Sequentiality in Lambda Calculus and CL

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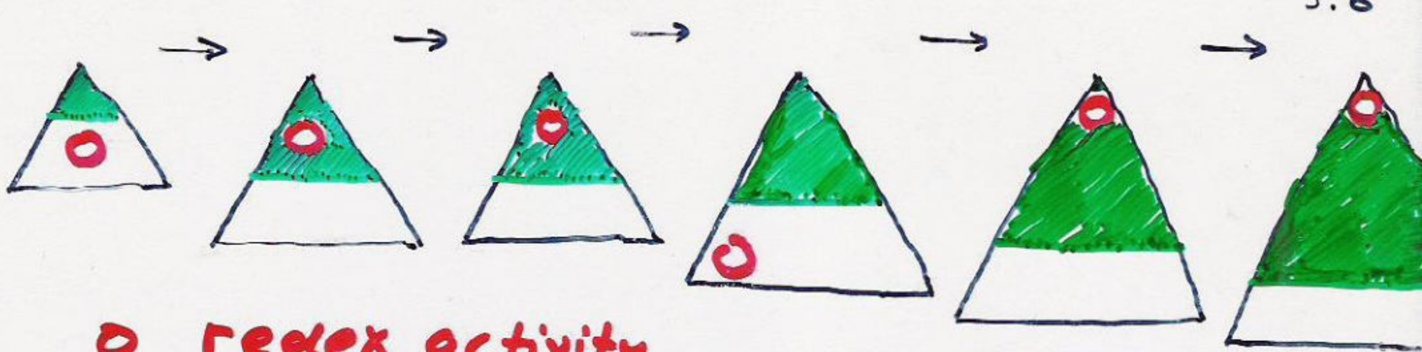
1. Infinitary rewriting

2. Infinitary lambda calculus

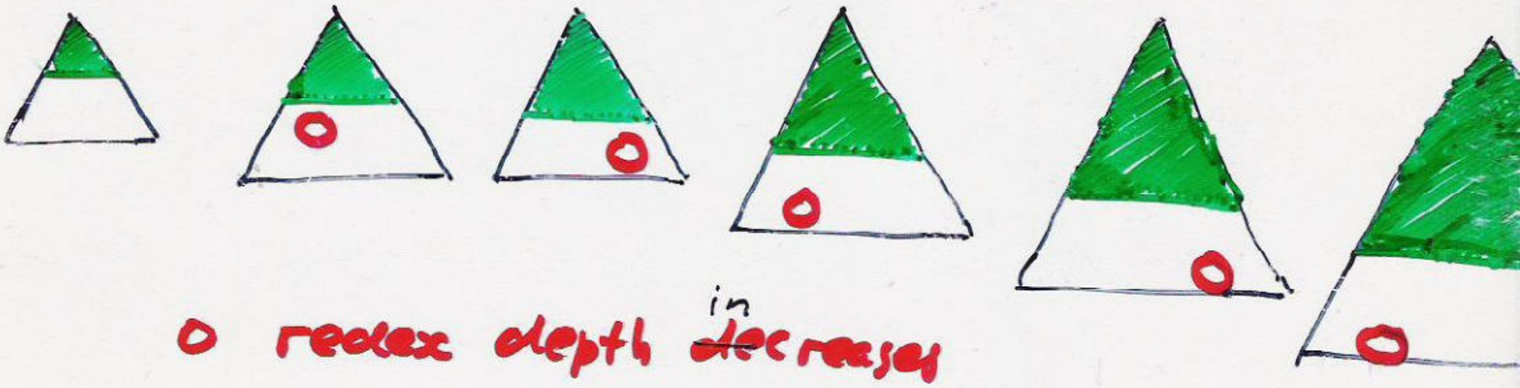
3. Berry's Sequentiality Theorem

4. Proving BST by origin tracking



o redex activity

(Cauchy) convergence



o redex depth ~~de~~increases

strong convergence

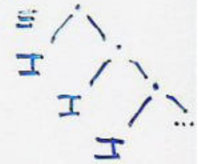
So $\omega\omega \not\to_\omega \omega\omega$

QUESTION.

? Relation between coinduction/coalgebra and infinitary term rewriting

Infinitary λ-calculus is not confluent:

$$\begin{array}{l} Y_T I \twoheadrightarrow I(Y_T I) \twoheadrightarrow I(I(Y_T I)) \twoheadrightarrow \cdots \\ \Downarrow \\ (\lambda x.xx)(\lambda x.xx) \end{array}$$

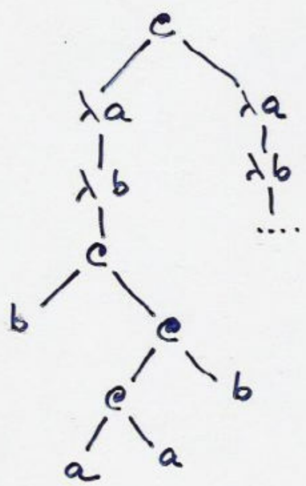
$I^\omega \equiv$ 

After identifying terms without hnf, infinitary confluence, CR_∞, holds.

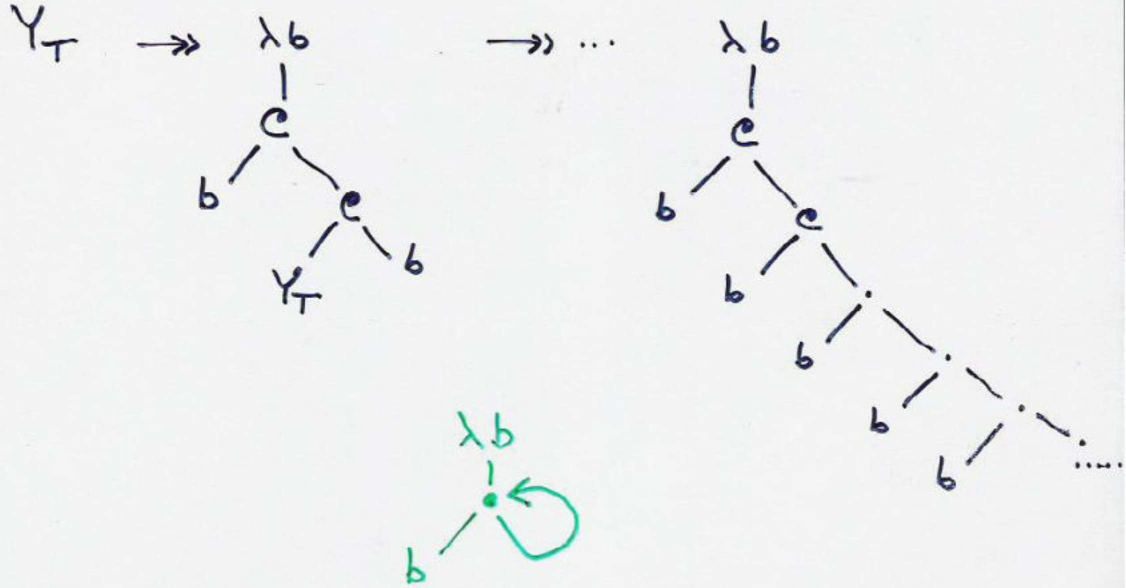
Infinitary λ-calculus and Böhm Trees

Finite λ-terms :

$$Y_T = (\lambda ab.\, b(aab)(\lambda ab.\, b(aab)$$



Infinite λ-terms:



cyclic λ-term

3 ways to define Böhm Trees.

(1) By direct approximations:

Direct approximation $\omega(M)$ of Ω-term M is obtained by replacing redexes $(\lambda x. P)Q$ by Ω and applying 'as much as possible' the

Ω-normalisation rules
$$\Omega M \to \Omega$$
$$\lambda x. \Omega \to \Omega$$

Now

$$BT(M) = \bigcup \{ \omega(N) \mid M \twoheadrightarrow_{\beta} N \}$$

(2) By coinduction (Barendregt [84])

(i) If M has no hnf then $BT(M) = \Omega$

(ii) If M has hnf $\lambda \vec{x}. y M_1 \cdots M_n$ then

$$BT(M) = \begin{array}{ccc} & \lambda \vec{x}.\, y & \\ / & & \backslash \\ BT(M_1) & \cdots & BT(M_2) \end{array}$$

(3) By infinitary rewriting employing β, $M \to \Omega$ if $M \not\equiv \Omega$ is unsolvable,

$$\Omega M \to \Omega$$
$$\lambda x. \Omega \to \Omega$$

Example:

$$\lambda zy . y(z\omega\omega)) I \to_\beta$$

$$\lambda y . y(I\omega\omega) \to_{uns}$$

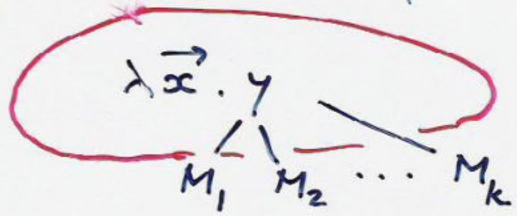
$$\lambda y . y\Omega$$

Every reduction will after α steps for some α a normal form : the BT. which is moreover unique (i.e. independent of actual reduction sequence).

If we want to reach the BT in at most ω steps, we have to impose a fairness assumption so that no redex will be infinitely often neglected.

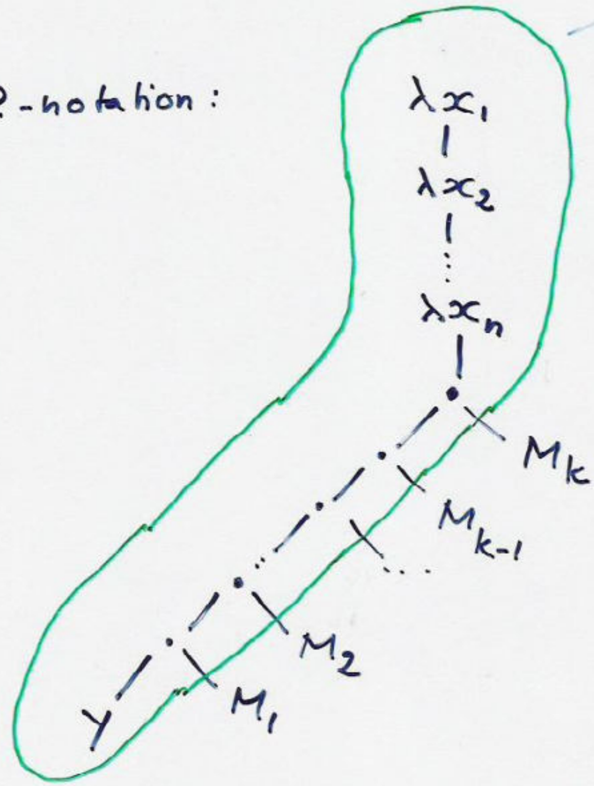
Applicative notation vs. Barendregt notation.

The elementary bits of information in a BT have the form



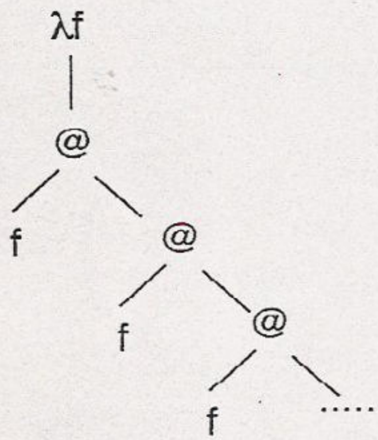
pinch together

in the @-notation:

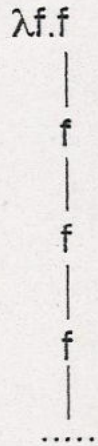


$Y = \lambda f.\ (\lambda x.\ f(xx))(\lambda x. f(xx))$

BT(Y) in applicative notation



BT(Y) in hnf notation

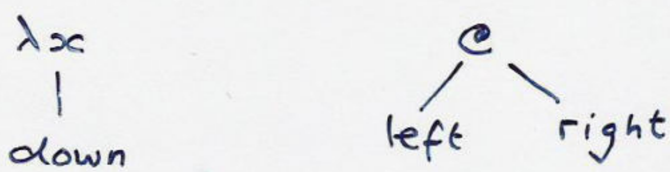


The set of infinite λ-terms (λ-trees) arises by metric completion of $Ter(\lambda)$ with the usual distance metric:

$$d(M,N) = 2^{-\text{min. depth of difference}}$$

How to measure the depth? Length of path from root to that occurrence.

More refined way: λ-terms grow in 3 dimensions down, left, right ($d\,\ell\,r$)

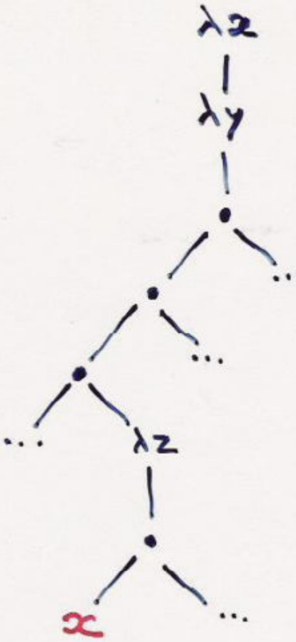


In the usual depth measure depth increases when going in any of the 3 directions.

Let the triple $d\ell r \in \{0,1\}^3$:

$$d\ell r = \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}$$

In the $d\ell r$ metric depth does <u>not</u> increase in directions with 0.



x has 111-depth 7
101-depth 4
ℓ-steps don't count

001-depth 1
d, ℓ-steps don't count

We now get by metric completion 8 completions $\mathrm{Ter}\,\lambda_{d\ell r}$

$$\mathrm{Ter}\,\lambda_{000} \subseteq \mathrm{Ter}\,\lambda_{001} \subseteq \mathrm{Ter}\,\lambda_{101} \subseteq \mathrm{Ter}\,\lambda_{111}$$

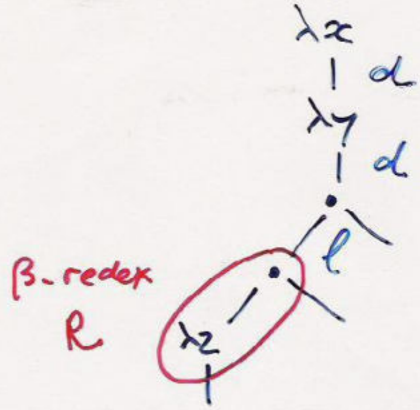
$\mathrm{Ter}\,\lambda_{000} = \mathrm{Ter}\,\lambda$

E.g. in $\mathrm{Ter}\,\lambda_{001}$ no trees with infinite $d\ell$-branches.

Adding β-reduction we have 8 λ-calculi, one being finitary λ-calculus, 7 infinitary λ-calculi. Only 3 of them are interesting:

λ_{001}	Böhm Trees
λ_{101}	Longo Trees
λ_{111}	Berarducci Trees

$d \ l \ r$
$0 \ 0 \ 1$ rules β, d, l

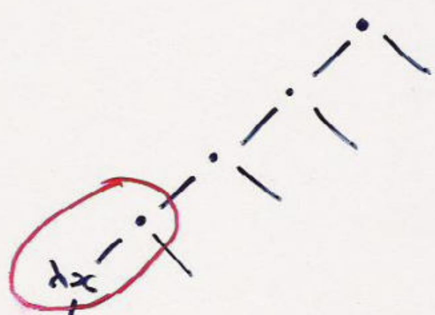


apply β, d, d, l

So redexes that may be contracted in 001-restriction of generating rules are the spine redexes.

$d \ l \ r$
$1 \ 0 \ 1$ rules β, l

Redexes that may be contracted: <u>the</u> lazy redex



A redex has dlr-depth 0 iff it can be contracted in the dlr-restriction of the generating rules.

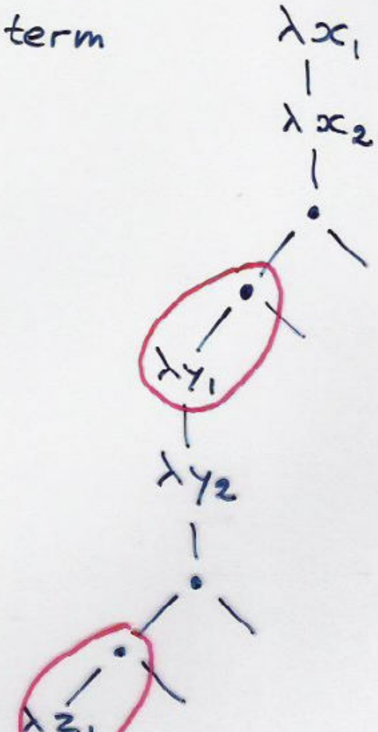
Evaluation to infinitary normal forms:

BT LT BeT

M is <u>dlr-unsolvable</u> if there is in λ_{dlr} an infinite reduction at dlr-depth 0.

For 001:

spine of a term



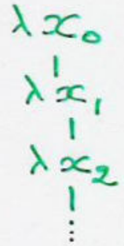
on the spine, the 001-depth is 0.

redexes at 001-depth 0: spine redexes. the uppermost is <u>the</u> head redex

A term is <u>unsolvable</u> or <u>has no head normal form</u> iff there is an infinite reduction contracting only spine redexes (Barendregt et al. 87)

	BT	LT	BeT
$(\lambda x.xx)(\lambda x.x)$	Ω	Ω	Ω
$(\lambda xy.xx)(\lambda xy.xx)$	Ω	$\top$	$\top$
$(\lambda x.xxz)(\lambda x.xxz)$	Ω	Ω	$(((\cdots)z)z)z$
$(\lambda x.z(xx))(\lambda x.z(xx))$	z^ω	z^ω	z^ω
$\lambda y.((\lambda x.xx)(\lambda x.xx))$	Ω	$\lambda y.\Omega$	$\lambda y.\Omega$
$(\lambda x.xx)(\lambda x.xx)y$	Ω	Ω	Ωy

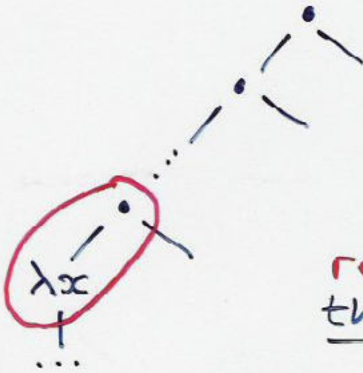
$\top$:



z^ω

$(((\ldots)z)z)z$:

For 101 :



redex at 101 - depth 0
<u>the</u> lazy redex

Another way of characterizing the 0-depth redexes suggested by F. J. de Vries :

β	$\dfrac{M \to N}{\lambda x.M \to \lambda x.N}$ ξ α
$\dfrac{M \to N}{MZ \to NZ}$ ν l	$\dfrac{M \to N}{ZM \to ZN}$ μ r

ξ, ν, μ in Ong's thesis
α, l, r here

0 0 1 : may use rules β, α, l

1 0 1 : may use rules β, l

1 1 1 : may use rules β

$\to_{\beta\ \alpha \ell r}$		normal forms
$\to_{\beta\ 001}$	: $\beta + \alpha + \ell$	hnf's
$\to_{\beta\ 101}$	: $\beta + \ell$	whnf's
$\to_{\beta\ 111}$	: β	$\overline{RED}$

hnf: $\lambda \vec{x} . y \vec{M}$

whnf: $\lambda x . N$, $y \vec{M}$

$$NF \subseteq HNF \subseteq WHNF \subseteq \overline{RED}$$

Berarducci Trees $BeT(M)$

$BeT(M) = y$ if $M \twoheadrightarrow y$

$BeT(M) = \lambda x.\ BeT(N)$ if $M \twoheadrightarrow \lambda x.N$

$BeT(M) = BeT(M_1)\ BeT(M_2)$

if $M \twoheadrightarrow M_1 M_2$ and M_1 is of order 0

(i.e. cannot reduce to $\lambda x.\ M_1'$)

$BeT(M) = \Omega$ in all other cases;

such M are called mute

A term M is mute if it is a term of order 0 which cannot be reduced to a variable or to an application of a term of order 0 to any term.

Equivalently: M has an infinite reduction with at the root infinitely many times a redex contraction.

	β	$M \to \Omega$ if M is dkr-uns	$\lambda x.\Omega \to \Omega$ Ω_d	$\Omega M \to \Omega$ Ω_l	
001	+	+	+	+	BT
101	+	+		+	LT
111	+	+			BeT

$$\Omega \qquad\qquad \Box\,\text{root}$$

$$\Downarrow$$

$BeT(M)$ mute Ω_{BeT} $\qquad \Box\Diamond\,\text{root}$

$\vee\!/$ $\qquad\qquad\qquad\qquad \Downarrow$

$LT(M)$ has no whnf Ω_{LT} $\qquad \Box\Diamond\,\text{lazy} \Leftrightarrow \Box\,\text{lazy}$

$\vee\!/$ $\qquad\qquad\qquad\qquad \Downarrow$

$BT(M)$ unsolvable, has no hnf Ω_{BT} $\qquad \Box\Diamond\,\text{head} \Leftrightarrow \Box\,\text{head}$

$\Leftrightarrow \Box\,\text{spine} \Leftrightarrow \Box\Diamond\,\text{spine}$

Ris: root $\Rightarrow$ lazy $\Rightarrow$ head $\Rightarrow$ spine

BST

Berry's Sequentiality Theorem

$M \in Ter(\lambda\Omega)$, $M \equiv C[\Omega, \dots, \Omega]$

$\equiv C[\Omega_1, \dots, \Omega_n]$.

Then each occurrence of Ω in $BT(M)$, say at σ, is in one of these cases:

(i) Ω at σ is independent of the Ω's in M. I.e. no refinement of M will give more 'output' information at position σ:

$$\forall M' \geq_\Omega M \qquad BT(M')/\sigma = \Omega$$

(ii) $\exists! \, i \in \{1, \dots, n\}$ such that the Ω at σ is "caused" by Ω_i.

I.e.: • Ω at σ is insensitive for increases at any of the $\Omega_1, \dots, \Omega_{i-1}, \Omega_{i+1}, \dots, \Omega_n$

• Ω at σ will be properly increased when Ω_i in M is refined to a fresh variable z.

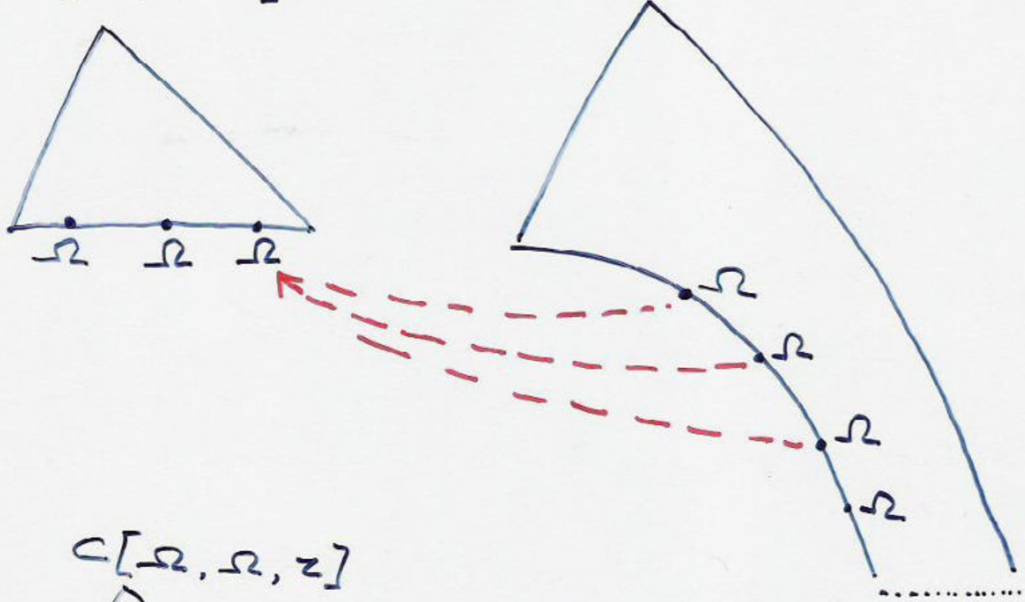
$M \equiv z\big((\lambda x.xx)(\lambda x.xx)\big)\Omega\,\Omega$

$BT(M) = z\ \Omega\ \ \Omega\ \Omega$

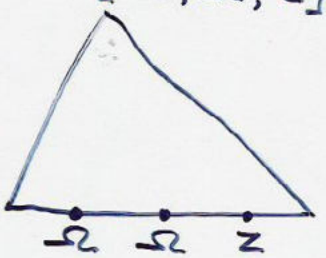
The first Ω is independent of refining M.

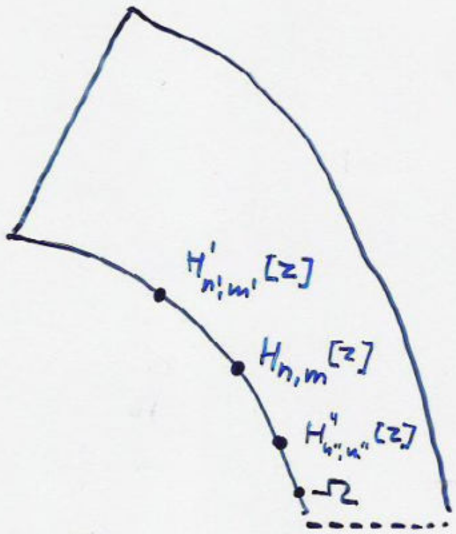
The 2nd, 3rd, are caused by the Ω's in M.

$C[\Omega, \Omega, \Omega]$

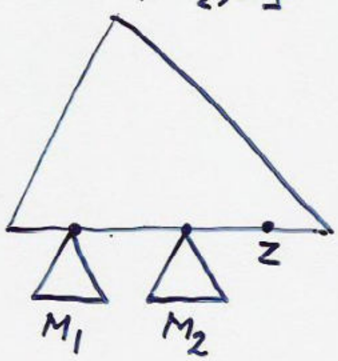


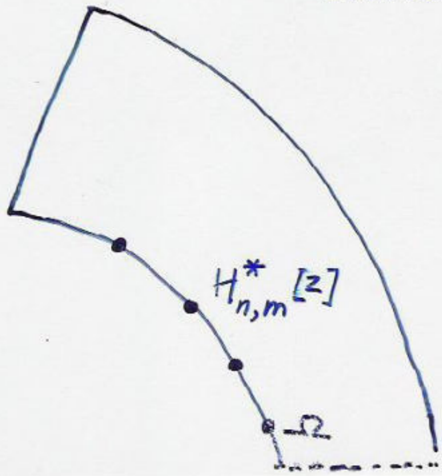
$C[\Omega, \Omega, z]$

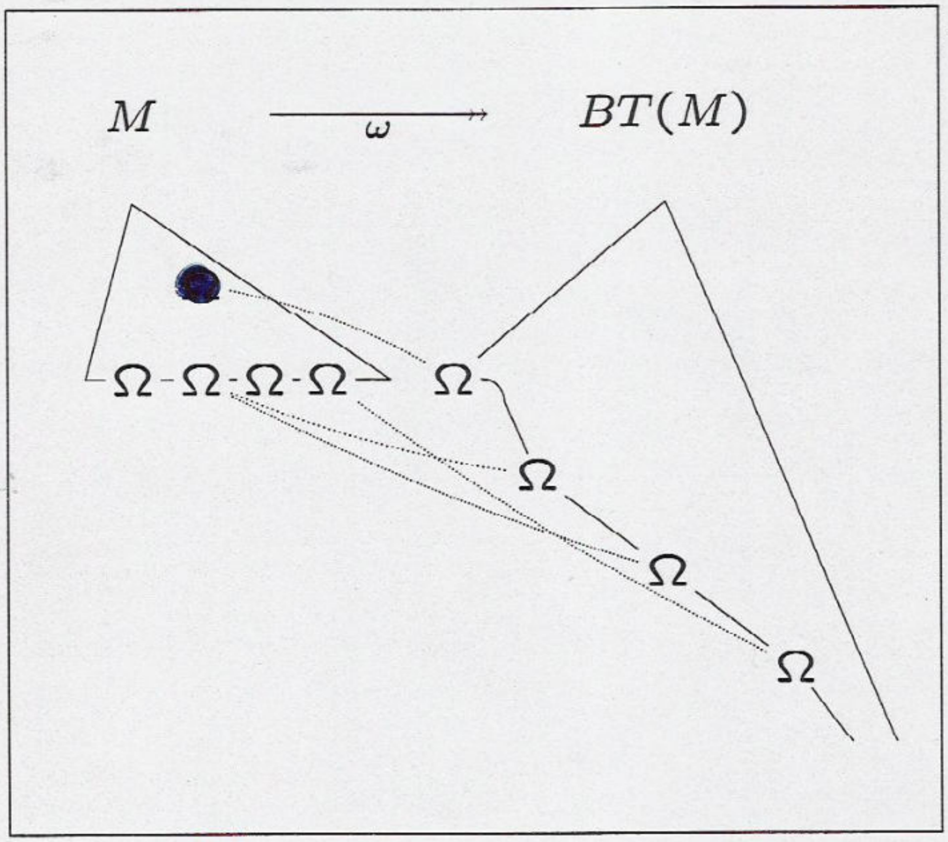




$C[M_1, M_2, z]$







Causal dependence of Ω's in Böhm tree from those in original term

Application:

- Parallel-or is not definable

$$PXT = T$$
$$PTX = T$$
$$PFF = F$$

- Berry-Kleene function with

$$FABX = 0$$
$$FXAB = 1$$
$$FBXA = 2$$

is not definable

$$P\,X\,T = T \qquad (1)$$
$$P\,T\,X = T \qquad (2)$$
$$P\,F\,F = F \qquad (3)$$

$$P\,\Omega\,T = T \qquad \text{by } (1)$$

$$P\,\Omega\,\Omega \leq T$$

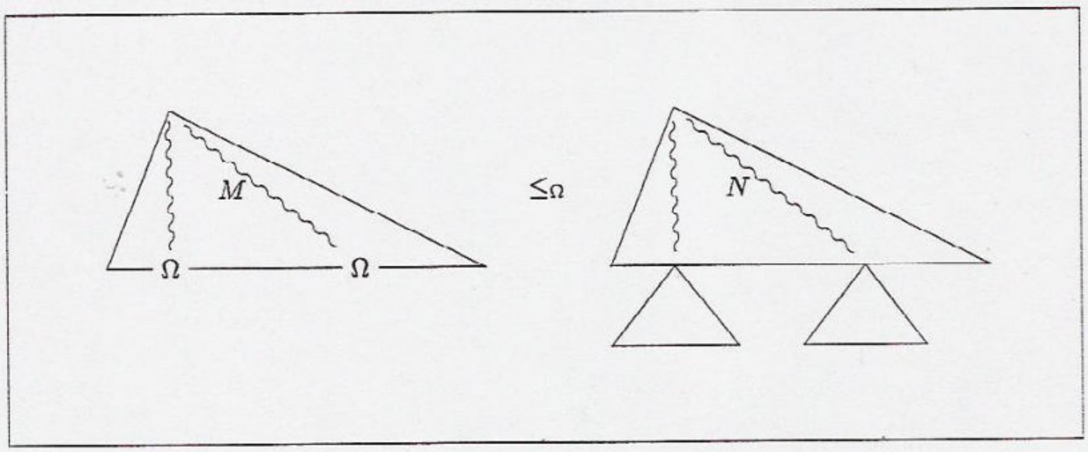
by monotonicity:
$M \leq_\Omega M' \Rightarrow$
$BT(M) \leq_\Omega BT(M')$

$$P\,\Omega\,\Omega \leq F$$
$$P\,\Omega\,\Omega = \Omega$$

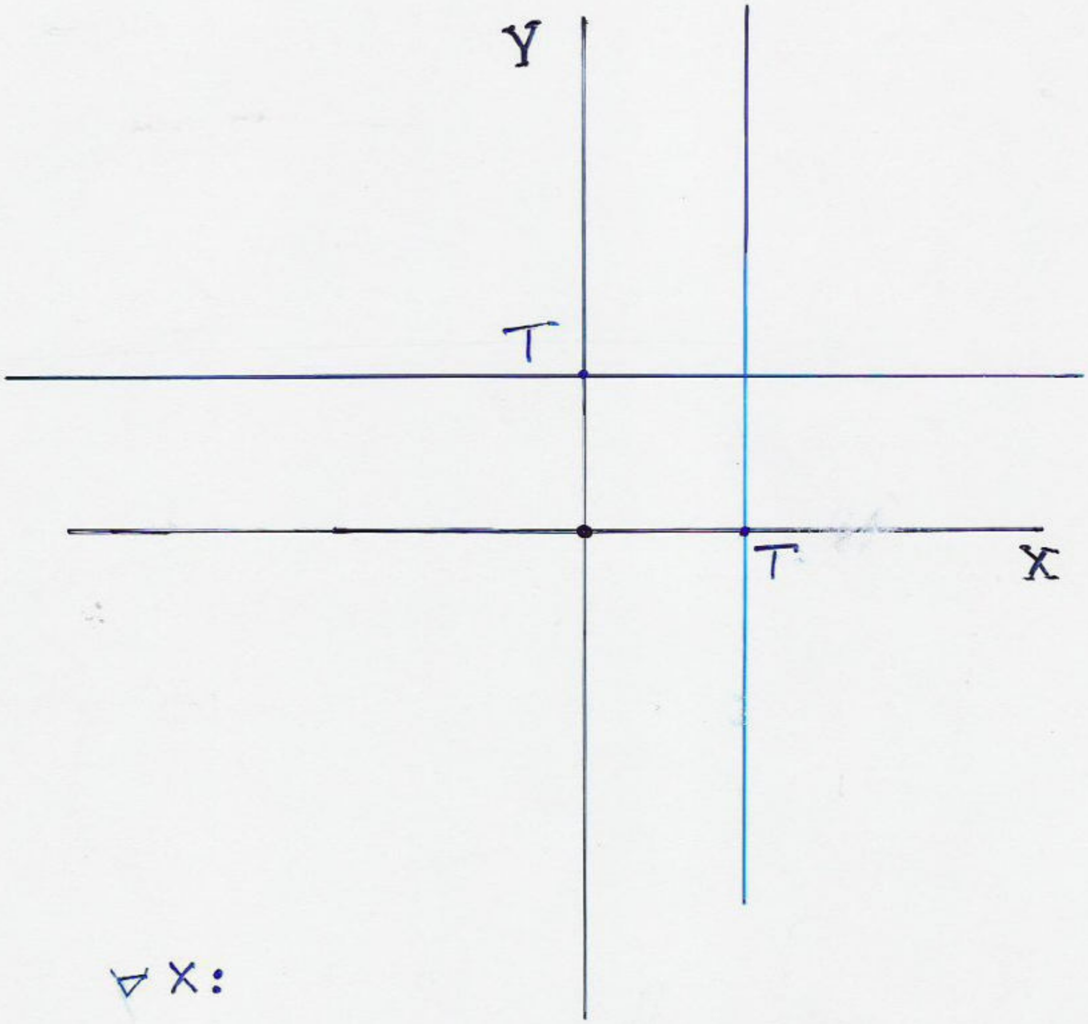
Case 1. $P\,\Omega\,\Omega = \Omega$ No.

Case 2. $P\,\Omega\,\Omega = \Omega$

then $P\,\Omega\,T = \Omega$ ↯
$\neq T$



Refining Ω's



$\forall X$:

$F X T_1 = \cancel{T}\ S_1$

$F T_2 X = \cancel{T}\ S_2$

Then $\forall X, Y \quad FXY = T$

If F is constant on the two perpendicular lines, it is constant every where.

$$M x N = N_1 \quad (1)$$
$$M N' x = N_2 \quad (2)$$

(1): $M \Omega N = N_1$

$$M \Omega \Omega \leqslant BT(N_1)$$

<u>Case I.</u> $BT(M\Omega\Omega) <_\Omega BT(N_1)$

<u>Case II.</u> $BT(M\Omega\Omega) = BT(N_1)$

<u>Case I.</u> $BT(M\Omega\Omega) <_\Omega BT(N_1)$. Say ~~the~~ a difference is at p:

$$BT(M\Omega\Omega)|p = \Omega$$
$$BT(N_1)|p \neq \Omega$$

$BT(M\Omega N) = N_1$, which is at $p \neq \Omega$,

so the Ω in $BT(M\Omega\Omega)|p = \Omega$ is not "robust". By BST it is caused by the first or the second Ω in $M\Omega\Omega$.
Say the first: So

$$\begin{array}{c} BT(M\Omega N)|p = \Omega \\ \| \\ BT(N_1)|p \neq \Omega \quad ↯ \end{array}$$

By symmetry, also not the second Ω.
We have proved that we are in case II:

$$BT(M\Omega\Omega) = BT(N_1) \quad (*)$$

We will now prove that $Mxy =_{BT} N_1$,

by proving that $Mxy =_{BT} M\Omega\Omega$ (then using (*).)

To prove: $Mxy =_{BT} M\Omega\Omega$

$$M\Omega\Omega \leqslant Mxy$$

<u>Case (i)</u> $M\Omega\Omega < Mxy$

Case (ii) $M\Omega\Omega = Mxy$

<u>Case (i)</u>. $M\Omega\Omega < Mxy$.

Say $\exists$ difference at p:

$$M\Omega\Omega \mid p = \Omega \qquad \phi$$

$$Mxy \mid p \neq \Omega$$

Apply BST on 'the' Ω in $M\Omega\Omega / p = \Omega$:

'Cause' in M is impossible: since $Mxy \mid p \neq \Omega$

cause is 1^{st} Ω: $M\Omega\Omega \mid p = \Omega$

Then $MxN \mid p \neq \Omega$ by BST (x fresh variable)

Also $MxN \underset{(1)}{=} N_1 \underset{(*)}{=} M\Omega\Omega$

Contradiction with ϕ.

Likewise the 2nd Ω cannot be the cause.

Hence case (i) is impossible.

Hence case (ii):

$$M\Omega\Omega = N_1 = Mxy.$$

Likewise $Mxy = N_2$. $\square$

$$F\,0\,1\,X = X$$
$$F\,X\,0\,1 = X$$
$$F\,1\,X\,0 = X$$

This F is not definable:

$$F\,0\,1\,\Omega = \Omega$$

$$F\,\Omega\,\Omega\,\Omega = \Omega$$

impossible : $F\,0\,1\,z = z > \Omega$

$$F\,\Omega\,\Omega\,\Omega = \Omega$$

$$F\,\Omega\,\Omega\,z = H_{n,m}[z]$$

$$F\,0\,1\,z = H^{*}_{n,m}[z] = z \quad \Rightarrow n = m = 0$$

So $F\,\Omega\,\Omega\,z = z$

F has to have a hnf, o.w. $\Omega = z$, ↯

say $F = \lambda abc.\, c\, M_1 \dots M_n$

then $F\Omega\Omega z = z\, M_1' \dots M_n' = z$. ↯

$F = \lambda abc.\, b\, M_1 \dots M_n$

then $F\Omega\Omega z = \Omega\, M_1' \dots M_n' = \Omega \neq z$ ↯

$F = \lambda abcd.\, d\, M_1 \dots M_n$

then $F\Omega\Omega z = \lambda d.\, d\, M_1' \dots M_n' \neq z$ ↯

So we cannot yet prove the non definability of A with

$$A \; X \; T = X$$
$$A \; T \; X = X$$

However, a refined version of BST does show this.

Likewise the Kleene - Berry function F s.t.

$$F \; A \; B \; X = X$$

$$F \; X \; A \; B = X$$

$$F \; B \; X \; A = X$$

is not definable in λ (nor in PCF).

This F corresponds to an <u>orthogonal</u> TRS, so the weak orthogonality of P and A is not the problem.

PCF

Types:

(i)	INT, BOOL are types (ground types)
(ii)	if σ, τ are types, then $(\sigma \rightarrow \tau)$ is a type

Constants:

<u>true</u>	:	BOOL
<u>false</u>	:	BOOL
$cond_{INT}$	:	$BOOL \rightarrow (INT \rightarrow (INT \rightarrow INT))$
$cond_{BOOL}$	:	$BOOL \rightarrow (BOOL \rightarrow (BOOL \rightarrow BOOL))$
$Y\sigma$	:	$(\sigma \rightarrow \sigma) \rightarrow \sigma$
$\underline{n}$ (for each $n \in \mathbb{N}$):		INT
<u>succ</u>	:	$INT \rightarrow INT$
<u>pred</u>	:	$INT \rightarrow INT$
<u>zero</u>	:	$BOOL \rightarrow INT$

Variables: $x_n{}^\sigma$ $(n \in \mathbb{N})$: σ

Terms:

(i) $x_n{}^\sigma$ is a term

(ii) constants are terms

(iii) if t, s are terms of type $\sigma \rightarrow \tau$ and σ respectively, then (ts) is a term of type τ

(iv) if t is a term of type τ, then $\lambda x_n{}^\sigma.t$ is a term of type $\sigma \rightarrow \tau$

Reduction rules:

$cond_{INT}$ <u>true</u> $Z_1\ Z_2$	$\rightarrow Z_1$	
$cond_{INT}$ <u>false</u> $Z_1\ Z_2$	$\rightarrow Z_2$	
$cond_{BOOL}$ <u>true</u> $Z_1\ Z_2$	$\rightarrow$ Z1	
$cond_{BOOL}$ <u>false</u> $Z_1\ Z_2$	$\rightarrow Z_2$	
$Y\sigma Z$	$\rightarrow Z(Y\sigma Z)$	
$(\lambda x^\sigma.Z_1(x^\sigma))Z_2$	$\rightarrow Z_1(Z_2)$	
<u>succ</u> <u>n</u>	$\rightarrow$ <u>n+1</u>	$(n \in \mathbb{N})$
<u>pred</u> <u>n+1</u>	$\rightarrow$ <u>n</u>	$(n \in \mathbb{N})$
<u>zero</u> <u>0</u>	$\rightarrow$ <u>true</u>	
<u>zero</u> n+1	$\rightarrow$ <u>false</u>	$(n \in \mathbb{N})$

Table 3

An extension of BST

A <u>head context</u> $H_{n,m}[\ ]$ is a context of the form

$$\lambda x_1 \ldots x_n . [\ ] M_1 \ldots M_m$$

So $\lambda x . \Omega = H_{1,0}[\Omega]$

$\Omega M = H_{0,1}[\Omega]$

THEOREM. Suppose $C[\Omega, \Omega, \Omega] =_{BT} \ldots\ldots \Omega \ldots\ldots$.

Let the Ω displayed in the RHS trace back (be caused by) say the third Ω in the LHS.

(i) Then, for a fresh variable z:

$$C[\Omega, \Omega, z] =_{BT} \text{---}\ H_{n,m}[z]\ \text{---}$$

where some Ω's in the --- --- context are replaced by $H'_{n',m'}[z]$, $H''_{n'',m''}[z]$.

Furthermore, if we instantiate (refine) the other Ω's with say M_1, M_2, then

(ii) $C[M_1, M_2, z] =_{BT} \sim\sim\sim\ H^*_{n,m}[z]$

with the same n, m as in (i)!

THEOREM.

Let M_{ij}, $N_i \in Ter(\lambda)$ for $i,j \in \{1,\dots,n\}$.

Let $C[\;,\dots,\;]$ be an n-ary context and suppose for all $Z \in Ter(\lambda)$:

$$C[M_{11}, M_{12}, \dots, M_{1,n-1}, Z] =_{BT} N_1$$

$$C[M_{21}, M_{22}, \dots, Z, M_{2n}] =_{BT} N_2$$

$$\dots$$

$$C[Z, M_{n2}, \dots\ M_{n,n-1}, M_{nn}] =_{BT} N_n$$

Then for all $\vec{Z} = Z_1, \dots, Z_n \in Ter(\lambda)$

we have

$$C[\vec{Z}] =_{BT} N_1 =_{BT} = \dots =_{BT} N_n .$$

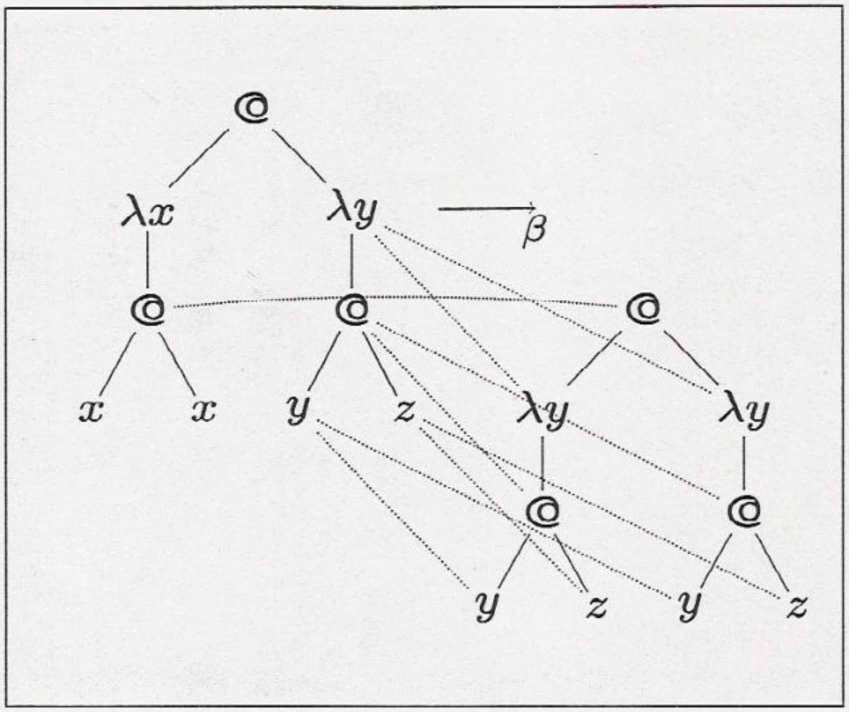
Metavariable Z is not allowed in RHSs N_i.

Counterexample:

$$K\,Z\,T =_{BT} Z$$

$$K\,T\,Z =_{BT} T$$

But the conclusion $\forall Z, Z'$
$K Z Z' =_{BT} Z =_{BT} T$ would be false.


@
λx
λy
→β
@
@
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x
x
y
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λy
λy
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y
z
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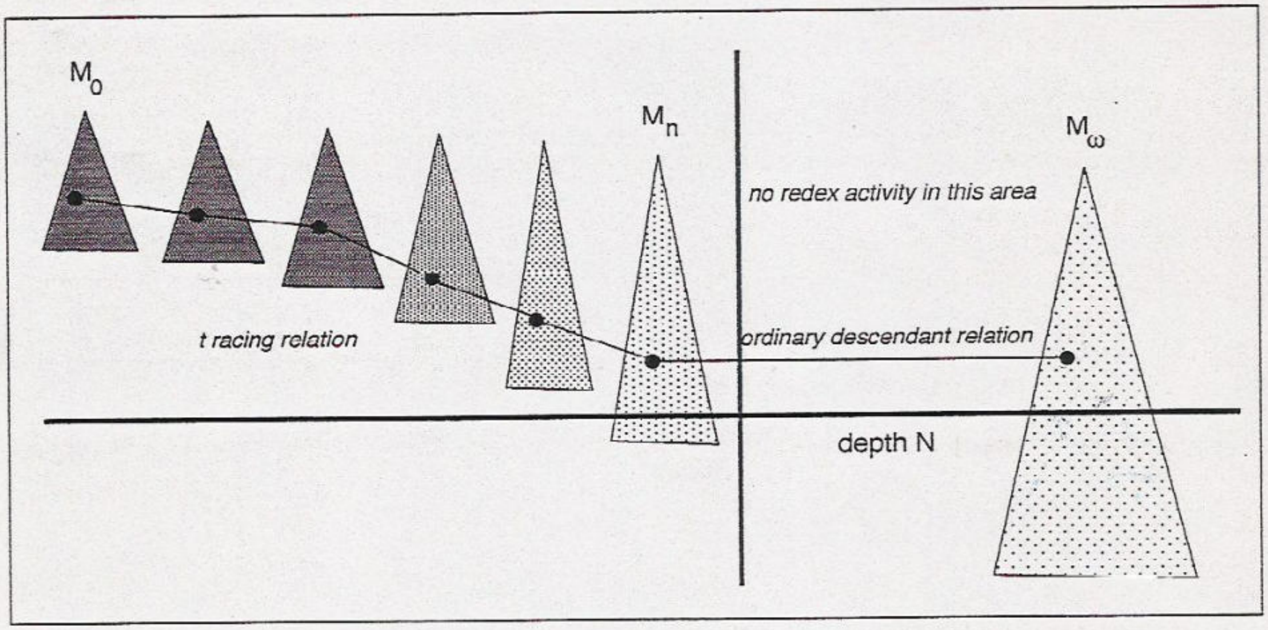
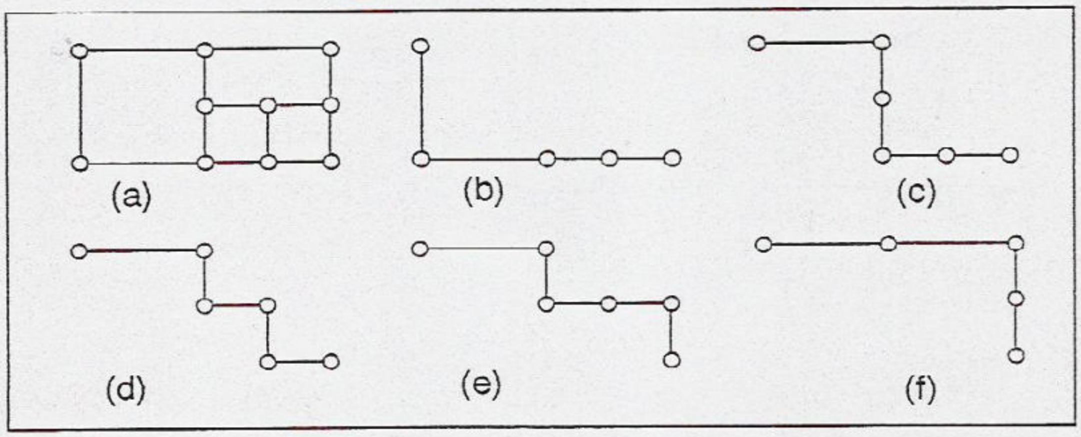
z

$$M \equiv [\lambda x.(x^{7}\, x^{8})^{20}]^{4}(y^{1}\, z^{0})^{2})^{37} \to_\beta$$

$$N \equiv ((y^{1}\, z^{0})^{2}(y^{1} z^{0})^{2})^{20}$$



M_0
M_n
M_ω
no redex activity in this area
t racing relation
ordinary descendant relation
depth N


(a)
(b)
(c)
(d)
(e)
(f)