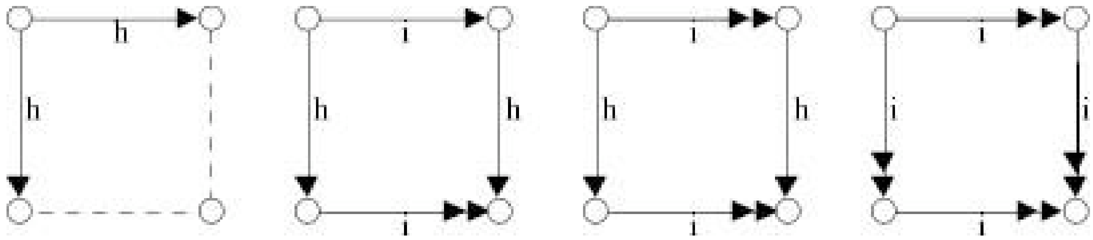
A2. PROPOSITION. Head steps and internal steps propagate through each other as in the diagrams in Figure A1. The proofs are elementary and omitted.

A3. THEOREM. (*Head Normalization Theorem*)

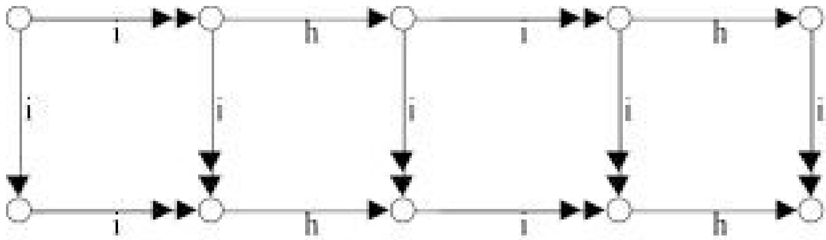
Let R *be an orthogonal TRS and let* t *be a term in* R *with a reduction* t → t'→ t"→ ... *containing infinitely many root steps.*

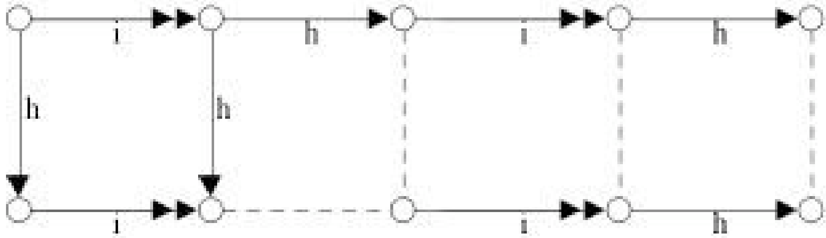
(i) *Then* t *has no head normal form.*

(ii) *A fortiori, t has no normal form.*



spot the error!





DEFINITION. (DEF. 2.3.10 in Kenn90a,b). A TRS is called *unifiable* if it contains a *unifiable* rule, that is a rule l → r such that for some substitution σ with finite and infinite terms for variables $l^\sigma \equiv r^\sigma$. Note that unifiability in the space of finite and infinite terms means unifiablity "without the occurs check": the terms I(x) and x are unifiable in this setting, and their most general unifier is the infinite term I^ω. Collapsing rules, i.e. rules which right hand side is a variable are unifiable.

LEMMA. (2.3.11 in Ken 90a,b) *The following are equivalent for an orthogonal TRS:*

(i) *The TRS is non-unifiable,*

(ii) *all Cauchy-convergent reductions of the TRS are strongly convergent,*

(iii) *all Cauchy-convergent reductions are top-terminating.*

THEOREM (2.3.12 in Ken90a,b). *Any non-unifiable orthogonal TRS has the infinite Church-Rosser Property for Cauchy convergent and for strongly convergent reductions.*

12.3.6. Exercise. Show that in any iTRS, a reduction sequence is strongly convergent if and only if for every natural number n, the number of steps of the sequence which reduce a redex at depth less than n is finite.

Use the preceding exercise to prove

12.3.7. Proposition. *Every strongly converging reduction has countable length.*

12.3.4. Example. Consider the rule $I(x) \to x$ and the term $I(I(I(\ldots)))$, which we will abbreviate to I^ω. Suppose we try to perform a complete development of each of the infinitely many redexes in this term, outermost first. We obtain the sequence $I^1(I^2(I^3(\ldots))) \to I^2(I^3(I^4(\ldots))) \to I^3(I^4(I^5(\ldots))) \to \cdots$, where we have attached labels to show how each subterm is derived from the subterms of the previous term. The sequence weakly converges to $I(I(I(\ldots)))$. In the process, every redex of the initial term gets reduced, yet in the limit, we appear to still have all of them left. Where did they come from?

For this reason we do not further consider weak continuity or convergence. When we talk of reduction sequences, they will be assumed to be strongly continuous, and if they have limits, they will be assumed to strongly converge to them. The condition on depths of reduction which strong convergence adds was first stated (for sequences of length up to ω^2) by Farmer and Watro [1990].

1.4.11. EXERCISE. (W. van der Poel) Write A ≡ (SS). Prove that the CL-term SASAS has a normal form, of length $\approx 10^6$.

1.4.12. EXERCISE. Give an example of a transfinite reduction of length $> \omega$ that cannot be compressed to length ω, but only to finite reductions.

1.4.13. EXERCISE.(i) Note that CL_I, the non-erasing variant of CL based on the combinators I and J, has the property CR^∞. The combinator J has the reduction rule
Jabcd → ab(adc).
(ii) Another basis for CL_I is given by the combinators B, S, C, I, with Bxyz → x(yz) and Cxyz → xzy.
Does this TRS have the property CR^∞ ?

EXERCISE (For Chapter 2) (i) Note that other than for CL, for lambda calculus both the ordinary version and the non-erasing version fail to have the property CR^∞ .
(ii) One can define collapsing steps for lambda calculus as being of the form (λy.y)x → x or (λy. x)M → x.
Strict hypercollapsing terms are those built from an infinite tower of such collapsing contexts, or reducing to that, or having an infinite reduction with infinitely many of those steps. Note that it is not sufficient to restore CR^∞ to work modulo these strict hc terms.
(iii) There are two definitions for hc terms. Built from an infinite tower of collapsing contexts, reducing to that, or having an infinite reduction with infinitely many collapsing root steps. Prove that these two definitions are equivalent.

NOTE. (J. Waldmann) SN for S-terms is decidable.

1.4.14. EXERCISE. (H. Zantema) Show that infinite S-terms do *no*t have the property CR^∞.

NOTE. Actually, the UN^∞ theorem can be stengthened as follows. See also Chapter 2, Lemma 2.15.

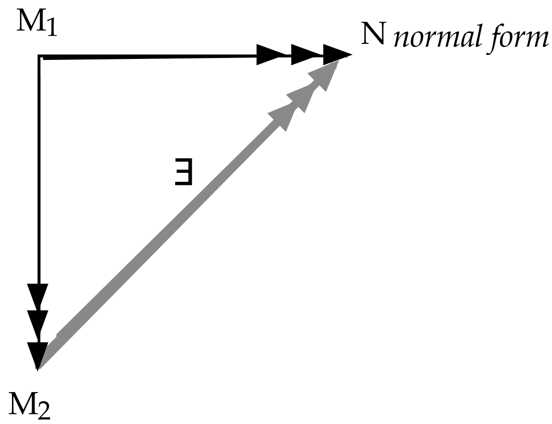


Figure 1.17

NOTE. We need a more precise definition of finitary TRS, iTRS, and connection between those, in order to make the comparison between SN and SN^∞ more precise.

(1) Let Σ be a signature, possibly infinite, and $\mathcal{R}$ be a set of reduction rules, possibly infinite. $\mathfrak{A}$ = (Σ, $\mathcal{R}$) is then a *finitary* TRS, with the usual definition of reduction rules and reduction relation. When the set of terms comprises all Σ-terms, we call $\mathfrak{A}$ a *full* TRS.
(2) A sub-TRS arises when we restrict the set of terms to a subset $\mathcal{T}$ of Ter(Σ), closed under the reduction relation. We define also sub-TRSs to be finitary TRSs. Typical examples are λI-calculus, or the fragment of CL consisting of finite S-terms, CL(S).
(3) *Infinitary* TRSs, or *iTRSs*, also can be full or sub-iTRSs. Full means that they contain all of

$\mathrm{Ter}^{\infty}(\Sigma)$, the set of finite and infinite terms over Σ . A sub-iTRS contains a subset $\mathcal{T}$ of $\mathrm{Ter}^{\infty}(\Sigma)$ which now is closed under $\twoheadrightarrow$, the infinitary reduction relation.

(4) Now we define canonical transformations from finitary TRSs to iTRSs and vice versa.

If $\mathfrak{A} = (\mathcal{T}, \mathcal{R})$ is a finitary TRS, then $\mathfrak{A}^{\infty}$ is the iTRS $(\mathcal{T}^{\infty}, \mathcal{R})$ where $\mathcal{T}^{\infty}$ is the closure of $\mathcal{T}$ under $\twoheadrightarrow$ in $\mathrm{Ter}^{\infty}(\Sigma)$.

Vice versa, we obtain from iTRS $\mathfrak{A} = (\mathcal{T}^{\infty}, \mathcal{R})$ a finitary TRS $\mathfrak{A}^{-\infty}$, by omitting the infinite terms from $\mathcal{T}^{\infty}$.

(5) Is it correct that $(\mathfrak{A}^{\infty})^{-\infty} = \mathfrak{A}$ and $(\mathfrak{A}^{-\infty})^{\infty} = \mathfrak{A}$?

(6) We can now state:

- CL(S) is not SN (Barendregt et al.), and
- $\mathrm{CL(S)}^{\infty}$ is SN^{∞} (Waldmann), but
- the iTRS with all possibly infinite S-terms as domain and the S-reduction rule, is not SN^{∞} (Zantema).

Appendix A1. Reduction diagrams

An important ingredient in finding a common reduct of the end points of two diverging reduction sequences consists of the *elementary diagrams*, see the examples in Figure 1.1. They are the 'atomic' or basic building blocks for constructing reduction diagrams. An non-trivial elementary diagram consists of two diverging steps (arrows), joined by two sequences of steps of arbitrary length. Note that in the e.d.'s we may use empty sides (the dashed sides, in some figures shaded), to keep matters orthogonal. This gives rise to some trivial e.d.'s as in the lower part of Figure 1.1. The e.d.'s are used as 'tiles' with the intention to obtain a completed reduction diagram as in Figure 1.2. Usually we will forget the direction of the arrows (second picture in Figure 1.2): they always are from left to right, or downwards (except the empty 'steps' that have no direction).

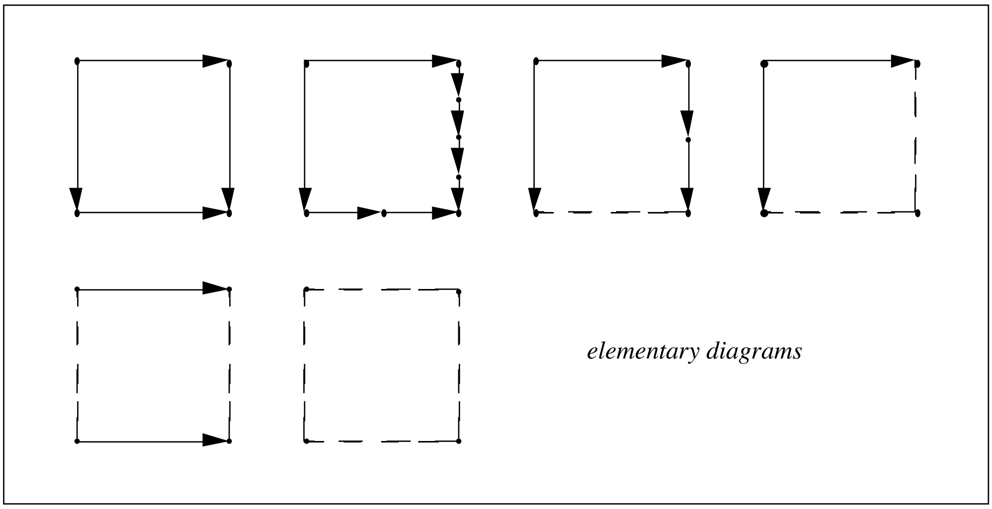


Figure A1.1

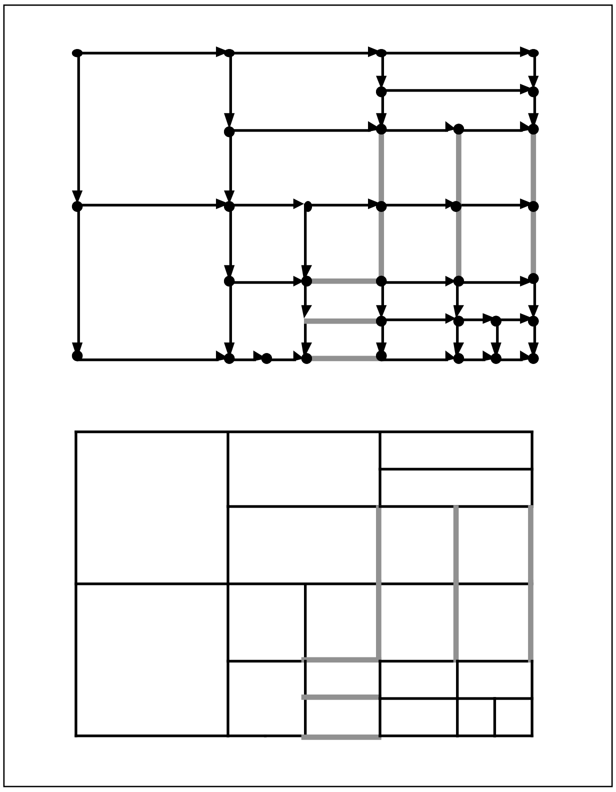


Figure A1.2

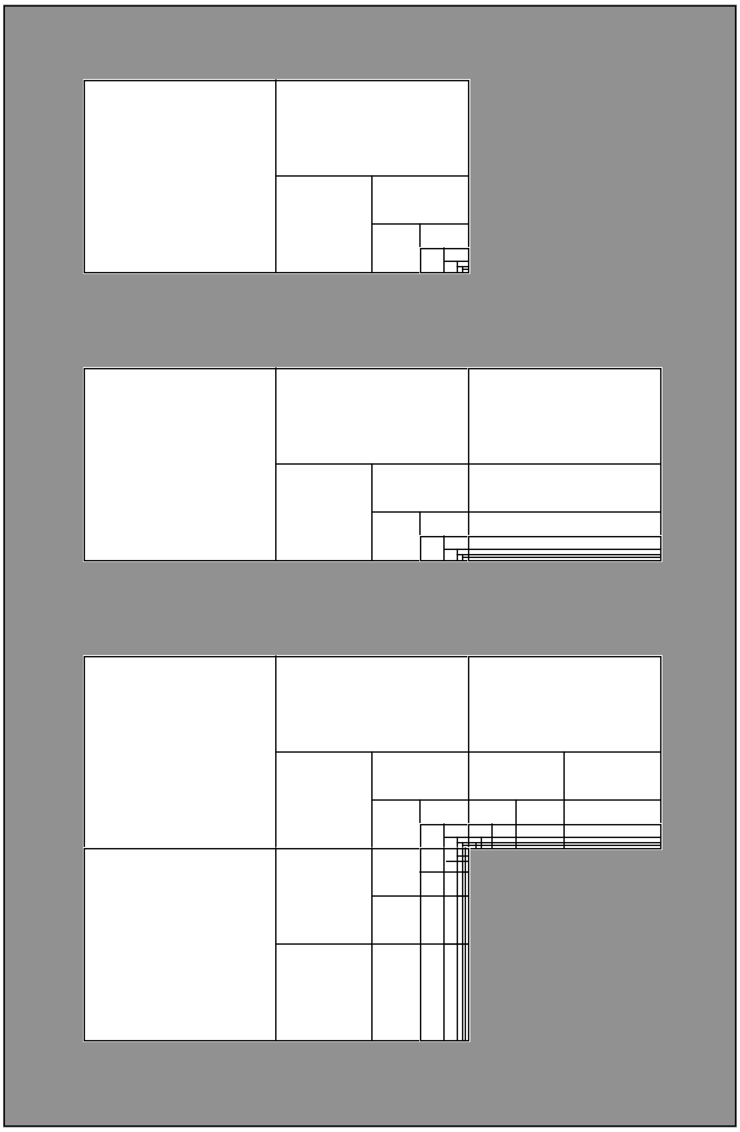


Figure A1.3

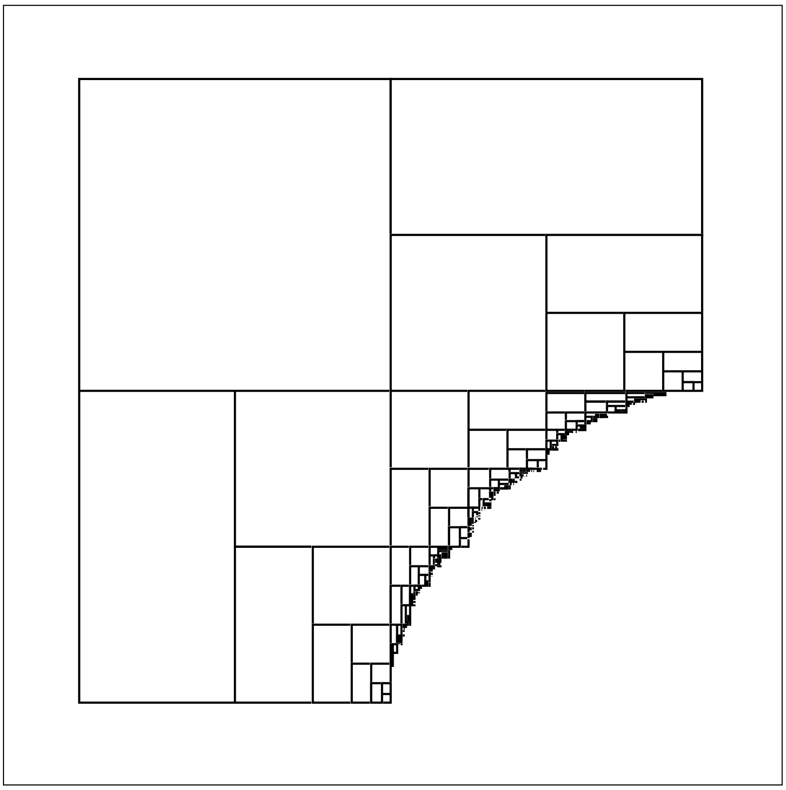


Figure A1.4

Appendix A2. The property UN^∞ for orthogonal TRSs.

Consider two infinite reductions

$$R: M_0 \to M_1 \to \ldots \to_\omega M_\omega,$$

$$R': M_0 \to M'_1 \to \ldots \to_\omega M'_\omega,$$

of a term M_0 to two infinite normal forms M_ω and M'_ω. We wish to prove that $M_\omega \equiv M'_\omega$, by showing that their finite approximations (prefixes) coincide:

$\forall n\ M_\omega/n \equiv M'_\omega/n$.

So consider M_ω/n and M'_ω/n. By definition of the limit notion in infinitary rewriting, we know that in reduction R the 'action' is after some stage N deeper than n, i.e. $\forall k>N\ d_k>n$, where d_k is the depth of the redex r_k contracted in the step $M_k \to M_{k+1}$.

Likewise in reduction R': after some N' all action is deeper than n. So we know that in the reduction $M_N \to \ldots \to_\omega M_\omega$ the prefix M_ω/n is 'untouched', and likewise M'_ω/n is untouched in the reduction $M'_{N'} \to \ldots \to_\omega M'_\omega$. Now consider the initial parts of R and R', up to N and N' respectively:

$$M_0 \to M_1 \to \ldots \to M_N,$$
$$M_0 \to M_1 \to \ldots \to M'_{N'},$$

and construct the reduction diagram determined by these two finite reductions, see Figure xx. Let M* be the common reduct of M_N and $M'_{N'}$ thus found. If we could now assume that the prefix M_ω/n also remained untouched in the reduction

$$M_N \to \ldots \to M^*,$$

and likewise for M'_ω/n in

$$M_N \to \ldots \to M^*,$$

we would have $M_\omega/n \equiv M'_\omega/n$ as desired. But unfortunately we cannot assume that.

A priori, the action may go 'upwards', during the reduction $M_N \to \ldots \to M^*$ in M_N. In fact it will not, by the assumption of orthogonality, as we will prove.

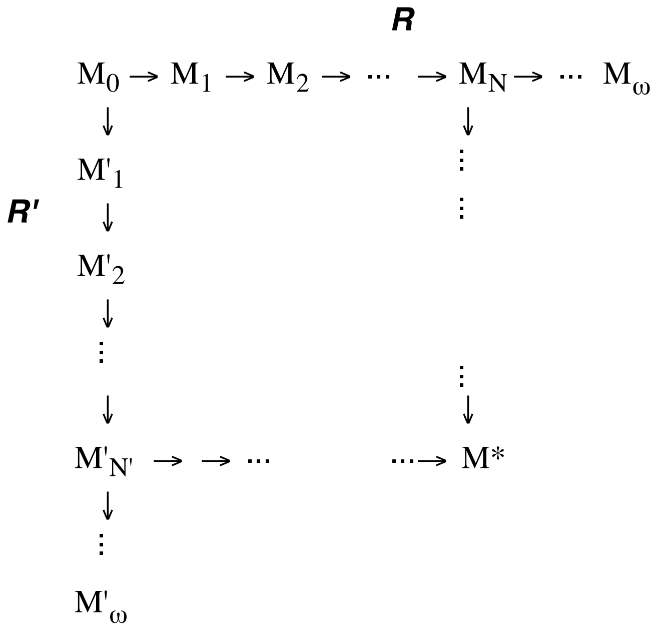


Figure. 1 Normal form evaluations.

DEFINITION 1. Prefix Π is *stable with respect to reduction* R if not only no reduction takes place in Π, but there is not even a redex in Π activated ('triggered') during R.

PROPOSITION 2. *Let* $M_0 \in Ter^\infty(R)$ *have prefix* Π, *and let* Π *be stable with respect to the infinite normalizing reduction* R: $M_0 \to M_1 \to \ldots \to_\omega M_\omega$, *a normal form. Then* Π *is stable with respect to any reduction.*

Proof. If the statement does not hold, there must be a symbol F in the prefix Π, such that the subterm headed by F is stable with respect to the normalizing reduction R, but such that F is triggered as the head of a redex in another reduction R'. So, without loss of generality, we can assume that $M_0 \equiv F(t_1,\ldots,t_n)$:

R

$t_0 \equiv F(t_1,...,t_p) \rightarrow \quad ... \quad \rightarrow^{\omega}$ t, normal form

$\downarrow$

R′ $\downarrow$

....

$\downarrow$

$t_n \equiv F(t'_1,...,t'_p)$

Figure 2. Proof of Proposition 2.

.

where in R, the F (or rather the context F(Ω,..., Ω)) is stable, but in R′ the F has become the head of a redex $F(t'_1,...,t'_p)$. Now we invoke the Parallel Moves Lemma for infinitary orthogonal rewriting, PML^{∞}, and construct the projection of R_0 along the first step of R′: result R_1.

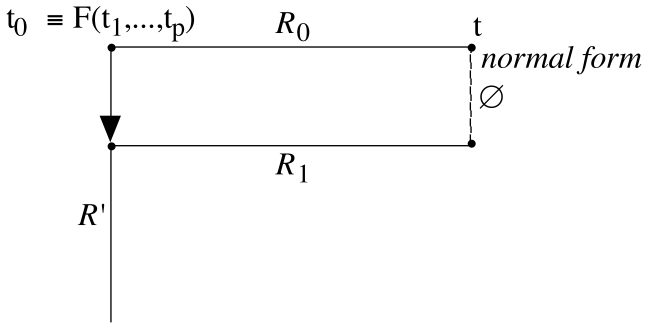


Figure 3. Using PML^{∞}.

The right-hand side of the strip determined by $t_0 \rightarrow t_1$ and R_0, is ∅, the empty reduction, by PML^{∞} (since there are no residuals of the redex contracted in $t_0 \rightarrow t_1$ present in t, a normal form). Now in R_0 there was no step at the root, by assumption. It follows that the same is true in R_1, by elementary reasoning with residuals in orthogonal reduction diagrams. Let us look at this fact somewhat closer:

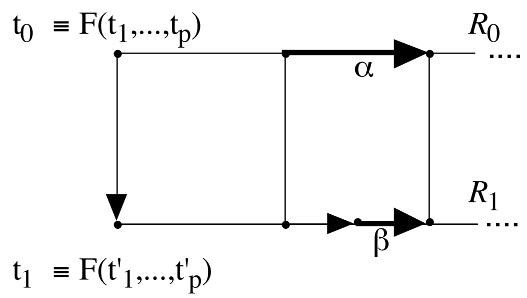


Figure 4. A closer look.

Paint the head F in the initial term $F(t_1,...,t_p)$ blue, all other symbol occurrences (including other F-occurrences) a different color. Call a redex blue, red, ... if that is the color of its head. So in R_0 no blue redex is contracted. Now in orthogonal rewriting, a step α contracts a redex with the same color as the redex in step β. So, R_1 cannot contract a blue redex; i.e. the F heading $F(t'_1,...,t'_p)$ is not triggered in R_1.

We now iterate this argument, projecting R_1 over the second step of R′, etc. Thus we arrive at R_n:

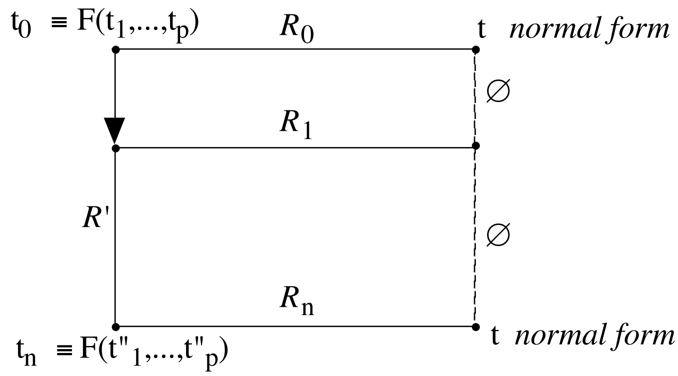


Figure 5. Iterating PML^∞ .

Now R_n starts with the term $t_n \equiv F(t'_1,...,t'_p)$, by assumption a redex, and proceeds to the normal form t, without ever contracting a root redex—i.e. the redex headed by F. But then, that root redex is still present in the normal form t, a contradiction.

This proves the Proposition and thereby also the theorem UN^∞ for orthogonal TRSs.

Appendix A3. Collapsing reductions (from Kennaway et al.)

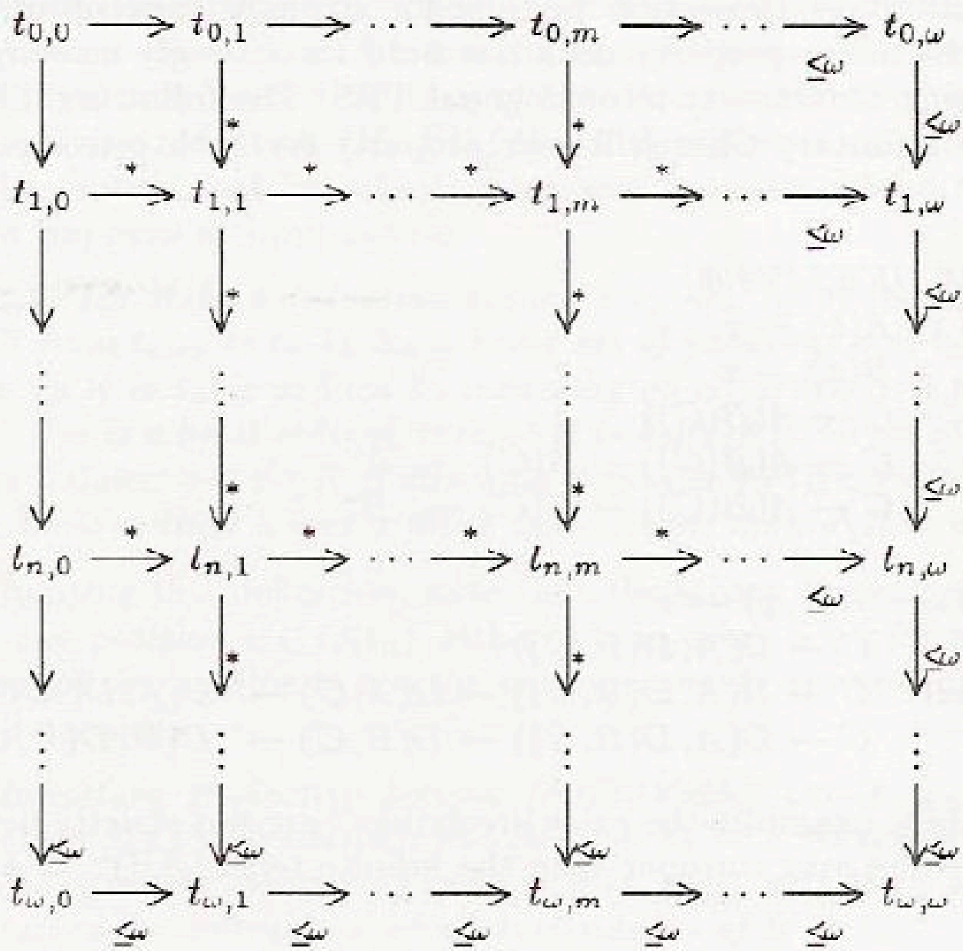


Figure 4.1

4.3 DEPTH-PRESERVING ORTHOGONAL TERM REWRITING SYSTEMS

In this section and the next we consider two natural classes of orthogonal TRS in which the infinitary Church-Rosser property holds for strongly converging sequences. The counter-examples suggest that collapsing rules are destroying the Church-Rosser properties. In the next section we will prove the Church-Rosser property for strongly converging reductions in orthogonal TRS without collapsing rules.

In this section however we will consider the more restricted but easier to deal with orthogonal TRS whose rules are depth-preserving.

DEFINITION 4.3.1 *A depth-preserving TRS is a left-linear TRS such that for all rules the depth of any variable in a right-hand side is greater than or equal to the depth of the same variable in the corresponding left-hand side.*

THEOREM 4.3.2 *Any depth-preserving orthogonal TRS has the infinitary Church-Rosser property for strongly converging sequences.*

PROOF. Let $t_{0,0} \to t_{0,1} \to \ldots \to_{\leq\omega} t_{0,\omega}$ and $t_{0,0} \to t_{1,0} \to \ldots \to_{\leq\omega} t_{\omega,0}$ be strongly convergent.

a. Using the Infinitary Projection Lemma for strongly convergent reductions we construct the horizontal strongly converging sequences $t_{n,0} \to^* t_{n,1} \to^* \ldots \to_{\leq\omega} t_{n,\omega}$ for $0 < n < \omega$, as depicted in figure 4.1. The vertical reductions are constructed similarly.

b. The construction of the Transfinite Projection Lemma also implies that the reduction $t_{n,\omega} \to_{\leq\omega} t_{n+1,\omega}$ is strongly converging.
By the depth-preserving property it holds for all $m, n \leq \omega$ that the depth of the reduced redexes in $t_{n,m} \to^* t_{n,m+1}$, which are all descendants of the redex $R_{0,m}$ in $t_{0,m} \to t_{0,m+1}$, is at least the depth of $R_{0,m}$ itself. Because $t_{0,0} \to t_{0,1} \to \ldots \to_{\leq\omega} t_{0,\omega}$ is strongly convergent we find by Lemma 4.2.6 that $t_{\omega,0} \to_{\leq\omega} t_{\omega,1} \to_{\leq\omega} t_{\omega,2} \ldots$ is strongly converging. Let us call its limit $t_{\omega,\omega}$.

c. In the same way the terms $t_{n,\omega}$ are part of a strongly converging sequence. The limit of this sequence is also equal to $t_{\omega,\omega}$, as can be seen with the following argument.
Let $\epsilon > 0$. Because $(t_{\omega,n})_{n\leq\omega}$ is a Cauchy sequence, there is an N_1 such that for all $m \geq N_1$ we have $d(t_{\omega,m}, t_{\omega,\omega}) < \frac{1}{3}\epsilon$.
Because $t_{0,0} \to t_{1,0} \to \ldots \to_{\leq\omega} t_{\omega,0}$ is strongly converging, there is an N_2 such that for $n \geq N_2$ we have that $2^{-d_n} < \frac{1}{3}\epsilon$ where d_n is the depth of the redex R_n reduced at step $t_{n,0} \to t_{n+1,0}$. Since the descendants of this redex R_n occur at least at the same depth, and since the TRS R is depth-preserving, we get $d(t_{n,m}, t_{\omega,m}) < \frac{1}{3}\epsilon$ for all $m \leq \omega$ and all $n \geq N_2$.
For similar reasons there is an N_3 such that for all $n \leq \omega$ and all $m \geq N_3$ we have that $d(t_{n,\omega}, t_{n,m}) < \frac{1}{3}\epsilon$.
Concluding: Let N be the maximum of N_1, N_2 and N_3. Then for $n \geq N$ we find using the triangle inequality for metrics that

$$\begin{aligned} d(t_{n,\omega}, t_{\omega,\omega}) &\le d(t_{n,\omega}, t_{n,n}) + d(t_{n,n}, t_{\omega,\omega}) \\ &\le d(t_{n,\omega}, t_{n,n}) + d(t_{n,n}, t_{\omega,n}) + d(t_{\omega,n}, t_{\omega,\omega}) \\ &\le \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon \\ &\le \epsilon. \end{aligned}$$

□

Observe that in this proof there are two places where it is essential that the reductions are strongly convergent. The first is the appeal to the Infinitary Projection Lemma. The second is in the argument that the sequences $(t_{\omega,n})_{n\in\omega}$ and $(t_{n,\omega})_{n\in\omega}$ have the same limit.

4.4 NON-COLLAPSING ORTHOGONAL TERM REWRITING SYSTEMS

DEFINITION **4.4.1** *A TRS R is non-collapsing if all its rewrite rules are non-collapsing, i.e. there is no rewrite rule in R whose right-hand side is a single variable.*

We will show that any non-collapsing orthogonal TRS satisfies the infinitary Church-Rosser property for strongly converging reductions. The proofs will use a variant of Park's notion of hiaton (cf. [Par83]). The idea is to replace a depth losing rule like $A(x, B(y)) \to B(y)$ by a depth-preserving variant $A(x, B(y)) \to B(\epsilon(y))$. In order to keep the rewrite rules applicable to terms involving hiatons, we also have to add more variants like $A(x, \epsilon^m(B(y))) \to B(\epsilon^{m+k+1}(y))$ for $k, m > 0$. By adding to a TRS all depth-preserving variants of its rewrite rules, we transform it into a depth preserving TRS.

DEFINITION **4.4.2** *Let R be a TRS based on the alphabet Σ. Let Σ_ϵ be the extension of Σ with a fresh unary symbol ϵ.*

a. *Let the ϵ-hiding function $\rho : Ter^\infty(\Sigma_\epsilon) \to Ter^\infty(\Sigma)$ be partially defined by induction as follows:*
   (1) $\rho(x) = x$,
   (2) $\rho(f(t_1, \ldots, t_n)) = f(\rho(t_1), \ldots, \rho(t_n))$ *for f in Σ and $t_i \in Ter^\infty(\Sigma_\epsilon)$ for $0 \le i \le n$,*
   (3) $\rho(\epsilon(t)) = \rho(t)$ *for $t \in Ter^\infty(\Sigma_\epsilon)$.*

   Hence ρ is well-defined on terms in $Ter^\infty(\Sigma_\epsilon)$ containing no infinite string of ϵs.

b. *A term $t \in Ter^\infty(\Sigma_\epsilon)$ is an ϵ-variant of a term $s \in Ter^\infty(\Sigma)$ if $\rho(t) = s$, that is, if hiding the ϵs in t results in s.*

c. *An ϵ-variant of a rule $l \to r$ is a pair of terms (l_ϵ, r_ϵ) such that*
   (1) $\rho(l_\epsilon) = l$.
   (2) $\rho(r_\epsilon) = r$.
   (3) *the root symbol of l_ϵ is not ϵ.*

(4) l_ϵ does not contain a subterm of the form $\epsilon(x)$ for any variable x.
(5) the root symbol of r_ϵ is not ϵ unless r is a variable,

d. *The ϵ-completion R^ϵ of R has alphabet Σ_ϵ. Its rules are the* depth-preserving *ϵ-variants of rules of R. We denote reduction in R^ϵ by $\to^\epsilon$.*

The proof of the following lemma is straightforward and omitted.

LEMMA 4.4.3 *The ϵ-completion of an orthogonal TRS is depth-preserving and orthogonal.* □

LEMMA 4.4.4 *Let R be a non-collapsing orthogonal TRS.*

a. *Let t_ϵ be an ϵ-variant of a term t of R. If t_ϵ strongly ϵ-converges in ω steps to some term s in R^ϵ, then s does not contain a branch ending in an infinite string of ϵs.*
b. *Let t_0 be the ϵ-variant of some term s_0. If $t_0 \to^\epsilon_\omega t_\omega$ is a strongly converging reduction in R^ϵ, then so is $s_0 \to_\omega s_\omega$ in R, where $s_i = \rho(t_i)$ for $0 \le i \le \omega$.*
c. *Let $t_0 \to_\omega t_\omega$ be a strongly converging reduction in R. Let s_0 be an ϵ-variant of t_0. Then there exists a strongly converging reduction $s_0 \to^\epsilon_\omega s_\omega$ in R^ϵ such that each s_i is an ϵ-variant of the corresponding t_i and similar for the reduction rules used.*

PROOF.

a. Since there are no collapsing rules, a string of ϵs can only be made longer by a reduction occurring at its top. Strong convergence implies that only finitely many such reductions can be made, and therefore that an infinite string of ϵs cannot be created.
b. Since t_0 is an ϵ-variant it does not contain an infinite string of ϵs. Neither does any of the t_i for $i \in \omega$, nor t_ω itself by the previous item 4.4.4(a). Hence, $\rho(t_n)$ is a well-defined term for all $0 \le n \le \omega$.
Because there are no infinite strings of ϵs in t_ω, every infinite path from the root of t_ω must contain infinitely many occurrences of members of Σ. Note also that t_ω is necessarily an infinite term.
Since by the previous item 4.4.4(a) t_ω contains no infinite string of ϵs, it must contain occurrences of members of Σ at arbitrarily great depth.
Given any finite number k, consider those occurrences v of t_ω, such that the path from the root to v contains at least k occurrences of symbols in Σ. By the preceding remarks, there must be at least one such occurrence. Let N_k be the minimum length of all such v. Because there are no infinite strings of ϵs, N_k must tend to infinity with k. Since $t_0 \to_\omega t_\omega$ is strongly converging there exists for any $k > 0$ an N such that for $n > N$, the depth of the redex reduced in $t_{n-1} \to t_n$ is at least N_k. This implies that the corresponding redex in $s_{n-1} \to s_n$ is at depth at least k, and hence $s_0 \to_\omega s_\omega$ is strongly convergent.
c. Trivial. The ϵ-variant s_0 of t_0 contains the corresponding ϵ-variant of the redex reduced in t_0. Apply an ϵ-variant of the corresponding rule. The resulting reduction satisfies the required properties. □

PROOF. Let R be an orthogonal TRS. Construct its ϵ-completion R^ϵ. By Theorem 4.3.2 the depth-preserving orthogonal TRS R^ϵ satisfies the infinitary Church-Rosser property. So if we start with two strongly converging reductions $t \to_{\leq\omega} s_1$ and $t \to_{\leq\omega} s_2$, then by Lemma 4.4.4(c) these reductions lift to two strongly converging reductions in R^ϵ, let us say $t \to^\epsilon_{\leq\omega} r_1$ and $t \to^\epsilon_{\leq\omega} r_2$. By Theorem 4.3.2 there exists a join u for the two lifted reductions such that $r_1 \to^\epsilon_{\leq\omega} u$ as well as $r_2 \to^\epsilon_{\leq\omega} u$. Erasing all ϵs using Lemma 4.4.4(b) we see that the term $\rho(u)$ is the join in R of $t \to_{\leq\omega} s_1$ and $t \to_{\leq\omega} s_2$. □

THEOREM 4.4.6 *An orthogonal TRS, each of whose rules is non-collapsing except for at most one rule of the form $I(x) \to x$, satisfies the infinitary Church-Rosser property for strongly converging reductions.*

PROOF. First, note that the proof of the previous theorem cannot be directly applied in the presence of the rule $I(x) \to x$. Consider the rules $A(x) \to I(x)$, $B(x) \to I(x)$, $I(x) \to x$. There are obvious reductions of the term $A(B(A(B(\cdots))))$ to both A^ω and B^ω. These lift to reductions ending with $A(\epsilon(A(\epsilon(\cdots))))$ and $\epsilon(B(\epsilon(B(\cdots))))$ respectively. If we now apply the Church-Rosser property of the depth-balanced system, we obtain reductions of these terms to $\epsilon(\epsilon(\epsilon(\epsilon(\cdots))))$, which cannot be lifted to strongly convergent reductions in the original system.

A simple modification of the previous proof establishes the present theorem. We modify the depth-preserving transformation by introducing two versions of ϵ: ϵ itself, and ϵ'. The rule $I(x) \to x$ is replaced by the depth-preserving version $I(x) \to \epsilon'(x)$. The other rules are transformed as before, except that wherever ϵ would appear on the left-hand side in the original transformation, either ϵ or ϵ' is used, in all possible combinations. On the right-hand sides, only ϵ is used. It is easy to see that the resulting system is depth-preserving and orthogonal, and hence that the infinite Church-Rosser property holds.

The distinction between ϵ and ϵ' can be thought of as labeling those occurrences of ϵ which arise from reductions of the I-rule.

Now consider two strongly converging reductions $t \to_{\leq\omega} s_1$ and $t \to_{\leq\omega} s_2$. As in the proof of the previous theorem, we obtain in R^ϵ a term u and two strongly converging reductions $r_1 \to^\epsilon_{\leq\omega} u$ and $r_2 \to^\epsilon_{\leq\omega} u$, where r_1 and r_2 are ϵ-variants of s_1 and s_2.

We cannot in general erase all the ϵs and ϵ's from these sequences to obtain a join for s_1 and s_2, since u may contain infinite branches of ϵs and ϵ's (which we shall call ϵ-branches for short). But we will show that we can transform these sequences in such a way as to eliminate such branches, after which the erasing process can be performed safely.

In every ϵ-branch in u, there must be infinitely many ϵ's. This follows for the same reason that in the non-collapsing case, no infinite branch of ϵs can arise.

Now consider an occurrence of ϵ' in an ϵ-branch of u. This must arise from a reduction by the rule $I(x) \to \epsilon'(x)$ at some point in each of the sequences $r_1 \to^\epsilon_{\leq\omega} u$ and $r_2 \to^\epsilon_{\leq\omega} u$. This reduction is performed on a subterm of the form $I(T)$, where T reduces to a ϵ-branch. By orthogonality, it is impossible for the reduction of the I-redex to be necessary for any later step of the sequence to be possible. If we omit it, the only effect is that certain occurrences of ϵ' later in the sequence are replaced by I.

We therefore omit from both $r_1 \to^{\epsilon}_{\leq\omega} u$ and $r_2 \to^{\epsilon}_{\leq\omega} u$ every I-reduction which gives rise to an occurrence of ϵ' in any ϵ-branch of u. This gives a term u' containing no such occurrences of ϵ', and reduction sequences $r_1 \to^{\epsilon}_{\leq\omega} u'$ and $r_2 \to^{\epsilon}_{\leq\omega} u'$. These sequences have the property that they contain no ϵ-branch anywhere. They may therefore be lifted to strongly convergent reductions in the original system, providing a strongly convergent joining of the original reduction sequences. □

4.5 CONCLUSION

The results of Dershowitz, Kaplan and Plaisted in [DKP91] imply that top-terminating orthogonal TRS satisfy the infinitary Church-Rosser property for Cauchy converging reductions which start from a finite term. (Cf. [DKP91]: combine their Theorem 3.3, Proposition 5.1, Theorem 6.4, and Theorem 6.3.) The property *top-termination*, that is, there are no derivations of infinite length starting from a finite term with infinitely many rewrites at topmost position, is rather strong and not very syntactic.

Our Theorems 4.4.5 and 4.4.6 show that for strongly converging reductions, orthogonal systems with no collapsing rules, other than possibly one of the form $I(x) \to x$, have the infinitary Church-Rosser property without conditions on the finiteness of the initial term. Theorem 4.4.6 is the best possible result for orthogonal TRS, since the counter-examples in 4.2.10 make it clear that no larger class of orthogonal TRS is Church-Rosser.

We do not know what the situation is for Cauchy converging reductions. For example, do non-collapsing orthogonal TRS have the infinitary Church-Rosser property for Cauchy converging reductions?

REFERENCES

[AGM92] S. Abramsky, D. Gabbay, and T. Maibaum (editors). *Handbook of Logic in Computer Science*, vol. II. Oxford University Press, 1992.

[AN80] A. Arnold and M. Nivat. The metric space of infinite trees: Algebraic and topological properties. *Fundamenta Informatica*, 4, pp. 445–476, 1980.

[Boo91] R. V. Book (editor). *Proc. 4th Conference on Rewriting Techniques and Applications*, Springer Verlag, Lecture Notes in Computer Science 488, Como, Italy, 1991.

[BvEG+87] H. P. Barendregt, M. C. J. D. van Eekelen, J. R. W. Glauert, J. R. Kennaway, M. J. Plasmeijer, and M. R. Sleep. Term graph rewriting. In J. W. de Bakker, A. J. Nijman, and P. C. Treleaven (editors), *Proc. PARLE'87 Conference, vol.II*, Springer-Verlag, Lecture Notes in Computer Science 259, pp. 141–158, Eindhoven, The Netherlands, 1987.

[DJ90] N. Dershowitz and J.-P. Jouannaud. Rewrite systems. In van Leeuwen [vL90], chapter 15.

[DK89] N. Dershowitz and S. Kaplan. Rewrite, rewrite, rewrite, rewrite, rewrite. In *Proc. ACM Conference on Principles of Programming Languages, Austin, Texas*, pp. 250–259, Austin, Texas, 1989.

[DKP89] N. Dershowitz, S. Kaplan, and D. A. Plaisted. Infinite normal forms (plus corrigendum). In G. Ausiello, M. Dezani-Ciancaglini, and S. Ronchi Della Rocca (editors), *Automata, Languages and Programming*, Springer-Verlag, Lecture Notes in Computer Science 372, pp. 249–262, Stresa, Italy, 1989.

[DKP91] N. Dershowitz, S. Kaplan, and D. A. Plaisted. Rewrite, rewrite, rewrite, rewrite, rewrite. *Theoretical Computer Science*, **83**, pp. 71–96, 1991. Extended version of [DKP89].

[FW91] W. M. Farmer and R. J. Watro. Redex capturing in term graph rewriting. In Book [Boo91], pp. 13–24.

[HL79] G. Huet and J.-J. Lévy. *Call-by-Need Computations in Non-ambiguous Linear Term Rewriting systems.* Technical report, INRIA, 1979.

[HL91] G. Huet and J.-J. Lévy. Computations in orthogonal rewrite systems I and II. In Lassez and Plotkin [LP91], pp. 394–443. (Originally appeared as [HL79].).

[Kel55] J. L. Kelley. *General Topology.* Van Nostrand, Princeton, 1955.

[KKSdV90a] J. R. Kennaway, J. W. Klop, M. R. Sleep, and F. J. de Vries. *An Infinitary Church-Rosser property for Non-collapsing Orthogonal Term Rewriting Systems.* Technical Report CS-9043, CWI, Amsterdam, 1990.

[KKSdV90b] J. R. Kennaway, J. W. Klop, M. R. Sleep, and F. J. de Vries. *Transfinite Reductions in Orthogonal Term Rewriting Systems.* Technical Report CS-R9041, CWI, Amsterdam, 1990.

[KKSdV93] J. R. Kennaway, J. W. Klop, M. R. Sleep, and F. J. de Vries. The adequacy of term graph rewriting for simulating term rewriting. In *this volume*, 1993.

[Klo92] J. W. Klop. Term rewriting systems. In Abramsky et al. [AGM92], pp. 1–116.

[LP91] J.-L. Lassez and G. D. Plotkin (editors). *Computational Logic: Essays in Honor of Alan Robinson.* MIT Press, 1991.

[Par83] D. Park. The "fairness problem" and nondeterministic computing networks. In J.W. de Bakker and J. van Leeuwen (editors), *Foundations of Computer Science IV, Part 2*, vol. 159 of *Mathematical Centre Tracts*, pp. 133–161. CWI, Amsterdam, 1983.

[Ros73] B. K. Rosen. Tree-manipulating systems and Church-Rosser systems. *Journal of the ACM*, **20**, pp. 160–187, 1973.

[vL90] J. van Leeuwen (editor). *Handbook of Theoretical Computer Science*, vol. B: Formal Models and Semantics. North-Holland, Amsterdam, 1990.

REFERENCES.

[Bar84] H.P. BARENDREGT. *The Lambda Calculus, its Syntax and Semantics*. 2nd ed., 1984. North-Holland 1984.

[DKP89] N. Dershowitz, S. Kaplan, and D.A. Plaisted. Infinite normal forms. In ICALP '89, volume 372 of Springer LNCS, p. 249-262, 1989.

[FW89] W.M. Farmer and R.J. Watro. Redex capturing in term graph rewriting. Technical report, M89-59, MITRE, 1989.

[Hud88] P. Hudak. Report on the functional programming language Haskell. Draft Proposed Standard, 1988.

[KKSV93] J.R. Kennaway, J.W. Klop, M.R. Sleep, and F.J. de Vries. An infinitary Church-Rosser property for non-collapsing orthogonal term rewriting systems. In M.R. Sleep, M.J. Plasmeijer, and M.C.J.D. van Eekelen, editors, Term Graph Rewriting - theory and practice, p. 47-59. Wiley, 1993.

[KKSV95a] J.R. Kennaway, J.W. Klop, M.R. Sleep, and F.J. de Vries.Infinitary lambda calculus and Böhm models. In Proceedings of RTA95, Kaiserslautern, Springer LNCS 914, p.257-270, 1995.

[KKSV95b] J.R. Kennaway, J.W. Klop, M.R. Sleep, and F.J. de Vries. Transfinite reductions in orthogonal term rewriting systems. Information and Computation, 119(1):18-38, 1995.

[KKSV97] J.R. Kennaway, J.W. Klop, M.R. Sleep, and F.J. de Vries. Infinitary lambda calculus. Theoretical Computer Science, 175(1):93-125, 1997.

[KV91] J.W. Klop and R.C. de Vrijer. Extended rewriting systems. In S. Kaplan and M. Okada, editors, Proceedings of the Workshop on Conditional and Typed Rewriting systems, Springer LNCS 516, p.26-50, Montreal, 1991.

[PvE93] M.J. Plasmeijer and M.C.J.D. van Eekelen. Functional Programming and Parallel Graph Rewriting. Addison Wesley, 1993.

[Tur85] D.A. Turner. Miranda: a non-strict functional language with polymorphic types. In J.-P. Jouannaud, editor, Proceedings of the ACM Conference on Functional Programming Languages and Computer Architecture, Springer LNCS 201, p.1-16, 1985.