Symposium for Henk Barendregt

Detachment: mathematics and meditation

Clocks, black holes and white holes in the lambda calculus

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Charles Rennie Mackintosh

1. *An inkling of lambda calculus and combinatory logic*

2. *Black holes in the lambda calculus: Henk's notion of unsolvables*

3. *Fixed point combinators*

4. *Clocks in the lambda calculus*

5. *Rivers of knowledge*

1. An inkling of lambda calculus and combinatory logic

The following is a mini-tutorial on lambda calculus, possibly to be skipped.

1.1. *An easy string rewrite system*

the replacement rules

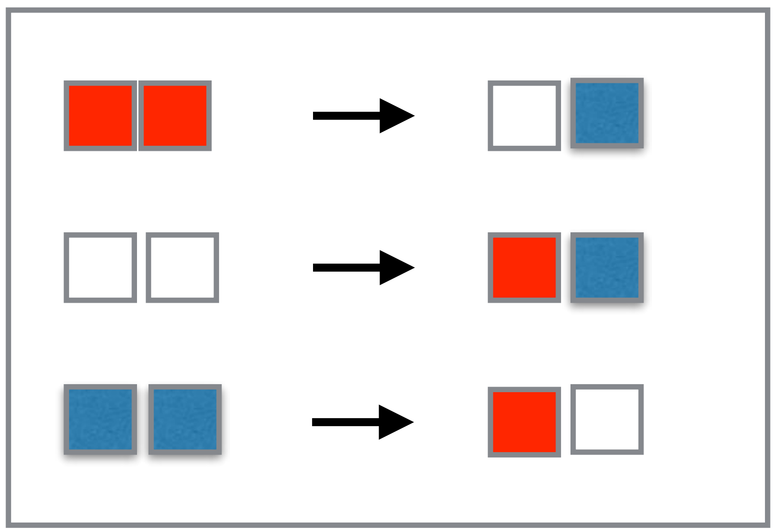
$$ab \longrightarrow bbba$$
$$ba \longrightarrow a$$
$$bb \longrightarrow b$$

$$abba \longrightarrow bbbaba$$
$$abba \longrightarrow aba$$
$$abba \longrightarrow aba$$

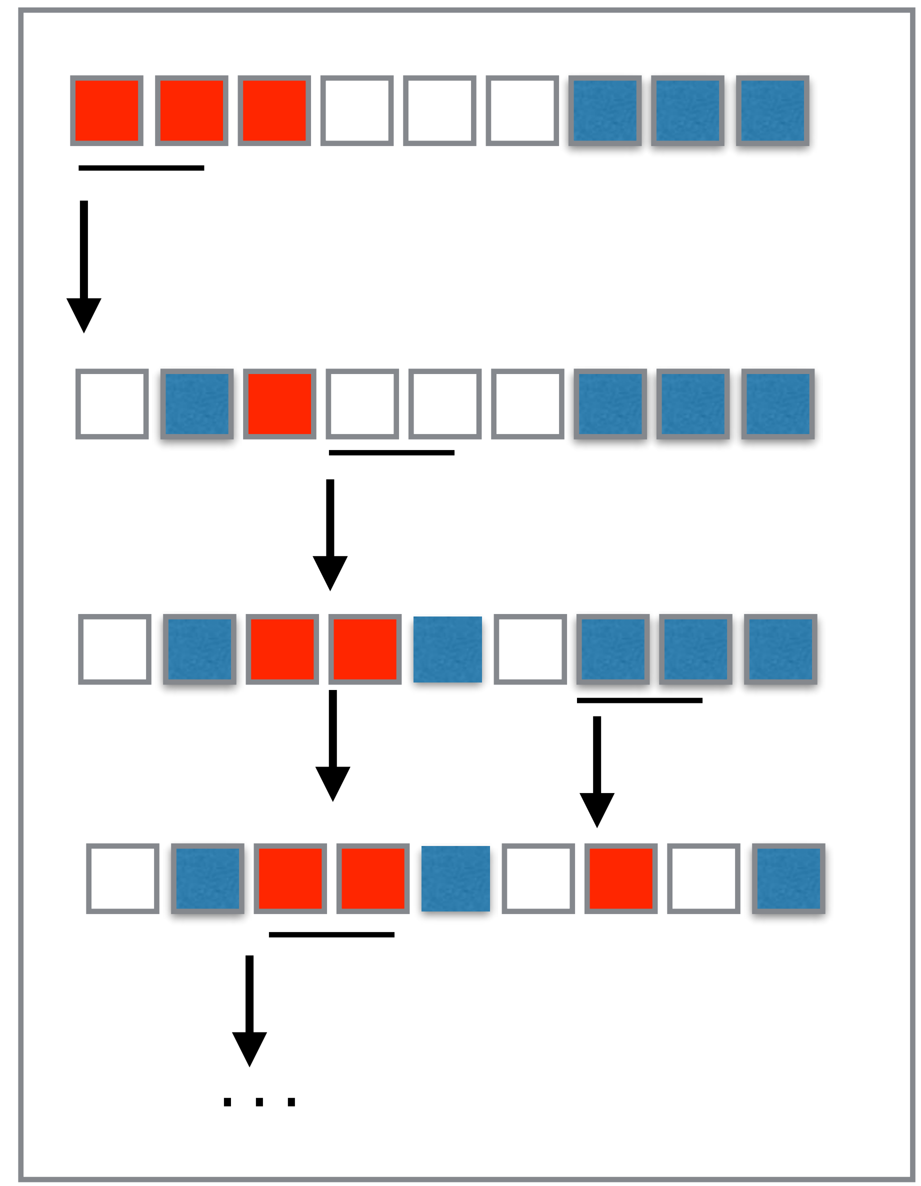
$$ab \longrightarrow bbba \longrightarrow ba \longrightarrow a$$

1.2. A difficult string rewrite system

the replacement rules



a play



must every play terminate, or are there infinite plays?

From inaugural lecture Hans Zantema,
'Op zoek naar bewijs', TUE, 2008

λ-calculus is a rewrite system , or a replacement system

The **λ**-calculus is the simplest possible programming language for information processing - the ‘ur-programming language’. The words in this language are called **λ**-terms. They are step by step transformed until a **λ**-term is reached representing an ‘answer’. A single rule giving these transformation steps suffices, the beta-rule (**β**-rule). It simply states how we can fill in a woord in a second word in the indicated places. The word that is filled in is colored red, the world , the word in which we fill in, is blue. the places where we fill in, are black symbols, officially called variables, and the variable for which we fill in is right after the **λ**.

Example. Fill in for each letter letter z in the word ziezo the word aap.
Result: aapieaapo. This rewrite step or **β**-reduction step is in mathematical notation written as

(**λ**z. ziezo) aap → aapieaapo

Another example of a **β**-reduction step is

(**λ**a. aap) ziezo → ziezoziezop

1.3. Calculemus!

maal: $M = \lambda fgx.f(gx)$
plus: $P = \lambda fgxy.fx(gxy)$
$1 = \lambda fx.fx$
$2 = \lambda fx.f(fx)$
$3 = \lambda fx.f(f(fx))$
$4 = \lambda fx.f(f(f(fx)))$

3 maal 2 $= M32 \rightarrow$
$(\lambda gx.3(gx))2 \rightarrow$
$\lambda x.3(2x) \rightarrow$
$\lambda xx'.2x(2x(2xx')) \rightarrow$
$\lambda xx'.(\lambda x'.x(xx'))(2x(2xx')) \rightarrow$
$\lambda xx'.x(x(2x(2xx'))) \rightarrow$
$\lambda xx'.x(x((\lambda x'.x(xx'))(2xx'))) \rightarrow$
$\lambda xx'.x(x(x(x(2xx')))) \rightarrow$
$\lambda xx'.x(x(x(x((\lambda x'.x(xx'))x')))) \rightarrow$
$\lambda xx'.x(x(x(x(x(xx'))))) = 6$

2 maal 3 $= M23 \rightarrow$
$(\lambda gx.2(gx))3 \rightarrow$
$\lambda x.2(3x) \rightarrow$
$\lambda xx'.3x(3xx') \rightarrow$
$\lambda xx'.(\lambda x'.x(x(xx')))(3xx') \rightarrow$
$\lambda xx'.x(x(x(3xx'))) \rightarrow$
$\lambda xx'.x(x(x((\lambda x'.x(x(xx')))x'))) \rightarrow$
$\lambda xx'.x(x(x(x(x(xx'))))) = 6$

2 plus 3 $= P23 \rightarrow$
$(\lambda gxy.2x(gxy))3 \rightarrow$
$\lambda xy.2x(3xy) \rightarrow$
$\lambda xy.(\lambda x'.x(xx'))(3xy) \rightarrow$
$\lambda xy.x(x(3xy)) \rightarrow$
$\lambda xy.x(x((\lambda x'.x(x(xx')))y)) \rightarrow$
$\lambda xy.x(x(x(x(xy)))) = 5$

3 tot de macht 2 = $3^2 = 23$

$(\lambda fx.f(fx))3 \to$

$\lambda x.3(3x) = \lambda x.(\lambda fx'.f(f(fx')))(3x) \to$

$\lambda xx'.3x(3x(3xx')) \to$

$\lambda xx'.(\lambda x'.x(x(xx')))(3x(3xx')) \to$

$\lambda xx'.x(x(x(3x(3xx')))) \to$

$\lambda xx'.x(x(x((\lambda x'.x(x(xx')))(3xx')))) \to$

$\lambda xx'.x(x(x(x(x(x(3xx')))))) \to$

$\lambda xx'.x(x(x(x(x(x((\lambda x'.x(x(xx')))x')))))) \to$

$\lambda xx'.x(x(x(x(x(x(x(x(xx')))))))) = 9$

Exercise (Barendregt)

The *length* of a term is its number of symbols times 0.5 cm. Write down a λ-term of length < 30 cm with normal form $> 10^{10^{10}}$ light year.
[*Hint.* The speed of light is $c = 3 \times 10^{10}$ cm/s.]

We laten de kleuren verder weg.
Een woord (officieel, een λ-term) kan ook uit een enkele letter bestaan.
Dus we hebben ook bijvoorbeeld de stappen
$(\lambda x.x)aap \rightarrow aap$
$(\lambda x.yxy)z \rightarrow yzy$
Een meer zinvol voorbeeld is het verdubbelen van woorden:
$(\lambda x.xx)aap \rightarrow aapaap$
Dus $(\lambda x.xx)$ is een woordverdubbelaar, elk woord waarop deze λ-term wordt 'toegepast', wordt in één stap verdubbeld. Een woord uitwissen kan ook:
$(\lambda a.goed)slecht \rightarrow goed$
Want goed bevat geen a!
λ-termen kunnen ook op zichzelf worden toegepast: zelfapplicatie. Voor de verdubbelaar levert dit een interessant verschijnsel op:
$(\lambda x.xx)(\lambda x.xx) \rightarrow (\lambda x.xx)(\lambda x.xx)$

1.4. A swiss pocket knife for lambda calculus

Lamda calculus and CL reduction tool by Freek Wiedijk

http://www.cs.vu.nl/~terese/lambda.html

Lambda calculus and CL reduction tool by Freek Wiedijk

Freek Wiedijk developed a tool for λ- and CL-reduction. It is currently used in the course Term Rewiting Systems at the VU.

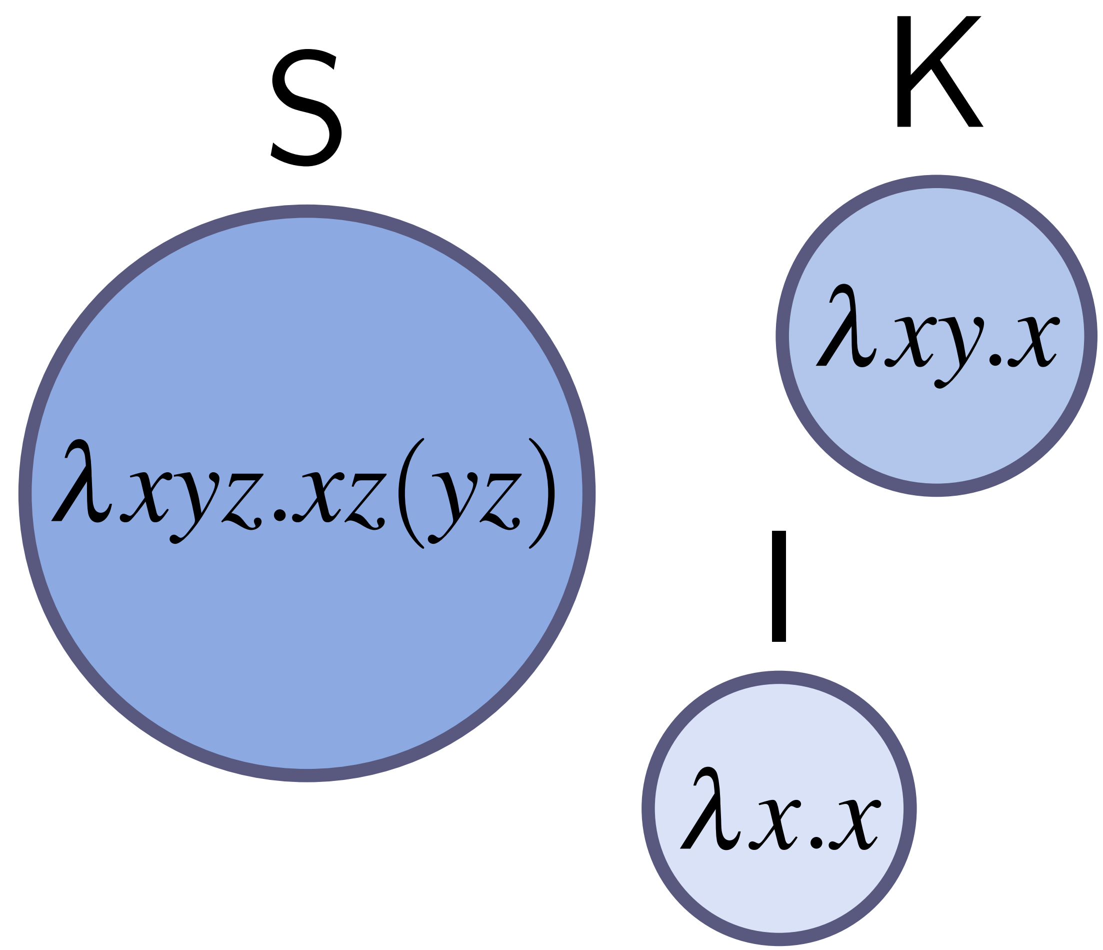
- Mac OS X executable file
  Instructions for installation on a Mac:
  - Transfer the file to your Mac OS X machine as lambda1, in your home directory
  - In a terminal, give the command "chmod a+x lambda1".
  - For execution of the tool type "./lambda1".
- source code
  In the presence of OCAML this source code file can be compiled, under UNIX, for example by the command
  - ocamlopt lambda1.ml -o lambda1

How to use the λ- and CL-tool

After starting, the tool prints

```
I=\x.x
K=\xy.x
S=\xyz.xz(yz)
B=\xyz.x(yz)
C=\xyz.xzy
W=\xy.xyy
1=\xy.xy
Y=\f.(\x.f(xx))\x.f(xx)
T=\xy.x
F=\xy.y
D=\x.xx
J=\abcd.ab(adc)
C'=JII
.li leftmost innermost
.lo leftmost outermost [default]
.po parallel outermost
```

1.5. Combinatory Logic, the twin of lambda calculus

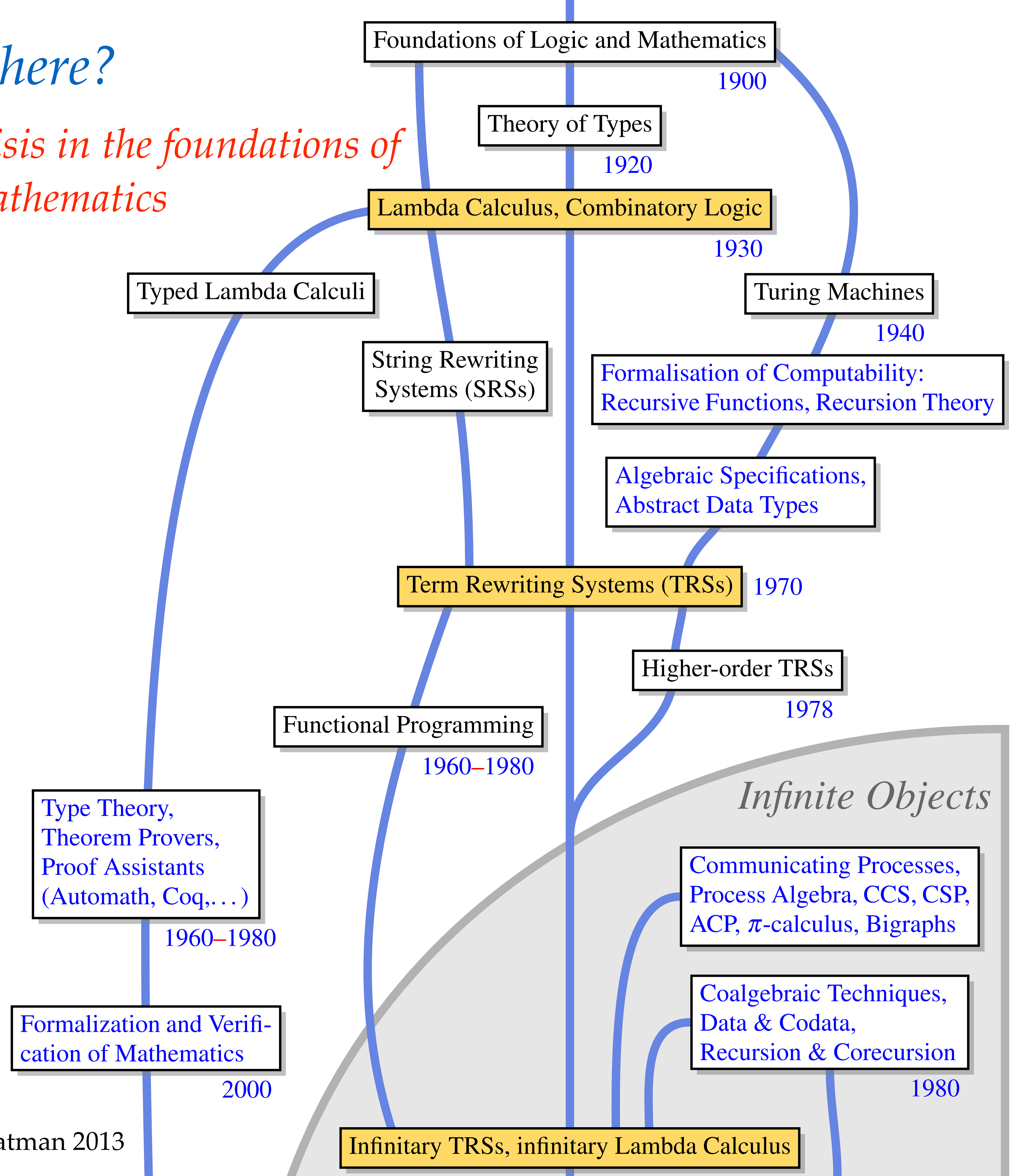


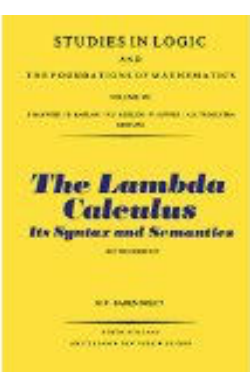
1924. "Über die Bausteine der mathematischen Logik"

Moses Schönfinkel

1.7. How did we come here?

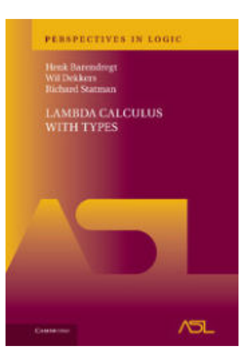
crisis in the foundations of mathematics





De Bruijn 1968...

Barendregt 1980,1984



Gonthier

Barendregt, Dekkers, Statman 2013

How did we come here? Zooming out three centuries

Gottfried Leibniz

1646 - 1716

calculus ratiocinator

Leibniz equality:
in proof checker Coq

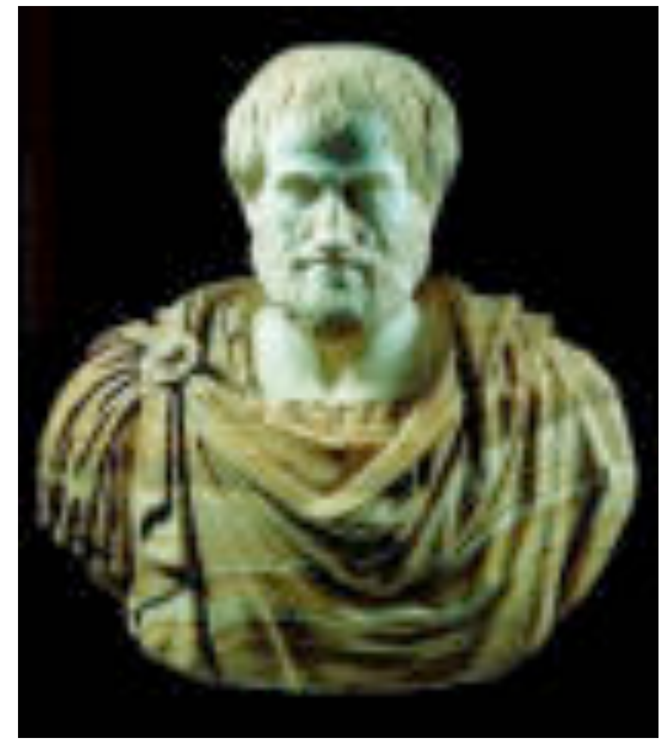
x = y
defined as
$\forall P: P(x) \leftrightarrow P(y)$

- binary number system, ...
- calculus ratiocinator: general system of a notation in which all the truths of reason should be reduced to a calculus. An ‘algebra of thoughts’.
- project proposal "I think that some chosen men could finish the matter within five years"
- No postdocs - If I had been less busy, or if I were younger or helped by well-intentioned young people, I would have hoped to have evolved a characteristic of this kind

How did we come here? Zooming out two milennia

Ἀριστοτελησ

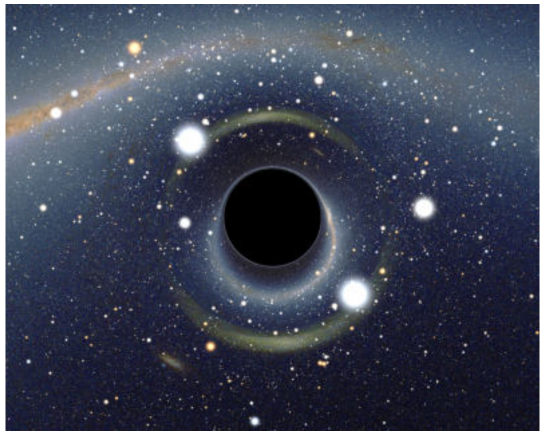
384 vC- 322 vC



coming back to this later

24 logische syllogismen, fragment van predikatenlogica

I. Barbara, Celarent, Darii, Ferio, Barbari, Celaront
II. Cesare, Camestres, Festino, Baroco, Cesaro, Camestros
III. Darapti, Disamis, Datisi, Felapton, Bocardo, Ferison
IV. Bamalip, Calemes, Dimatis, Fesapo, Fresison, Calemos



2. Black holes in the lambda calculus: Henk's notion of unsolvables

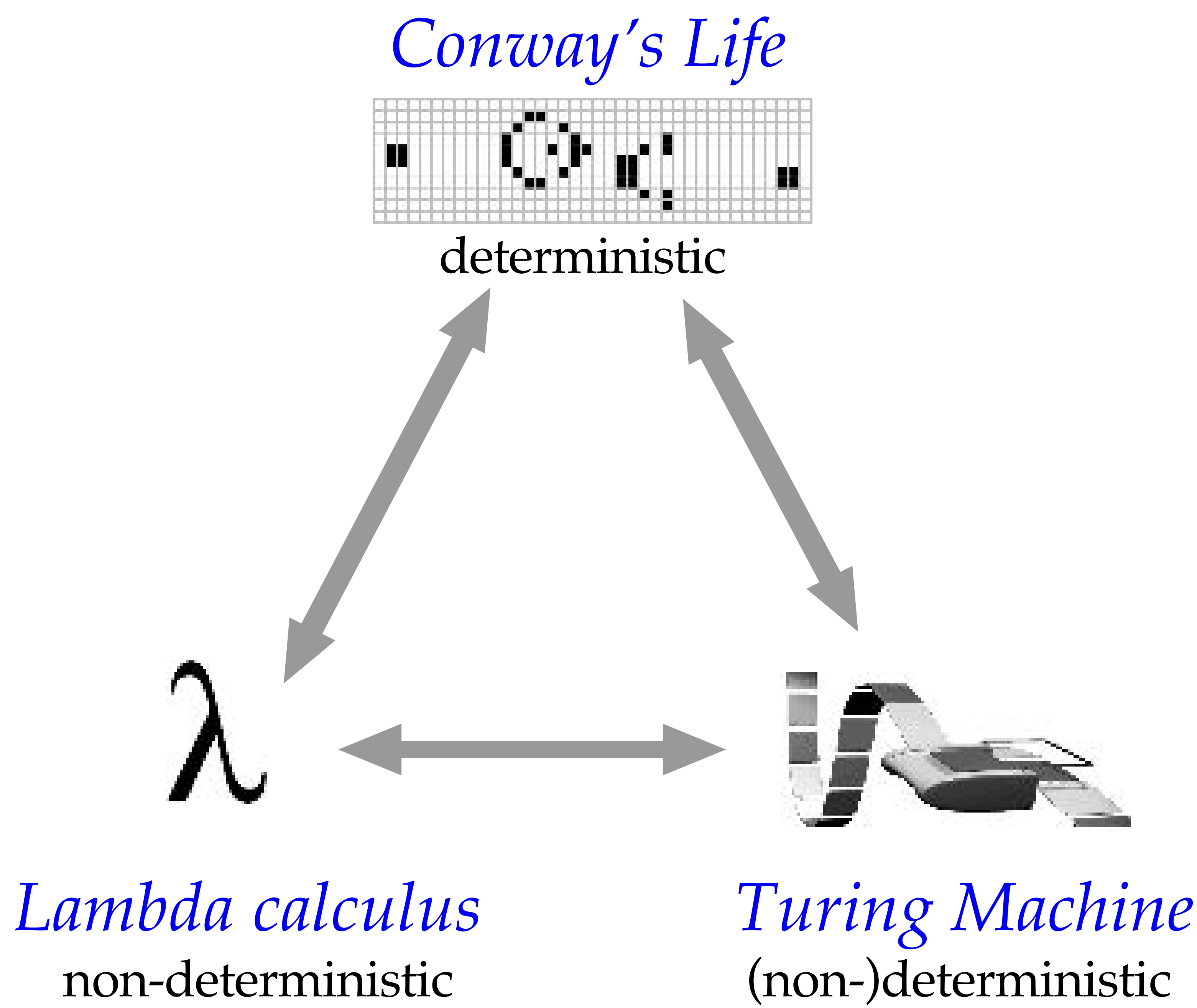
The straightforward extension of the space-time universe breaks down in the form of *black holes*, the astronomical witnesses of singularities in the mathematical theory of Einstein's equations, where some terms become infinite.

QUESTION:
Are black holes in space also somehow a consequence of space turned in itself? Is there something like reflection here?

In the computational universe there are several systems that can define and compute all computable functions. They are called *Turing complete*. They all show singularities where their 'normal functioning' breaks down. These singularities emerge due to self-reference, *reflection of the system in itself*. For Turing Machines this is known as the unsolvability of the Halting Problem; no TM can compute whether a TM plus input terminates or runs forever.

QUESTION:
see Chaitin: is there is a relation between incompleteness of PA with HP?

2.1. Turing complete systems



Also PA, Post Production systems,
tag systems, Markov systems, and so on.

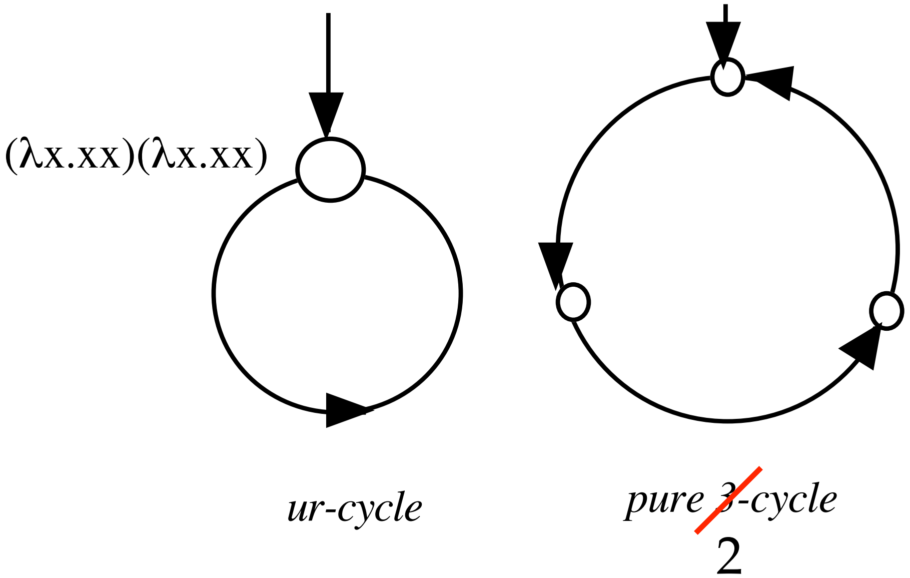
We have already encountered one black hole in lambda calculus:

$$(\lambda x.xx)(\lambda x.xx) \rightarrow (\lambda x.xx)(\lambda x.xx)$$

Another cyclic black hole:

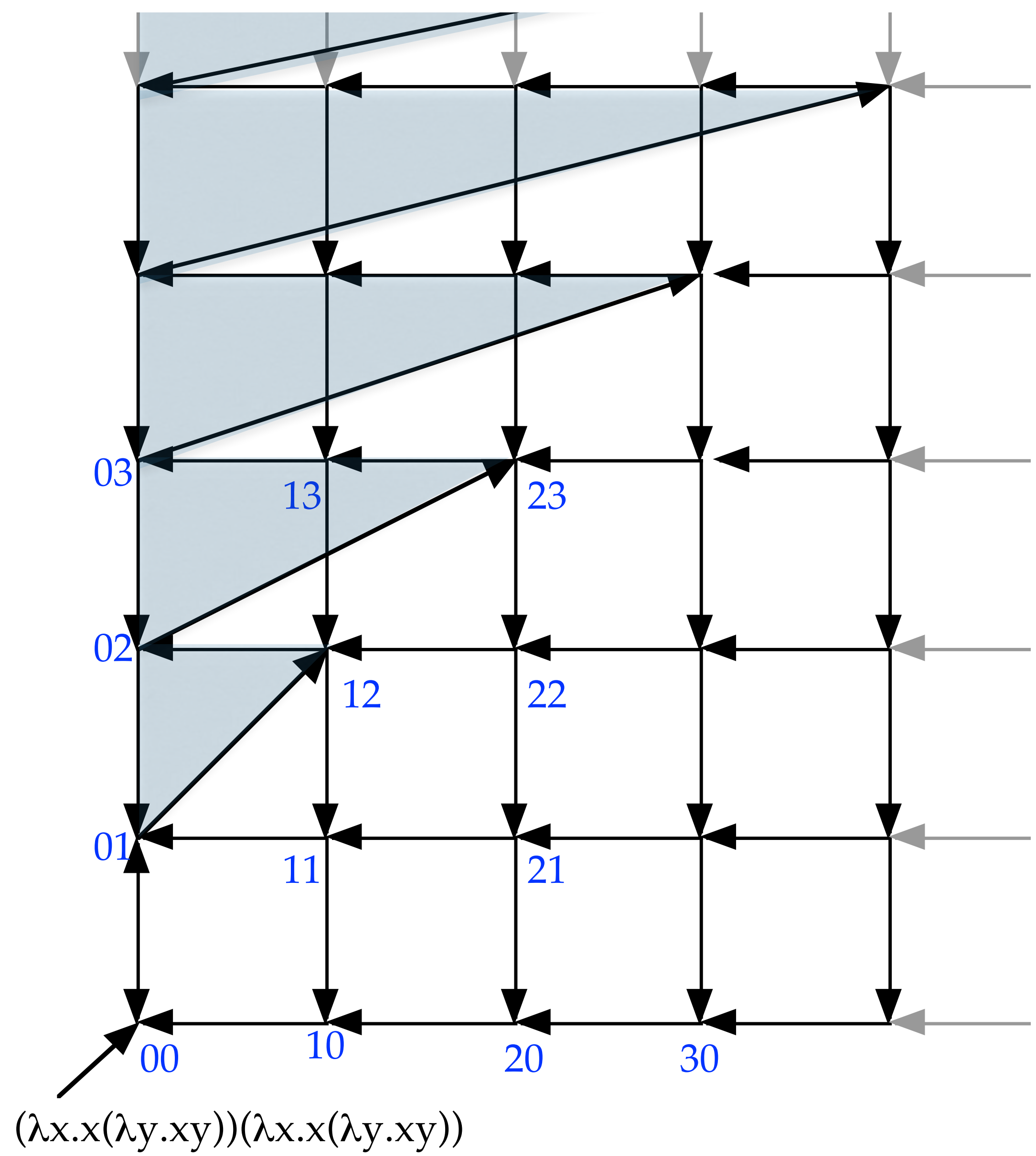
Let $A \equiv \lambda ab.baa$. Then

$$AAA \rightarrow (\lambda b.bAA)A \rightarrow AAA$$



Another black hole, whose reduction graph displays cycles of every length except 1:

$$(\lambda x.x(\lambda y.xy))(\lambda x.x(\lambda y.xy))$$



The unsolvable λ-terms M are those for which there is a sequence of λ-terms $A_1, \ldots A_n$ such that

$$M A_1 A_2 \ldots A_n =_\beta I$$

where $I \equiv \lambda x.x$

Equivalently, appealing to the Church-Rosser theorem

$$M A_1 A_2 \ldots M_n \twoheadrightarrow_\beta I$$

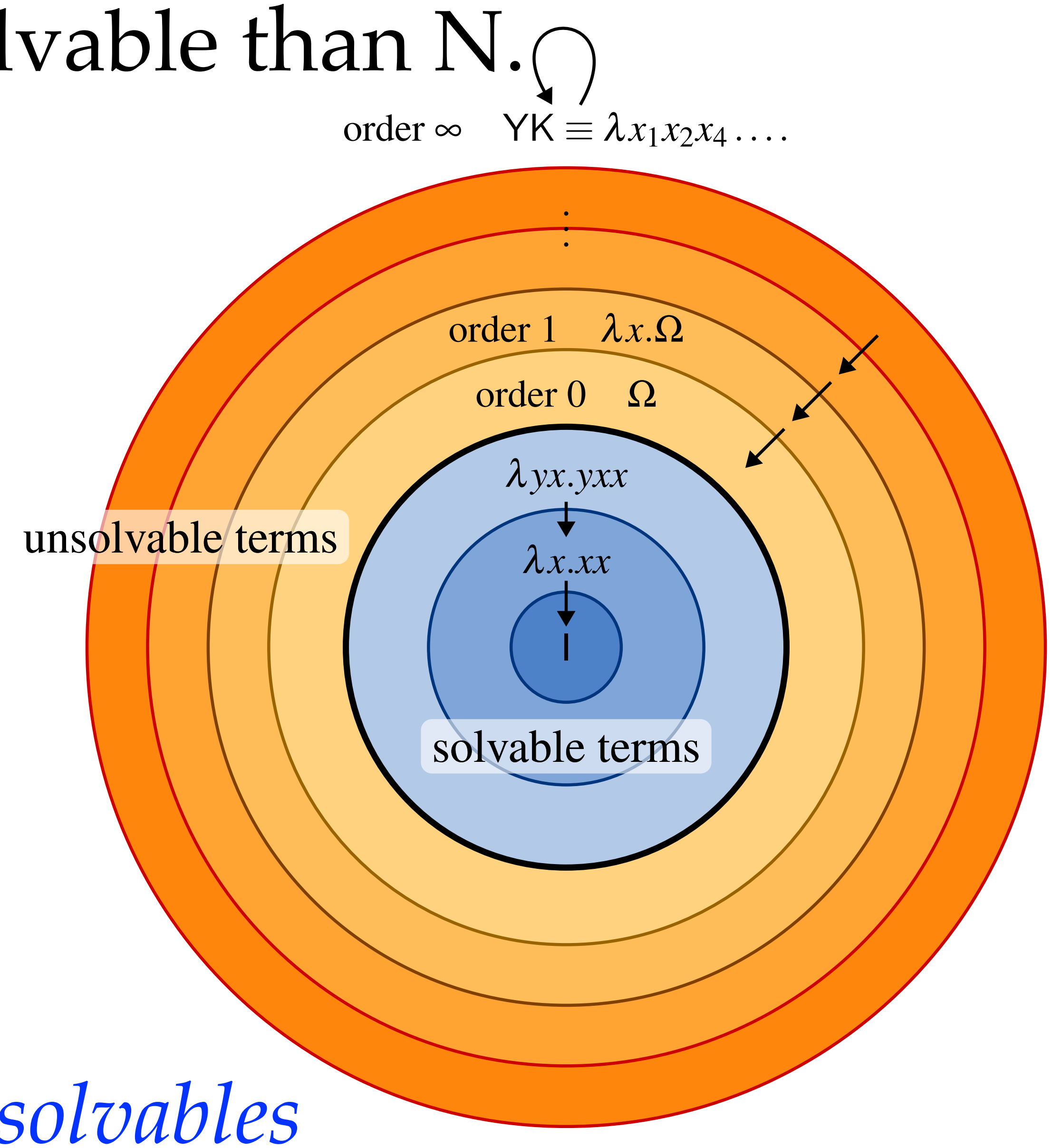
An equivalent definition was given by Wadsworth: unsolvable terms are those without *head normal form*.

2.1. *Black holes are hugely complicated*

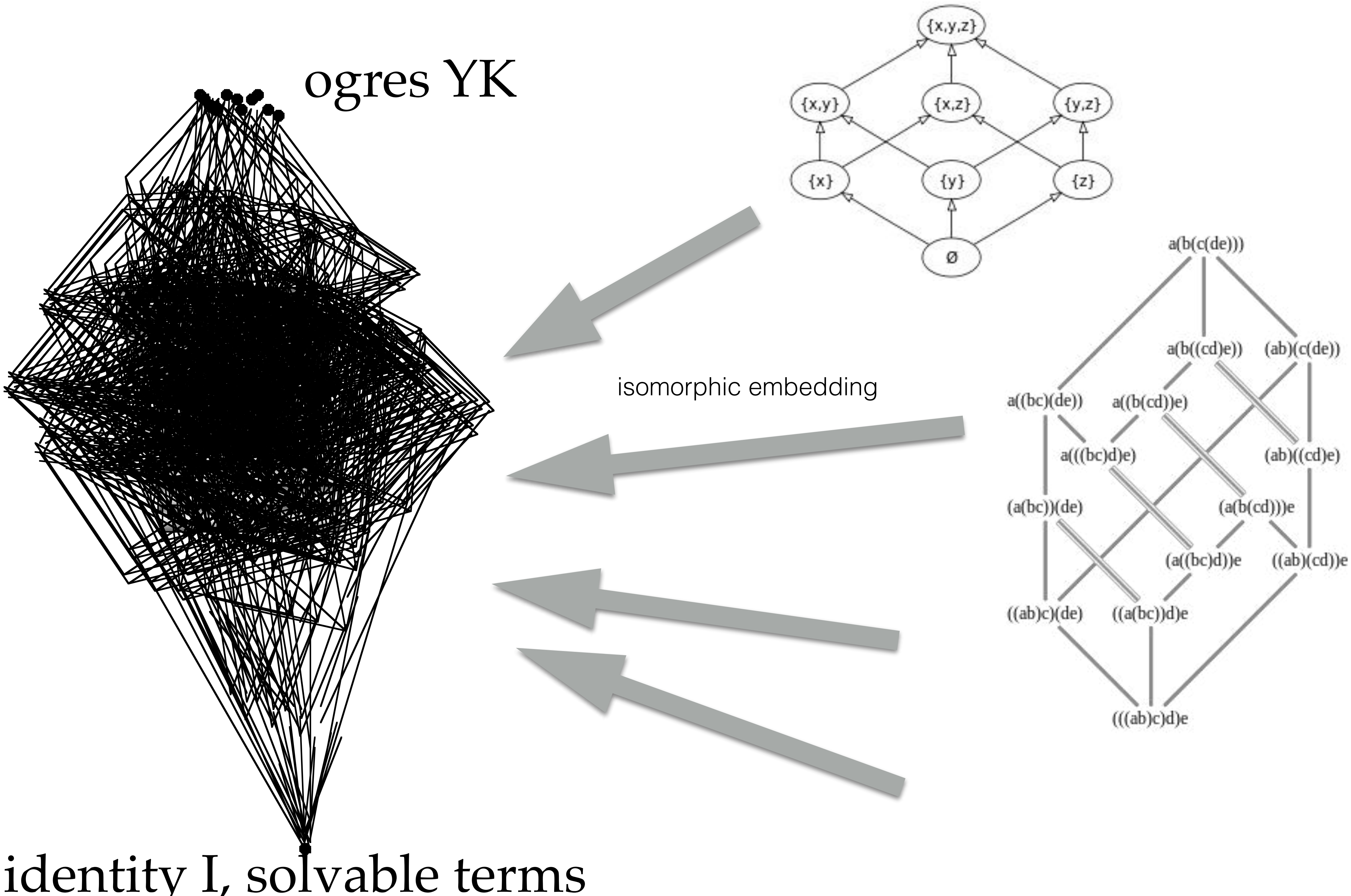
Statman 1978

instead of $MA \twoheadrightarrow_{\beta} N$, write $M \overset{A}{\Rightarrow} N$

$M \Rightarrow N$: M is more solvable than N.



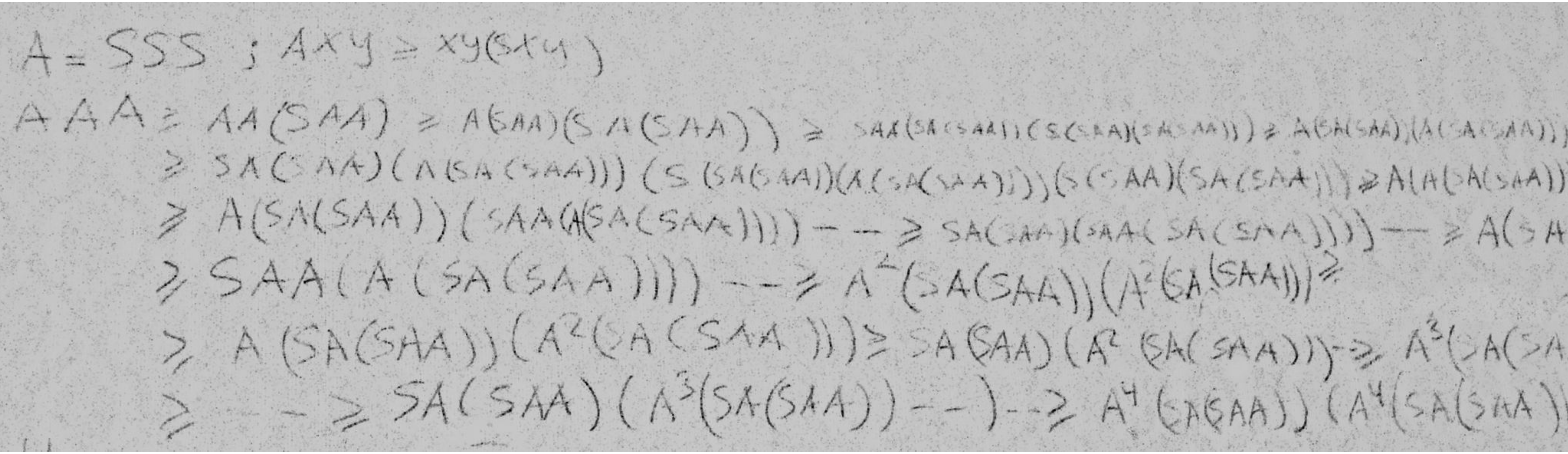
Every countable poset is embeddable in poset of unsolvables *countable, countably universal poset of unsolvables*



Rick Statman, 1986

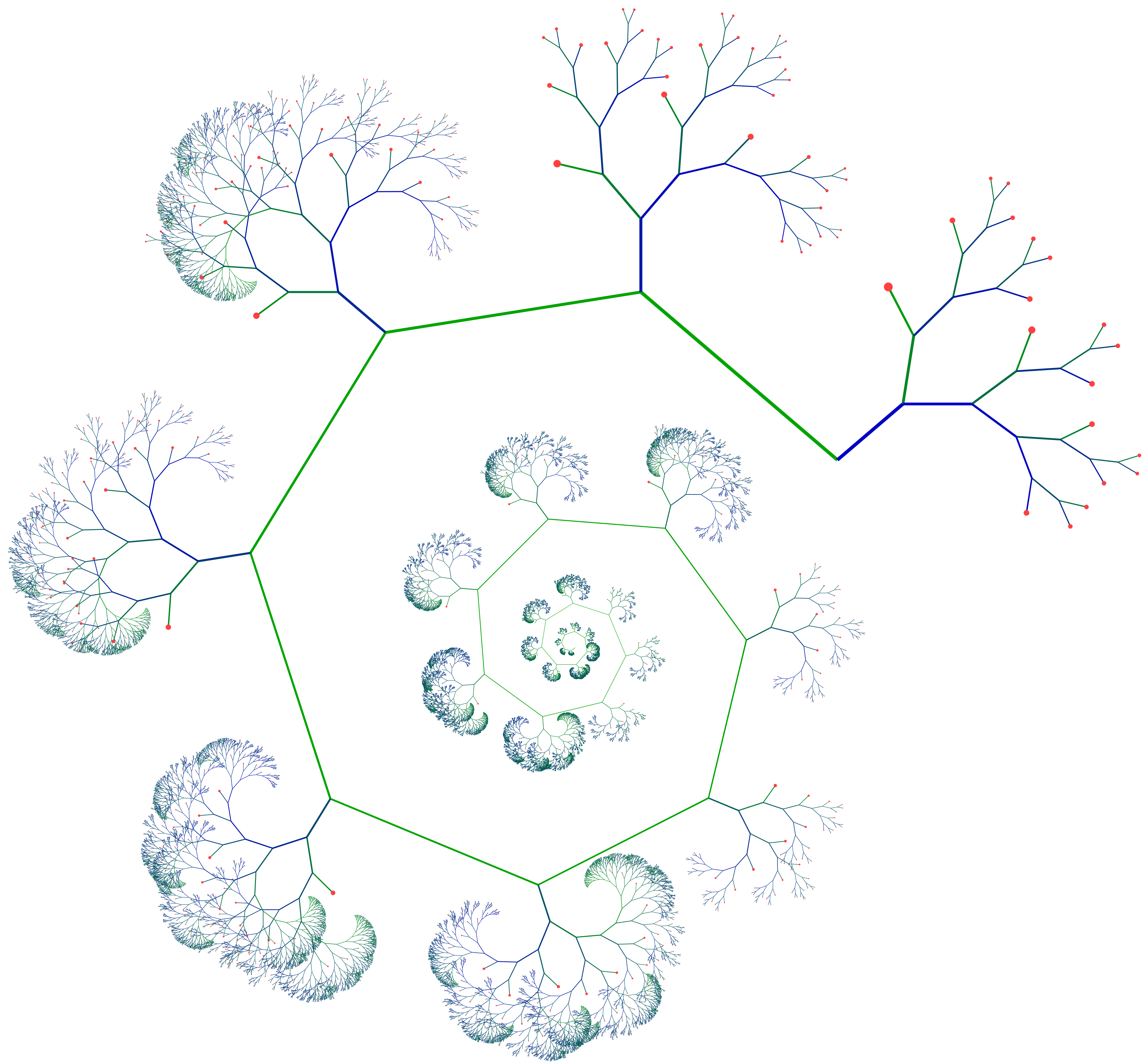
2.2. Once upon a time … Henk lived here (1975)

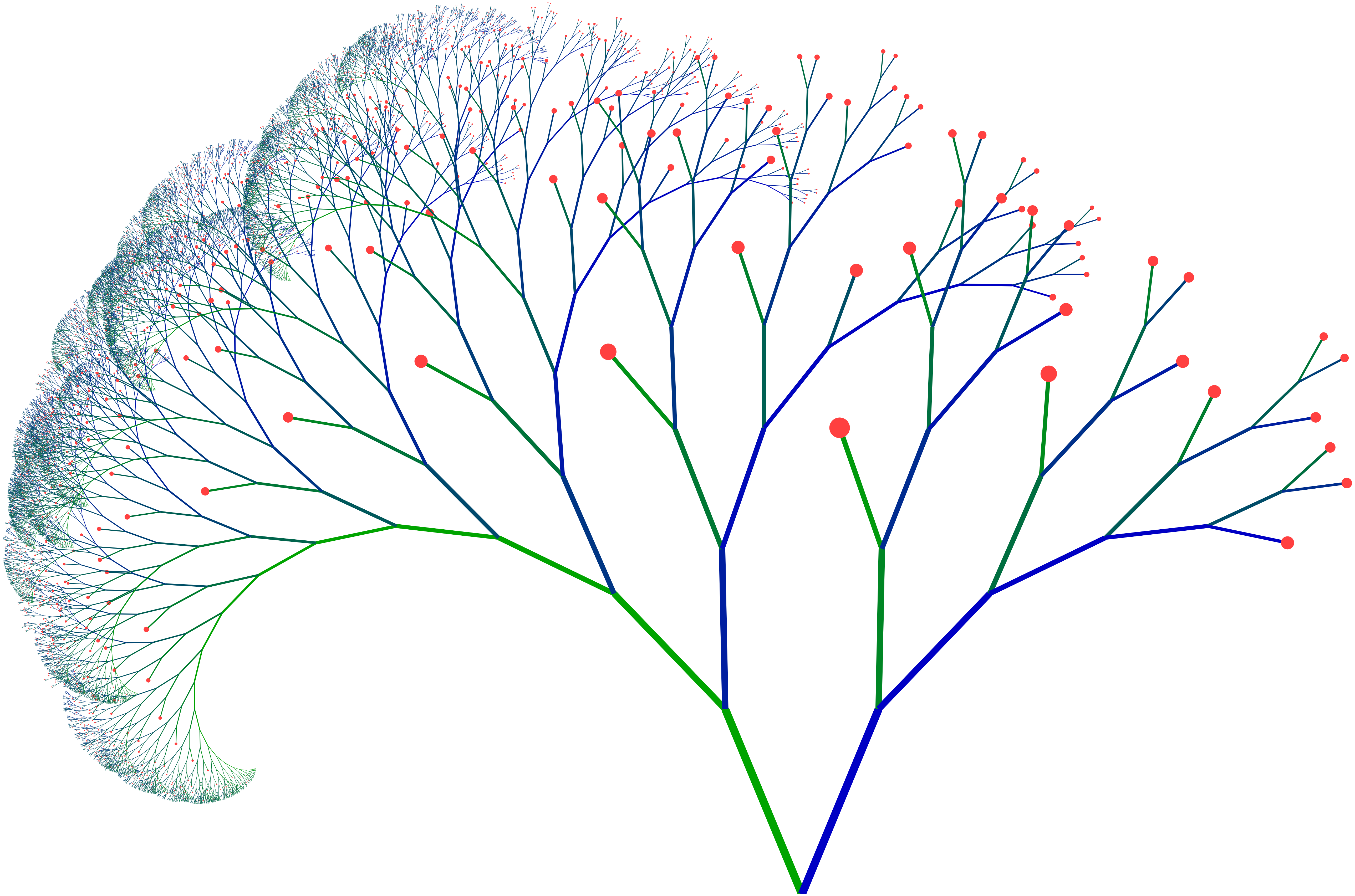
in the house there were writings on the wall

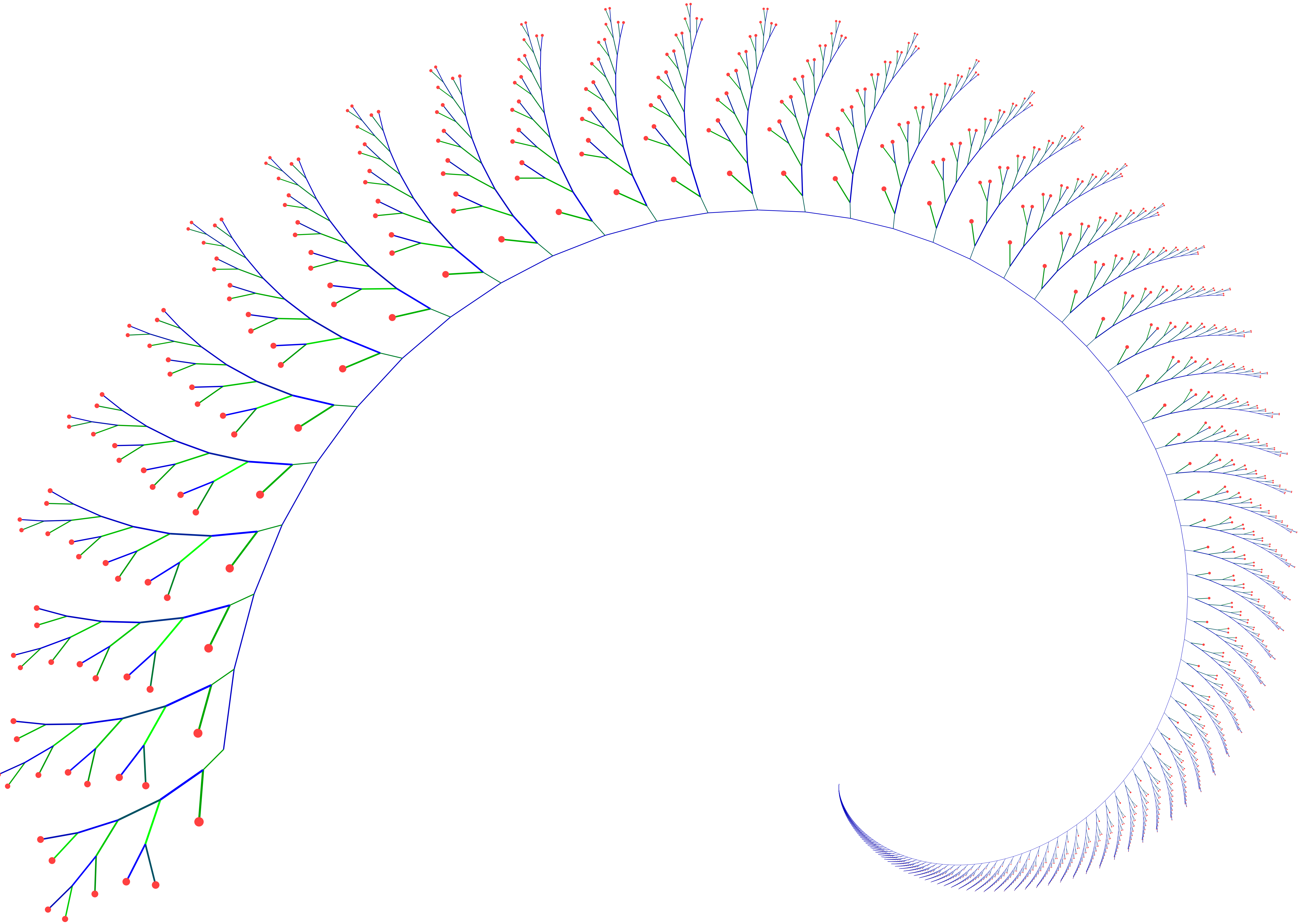


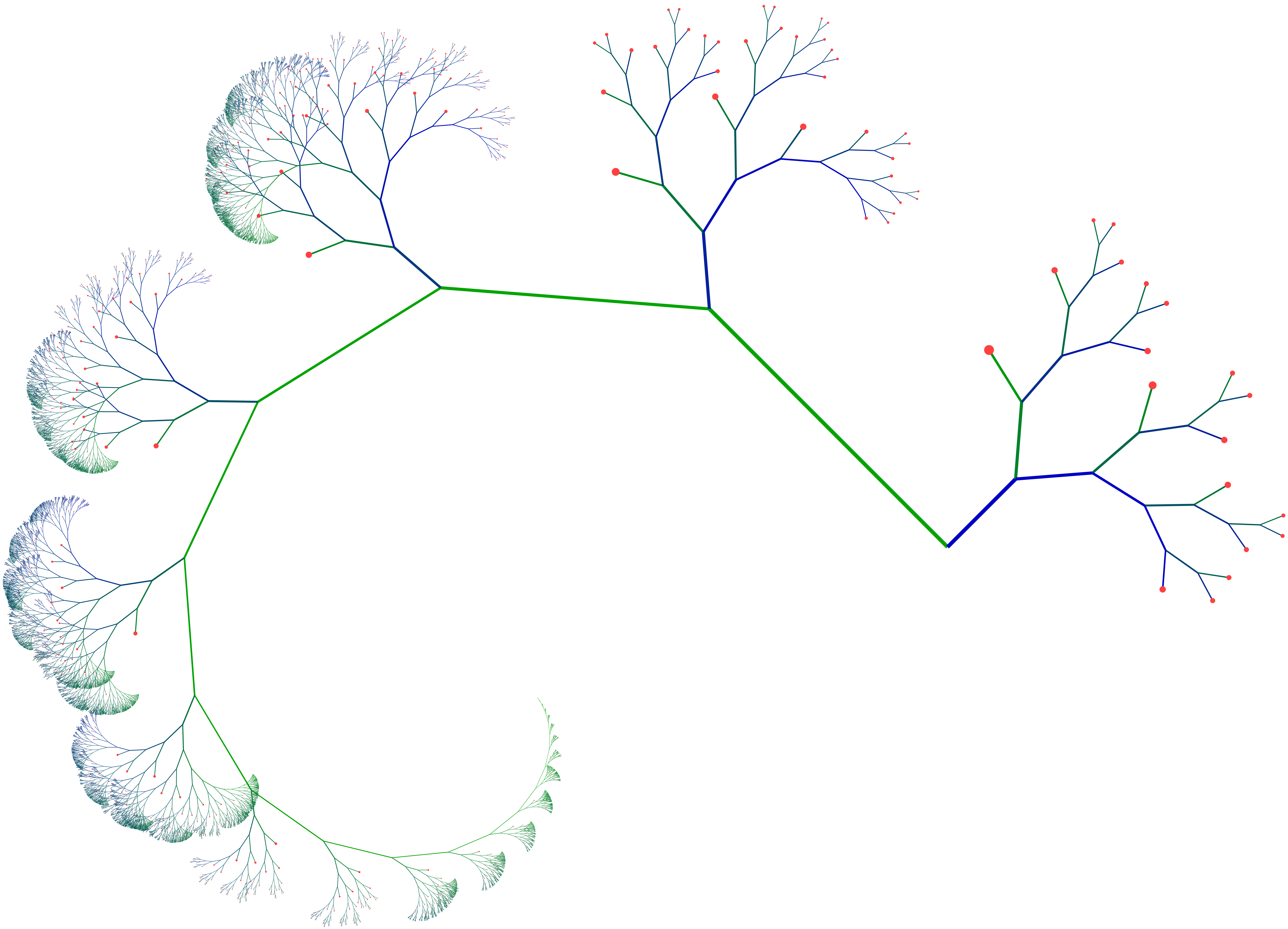
A=SSS
AAA
AAA
SSSAA
SA(SA)A
AA(SAA)
SSSA(SAA)
SA(SA)(SAA)
A(SAA)(SA(SAA))
SSS(SAA)(SA(SAA))
S(SAA)(S(SAA))(SA(SAA))
SAA(SA(SAA))(S(SAA)(SA(SAA)))
A(SA(SAA))(A(SA(SAA)))(S(SAA)(SA(SAA)))
SSS(SA(SAA))(A(SA(SAA)))(S(SAA)(SA(SAA)))
S(SA(SAA))(S(SA(SAA)))(A(SA(SAA)))(S(SAA)(SA(SAA)))
SA(SAA)(A(SA(SAA)))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))
A(A(SA(SAA)))(SAA(A(SA(SAA))))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))
SSS(A(SA(SAA)))(SAA(A(SA(SAA))))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))
S(A(SA(SAA)))(S(A(SA(SAA))))(SAA(A(SA(SAA))))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))
A(SA(SAA))(SAA(A(SA(SAA))))(S(A(SA(SAA)))(SAA(A(SA(SAA)))))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))
SSS(SA(SAA))(SAA(A(SA(SAA))))(S(A(SA(SAA)))(SAA(A(SA(SAA)))))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))
S(SA(SAA))(S(SA(SAA)))(SAA(A(SA(SAA))))(S(A(SA(SAA)))(SAA(A(SA(SAA)))))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))
SA(SAA)(SAA(A(SA(SAA))))(S(SA(SAA))(SAA(A(SA(SAA)))))(S(A(SA(SAA)))(SAA(A(SA(SAA)))))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))
A(SAA(A(SA(SAA))))(SAA(SAA(A(SA(SAA)))))(S(SA(SAA))(SAA(A(SA(SAA)))))(S(A(SA(SAA)))(SAA(A(SA(SAA)))))(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))

2.3. *Black holes can be beautiful*

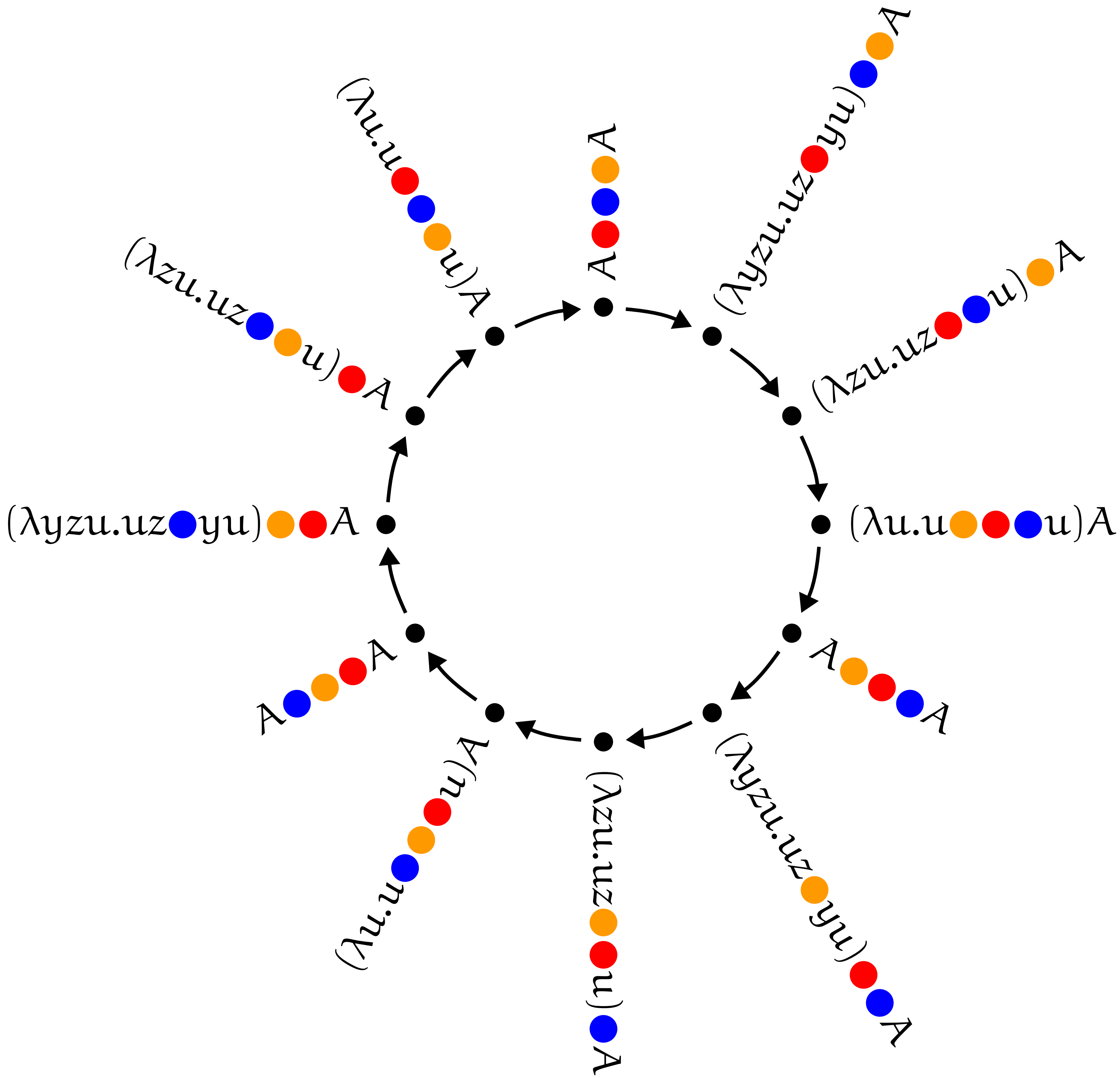




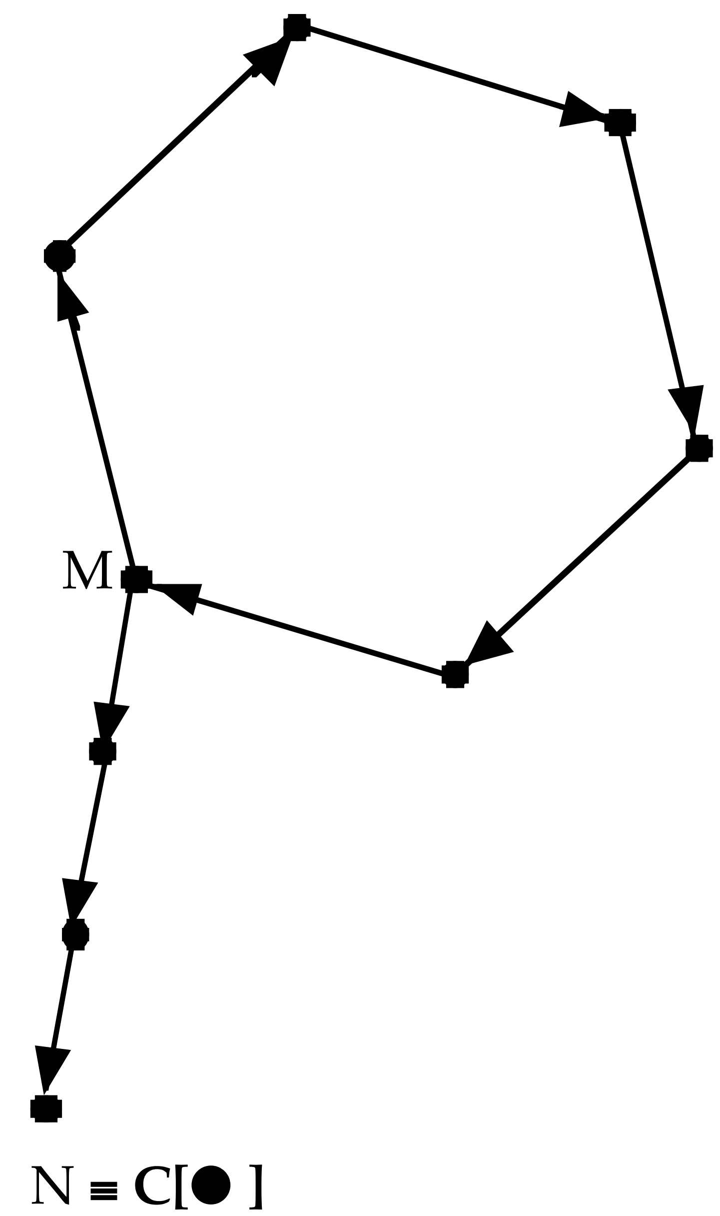




2.4. *Another nice black hole, a pure cycle*



2.5. *Let's have a theorem: reduction cycles entail black holes*



Theorem. Let the λ-term *M* admit a cyclic reduction. Then *M* is unsolvable, or else, a subterm of a reduct of *M* is unsolvable. In fact, even 'mute', the worst kind of unsolvable.

2.6. The threefold path of unsolvables

There is a natural hierarchy of unsolvable lambda terms, in three families, leading to Böhm Trees (BT), Lévy-Longo Trees (LLT),and Berarducci Trees (BeT).

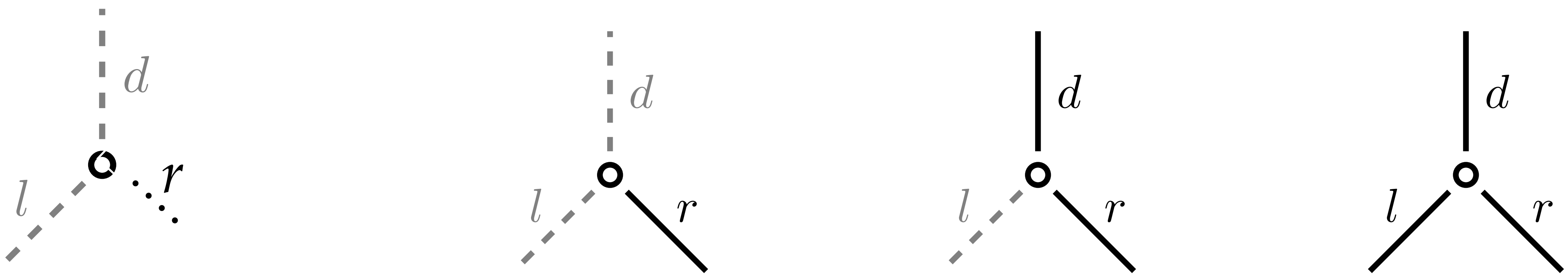
They embody different notions of undefined: having no *head normal form*, having no *weak head normal form*, having no *root normal form*.

This can be seen as three different semantics for lambda calculus, with BT-semantics as the coarsest (= most identifying), next LLT-semantics as intermediate, finally BeT-semantics as the finest (= most distinguishing)

From eightfold path to threefold path

Böhm trees provide a semantics of λ-calculus where terms without head normal form are considered meaningless. In fact, this semantical view is one of three canonical semantical frameworks that arise in a uniform way by considering the three dimensions d, l, r in which λ-terms can grow:

d *down*, in an abstraction;
l *left* in an application;
r *right* in an application,



λ_{000} finitary lambda calculus

λ_{001} BT

λ_{101} LLT

λ_{111} BeT

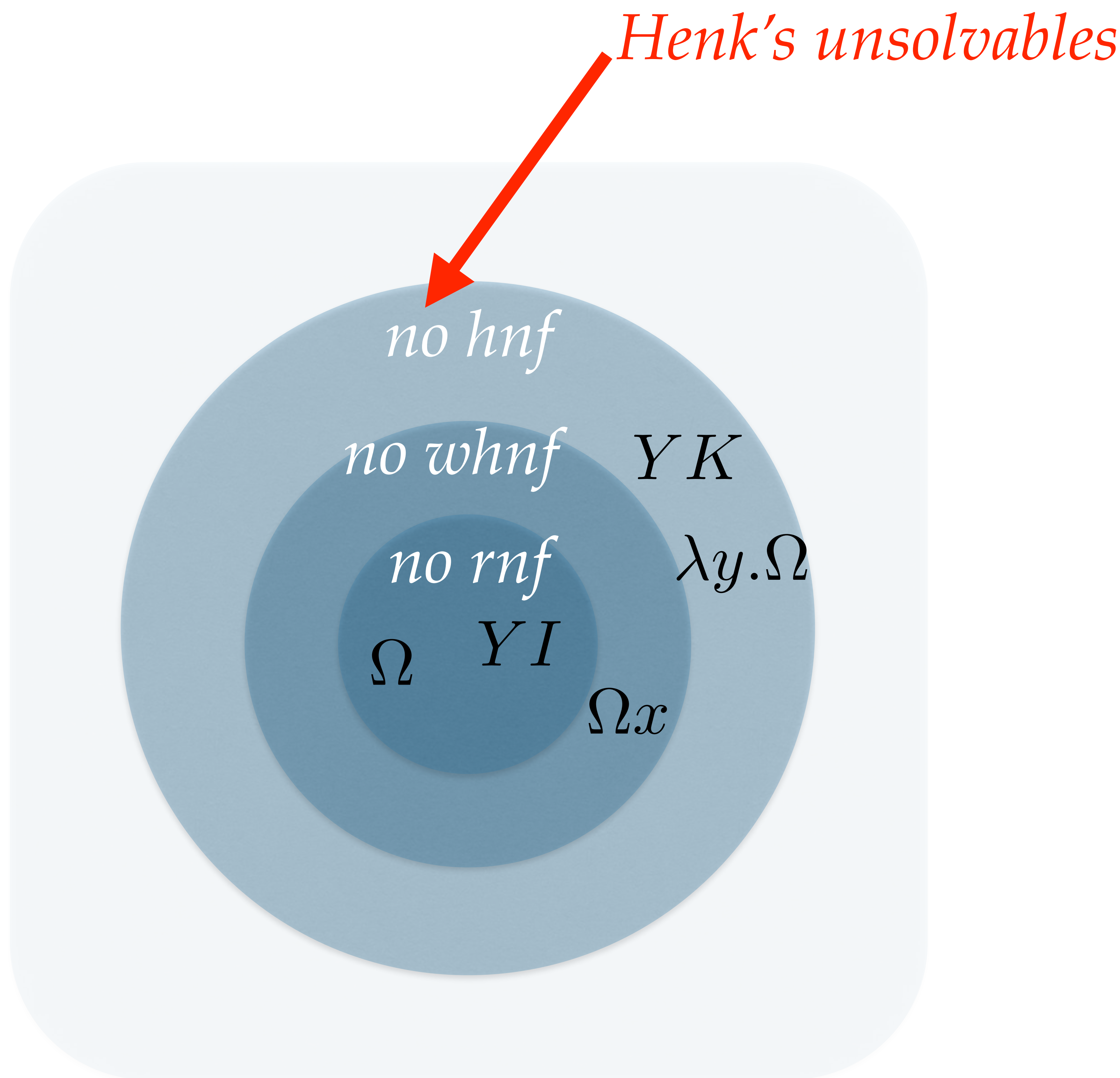
Suppressed dimensions.

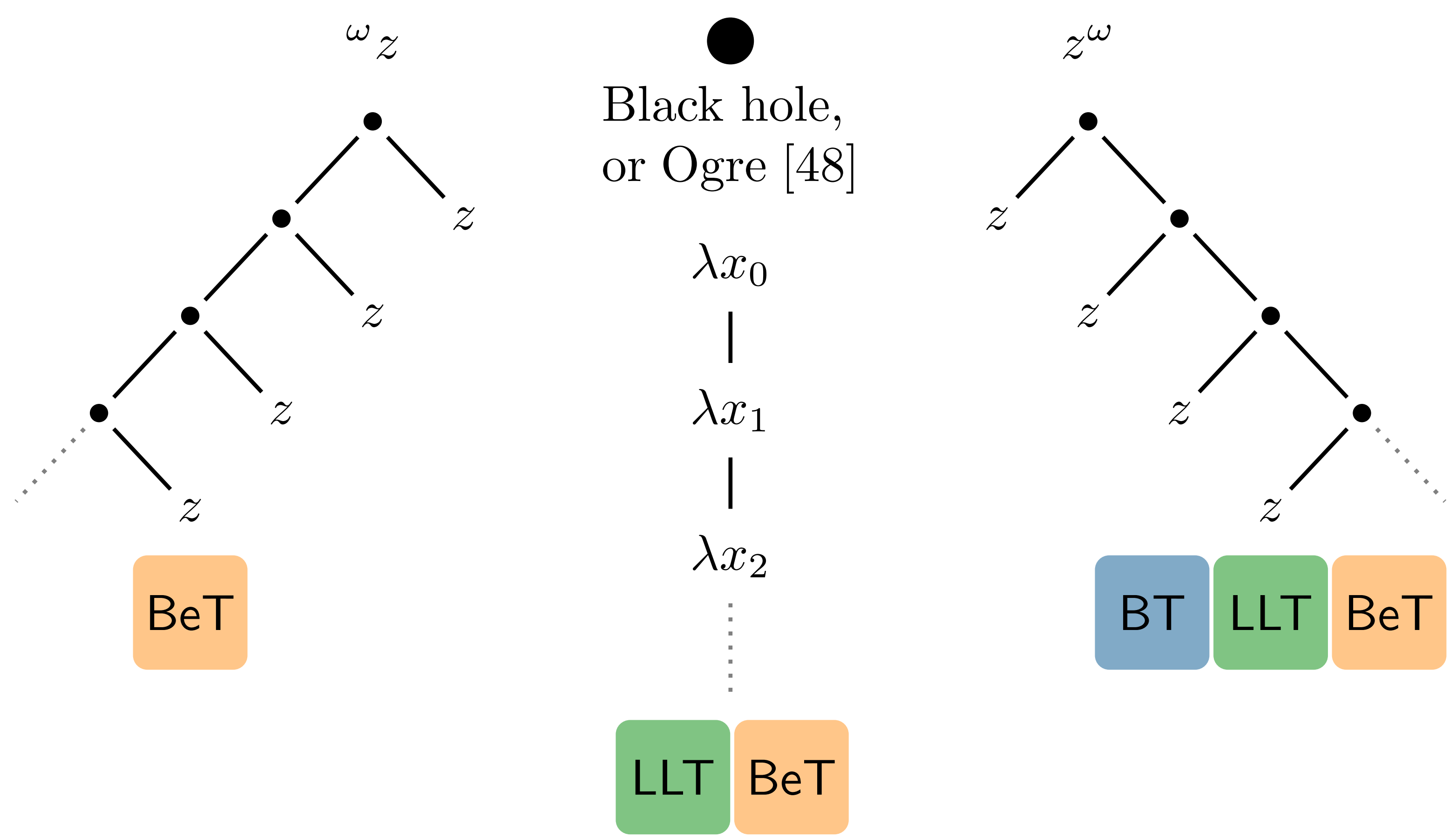
shades of undefinedness

undefined

more undefined

most undefined

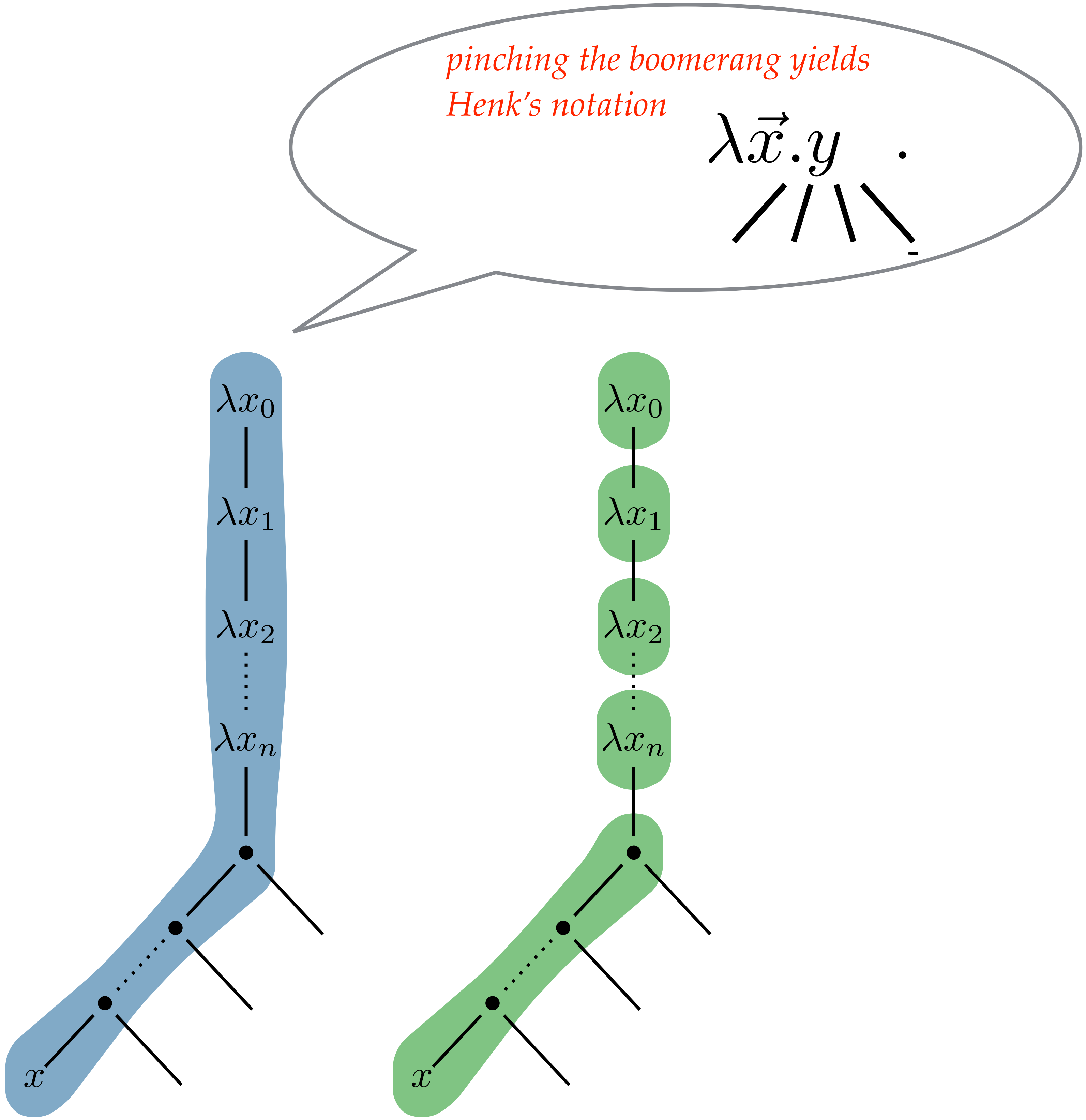




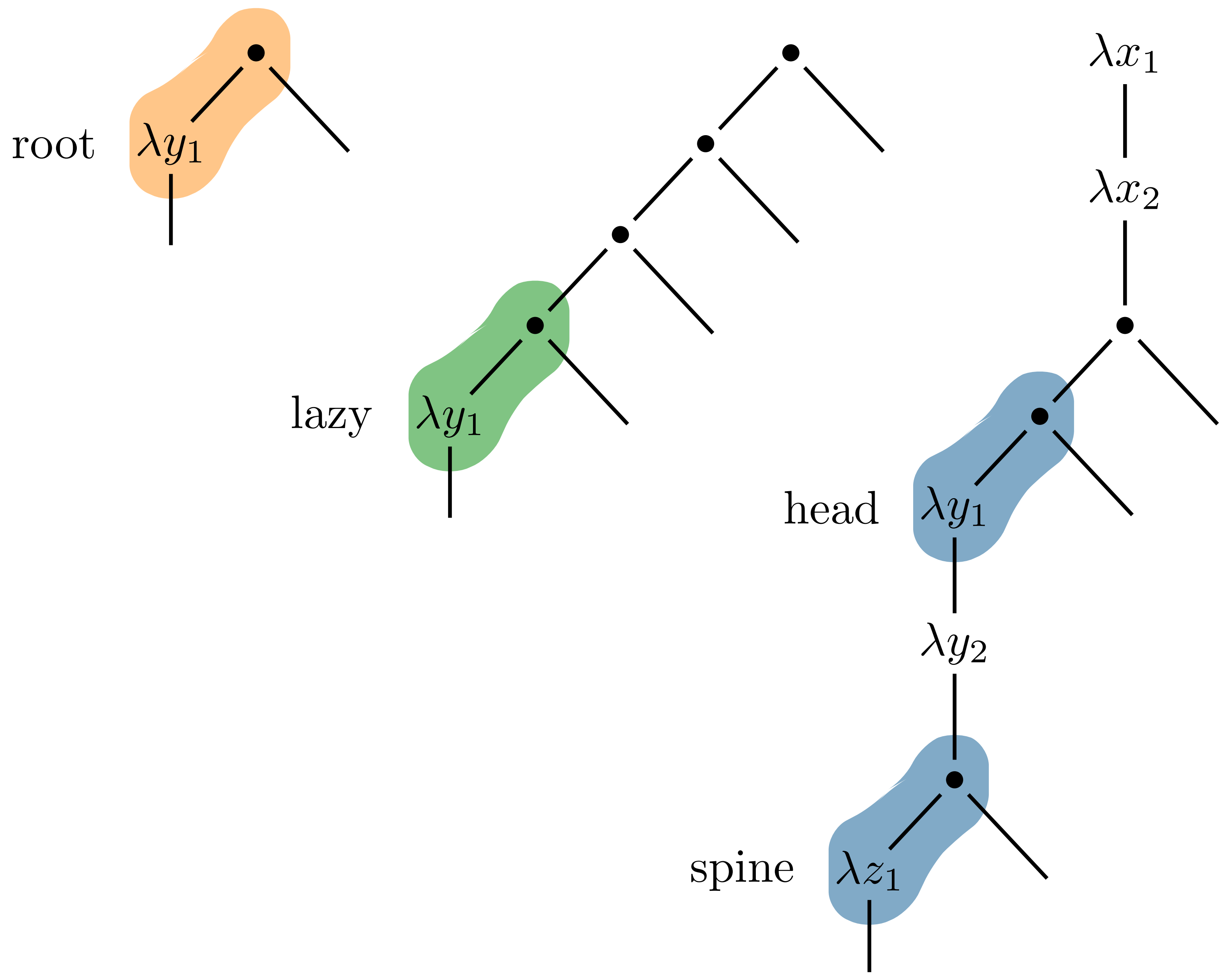
Three infinite λ-terms. The color flags mention to which families of trees they belong.

M	$\mathsf{BT}(M)$	$\mathsf{LLT}(M)$	$\mathsf{BeT}(M)$
$(\lambda x.xx)(\lambda x.xx)$	Ω	Ω	Ω
$(\lambda xy.xx)(\lambda xy.xx)$	Ω	$\bullet$	$\bullet$
$(\lambda x.xxz)(\lambda x.xxz)$	Ω	Ω	${}^{\omega}z$
$(\lambda x.z(xx))(\lambda x.z(xx))$	z^{ω}	z^{ω}	z^{ω}
$\lambda y.((\lambda x.xx)(\lambda x.xx))$	Ω	$\lambda y.\Omega$	$\lambda y.\Omega$
$(\lambda x.xx)(\lambda x.xx)y$	Ω	Ω	Ωy

BT, LLT, BeT-*examples.*



A pair of socks: building blocks for BT*s and* LLT*s.*



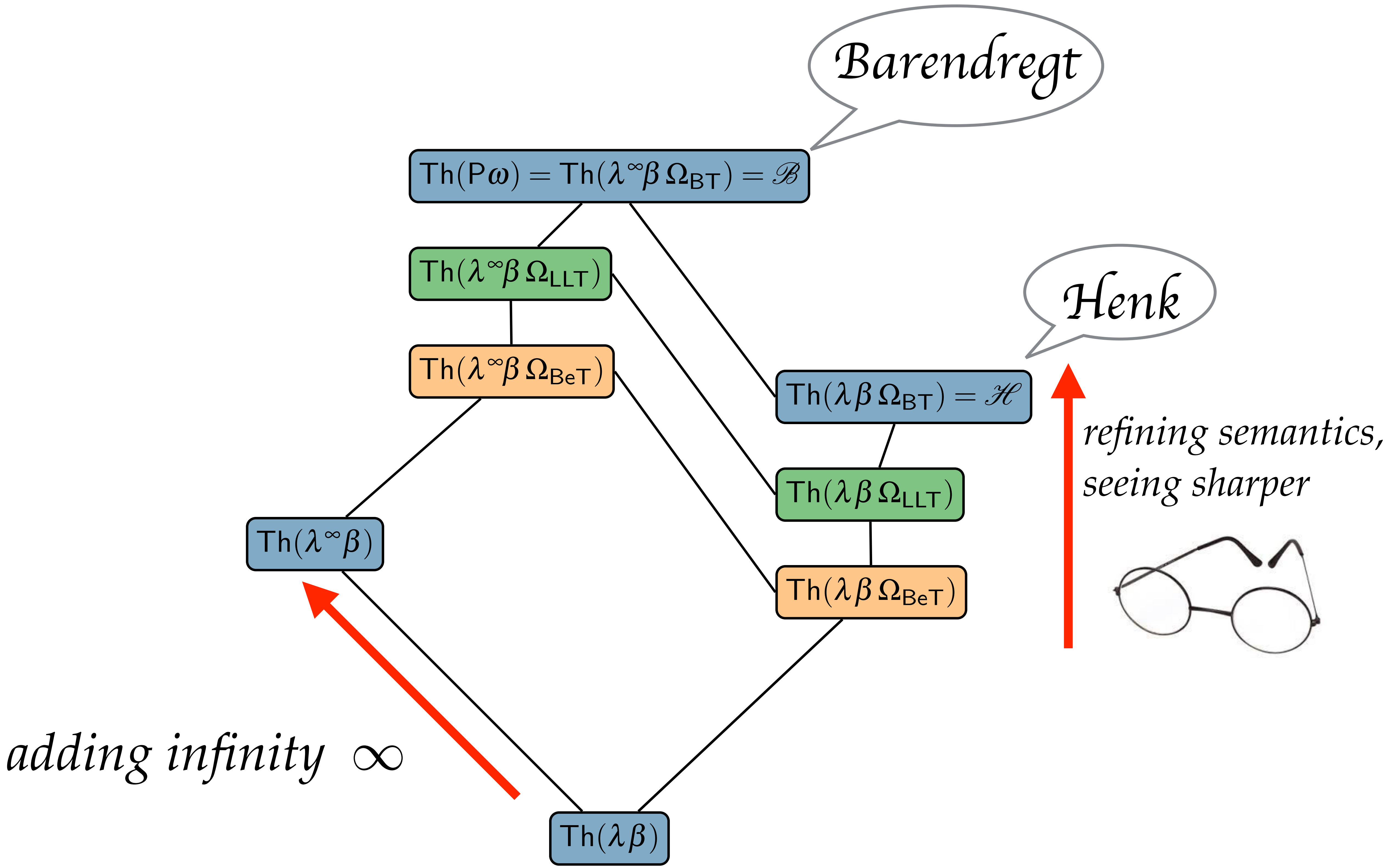
redex is root $\Longrightarrow$ lazy $\Longrightarrow$ head $\Longrightarrow$ spine

The strategic redexes: root, lazy, head and spine.

For BTs and LLTs, the ‘building blocks’ are as depicted in Figure 16. Note that in [2] another notation is used, which may be called the hnf-notation; there a ‘building block’ is obtained by pinching together the boomerang shaped figure of the form d^*l^* ending in a variable. (The left sock in Figure 16.) Then we obtain the building block $\lambda\vec{x}.y$. The building blocks for LLTs are sub-blocks of those for BTs. And in turn, the building blocks for BeTs are even smaller sub-blocks, namely application nodes, abstraction nodes λx, variables, Ω. So the composition or decomposition of the building blocks parallels the refinement order in $\mathsf{BT}(M) \leq_\Omega \mathsf{LLT}(M) \leq_\Omega \mathsf{BeT}(M)$. In a slogan:

The finer the building blocks, the finer the semantics.

Partial order of lambda calculus theories



3. Fixed point combinators

The theory of sage birds (technically called fixed point combinators*) is a fascinating and basic part of combinatory logic; we have only scratched the surface.*

R. Smullyan, To Mock a Mockingbird, 1985.

Alan Turing
invented this
fixed point combinator

$$(\lambda ab.b(aab))(\lambda ab.b(aab))\ M \rightarrow$$

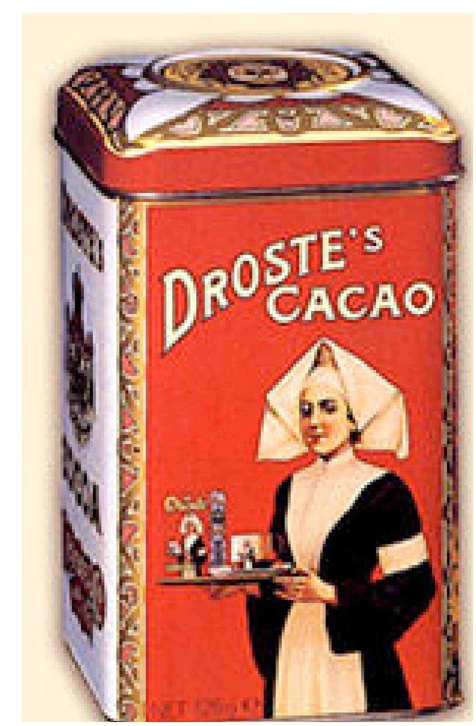
$$(\lambda b.b((\lambda ab.b(aab))(\lambda ab.b(aab))b))M \rightarrow$$

$$M((\lambda ab.b(aab))(\lambda ab.b(aab))M)$$

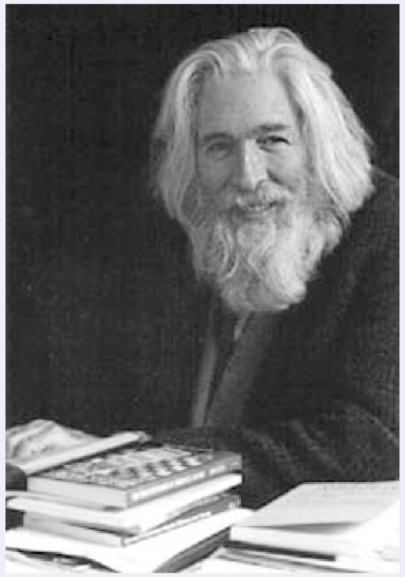
$$\blacksquare \rightarrow \rightarrow M(\ \blacksquare\ )$$

$$\blacksquare\ =\ M(\ \blacksquare\ )$$

$\blacksquare$ *is fixed point van M*



The owl δ



Smullyan: *An extremely interesting bird is the owl*

$$\delta \stackrel{\text{def}}{=} \lambda ab.b(ab) =_\beta \mathbf{SI}$$

- Y is an fpc if and only if $\delta Y =_\beta Y$
- If Y is a reducing fpc, then so is $Y\delta$
- There is no fpc Y such that $Y\delta =_\beta Y$ (Intrigila 1997)

3.0. Curry's and Turing's fixed point combinator

$Y_0 = \lambda f.\ (\lambda x.f(xx))\ (\lambda x.f(xx))$ Curry's fpc

$Y_1 = (\lambda ab.\ b(aab))\ (\lambda ab.\ b(aab))$ Turing's fpc

$\delta = \lambda xy.y(xy) = SI$, Smullyan's Owl

$\Delta = \delta^{\omega} = \delta(\delta(\delta(\delta \ldots = Y\delta$

$Y_0\ \delta = Y_1$

3.1. The Russell paradox

We form the set X of all sets x that are not member of themselves, i.e. $\neg x \ni x$, or simply $\neg(xx)$. So $X = \lambda x.\neg(xx)$. We compute:

$$XX \equiv (\lambda x.\neg(xx))(\lambda x.\neg(xx)) \to \neg(\lambda x.\neg(xx))(\lambda x.\neg(xx)) \equiv \neg(XX).$$

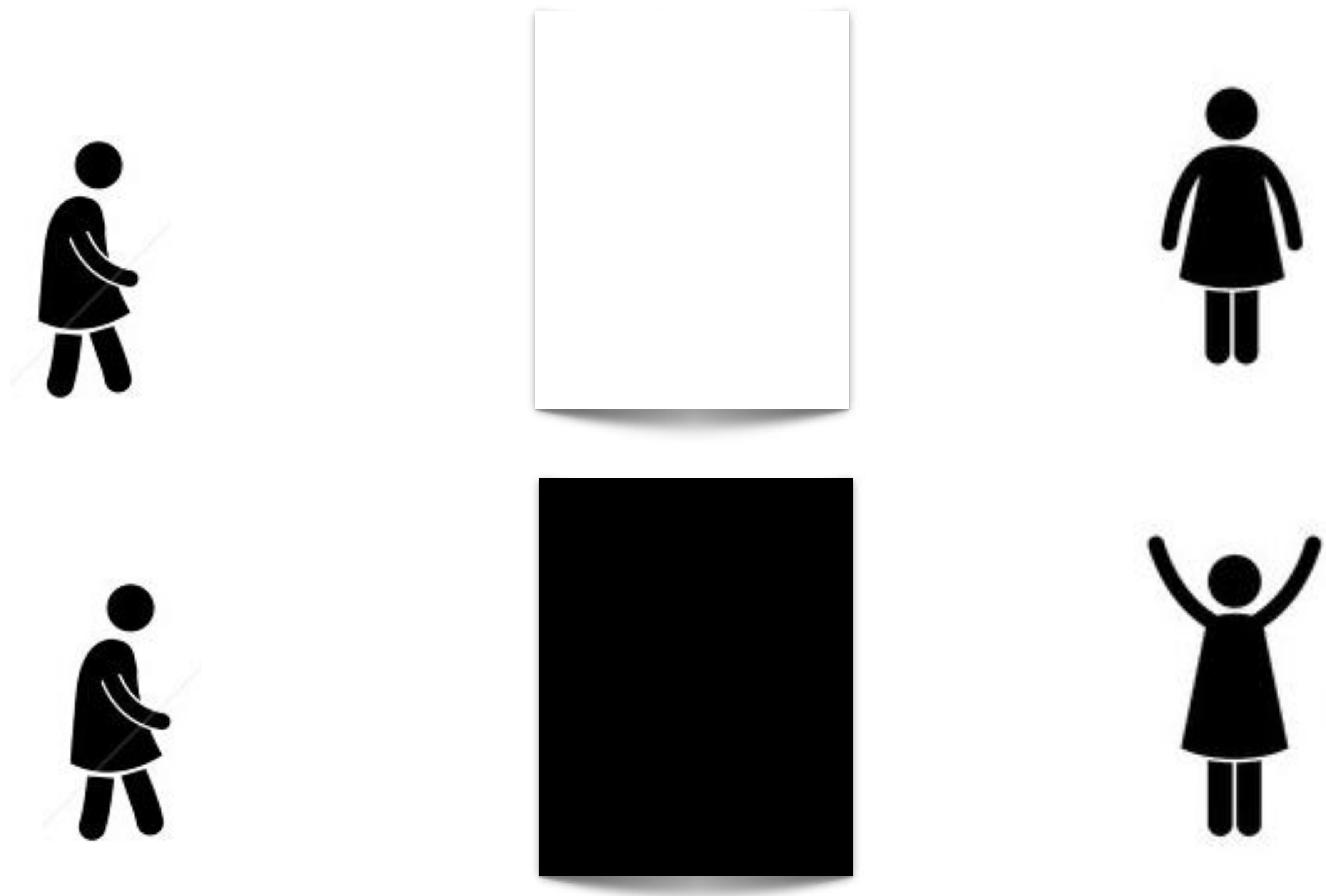
In fact, as we will see later, XX is the fixed point, or rather a fixed point, of negation $\neg$:

$$XX = Y\neg$$

where Y is Curry's fixed point combinator

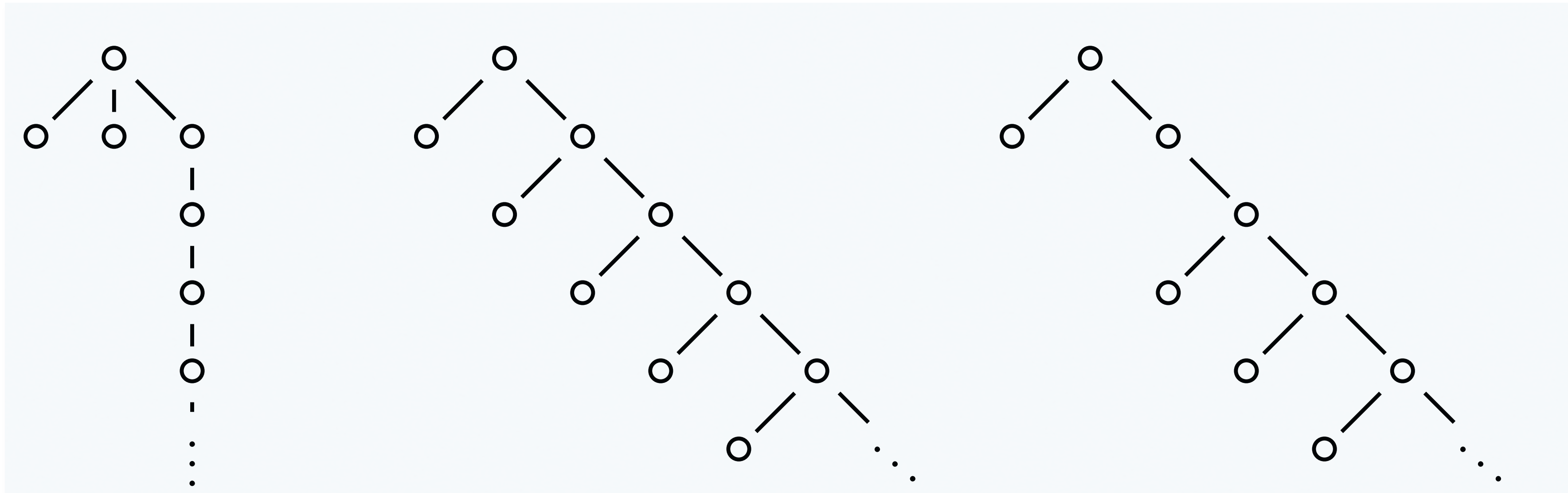
$$\lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

3.2. Henk's Russell-paradox hypnosis experiment (performed in Logic course 1975)



L can see R through the in-between screen (screen is white, transparent) iff L cannot see R raising hands. Screen turns black (non-transparent) iff L sees R with raised hands

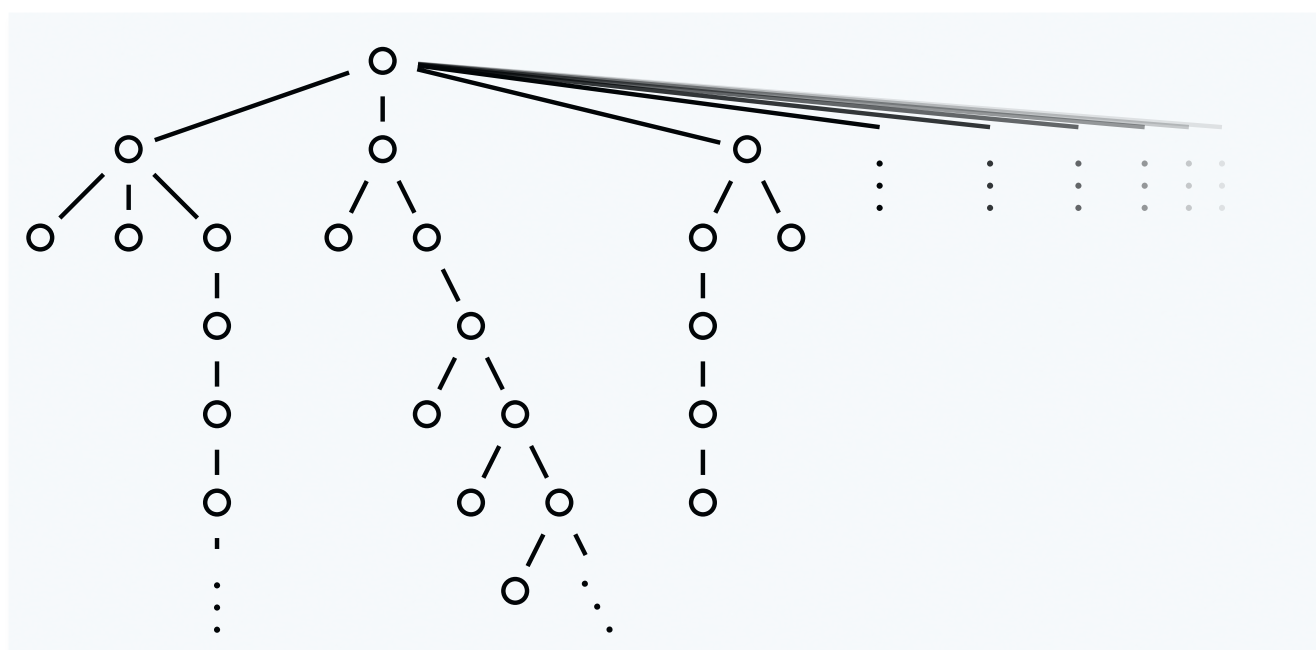
3.3. The Russell paradox by Dick de Bruijn told on a party



Two ordinary trees (left and right) and a special tree (middle)

ordinary: not containing itself as direct subtree

special: not ordinary

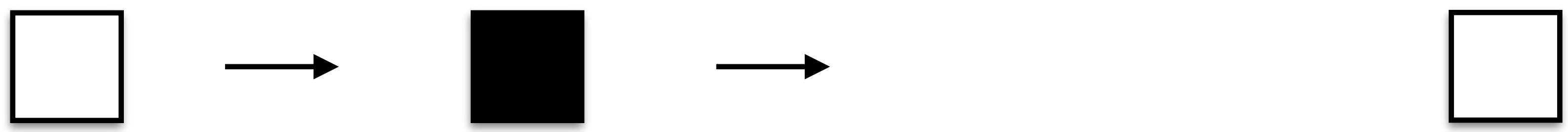


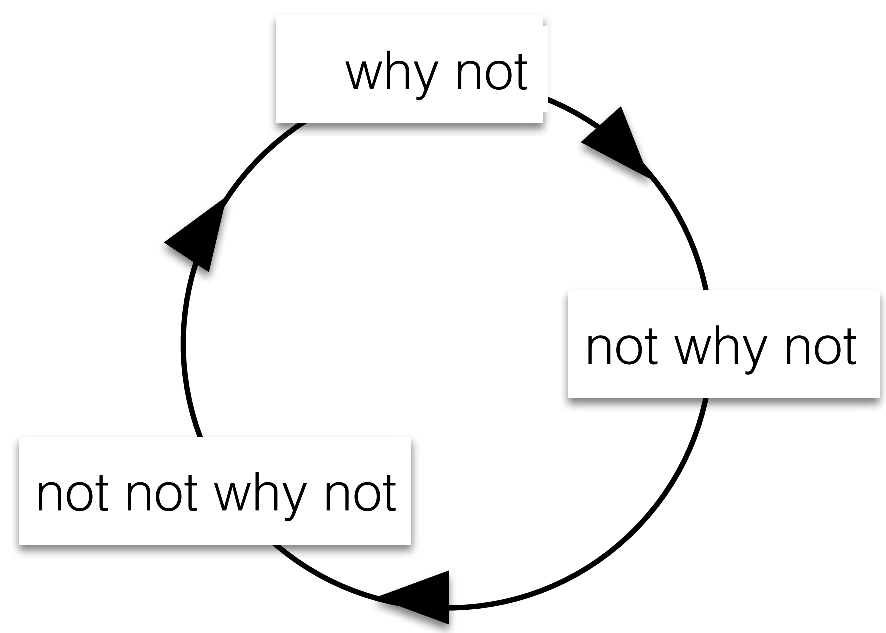
The super-tree with all ordinary trees as direct subtrees

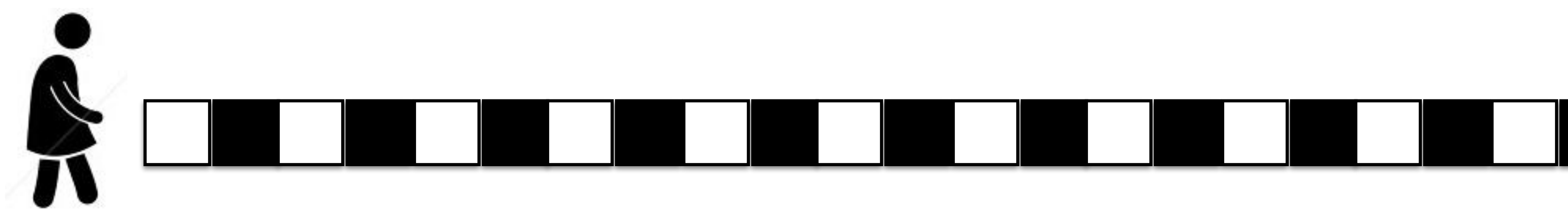
is ordinary iff it is not ordinary

3.4. The fixed point of negation ¬: why not

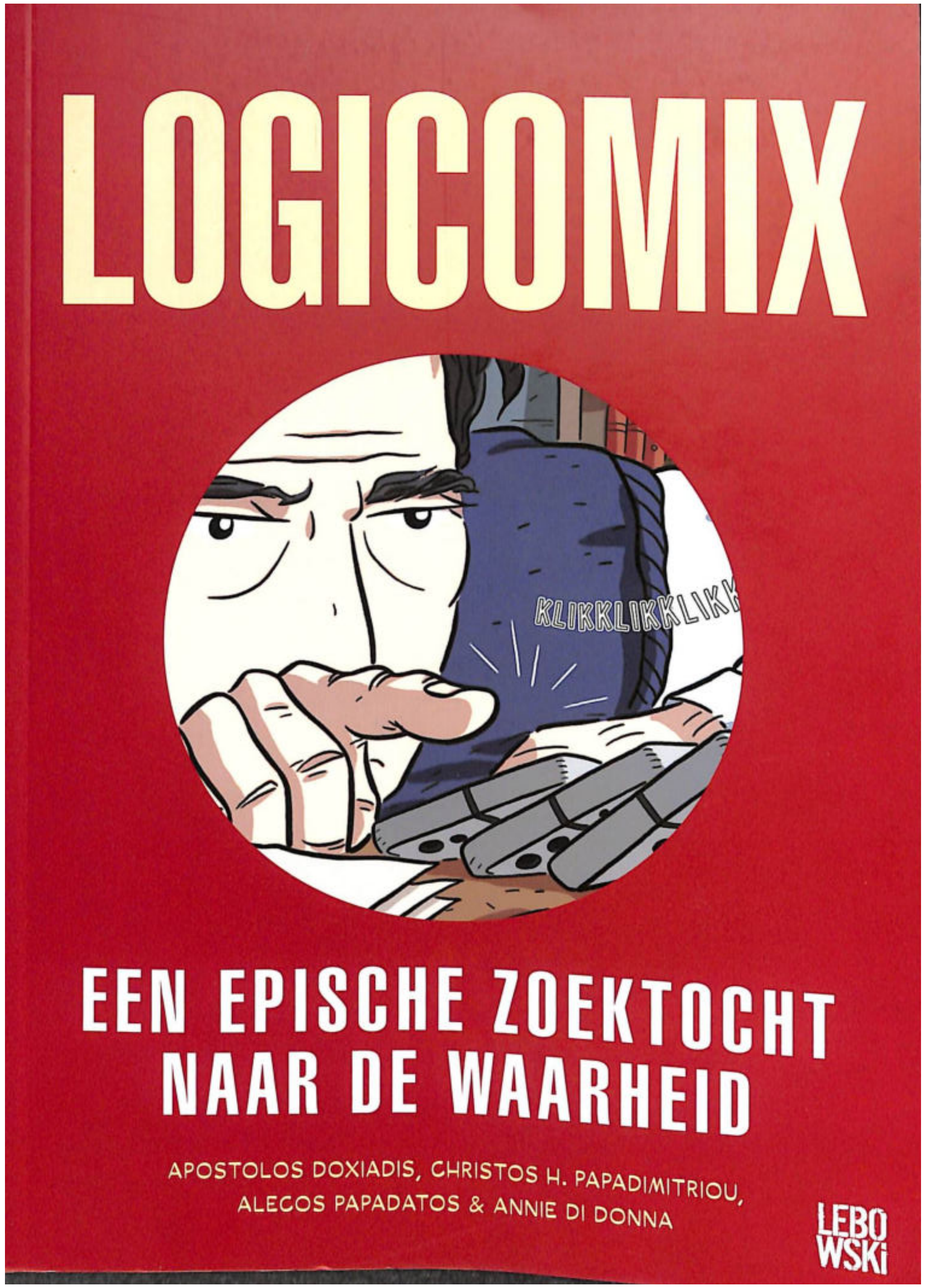
$$Y\neg \longrightarrow \neg(Y\neg) \longrightarrow \neg(\neg(Y\neg)) = Y\neg$$







3.5. The Russell paradox in a comicbook



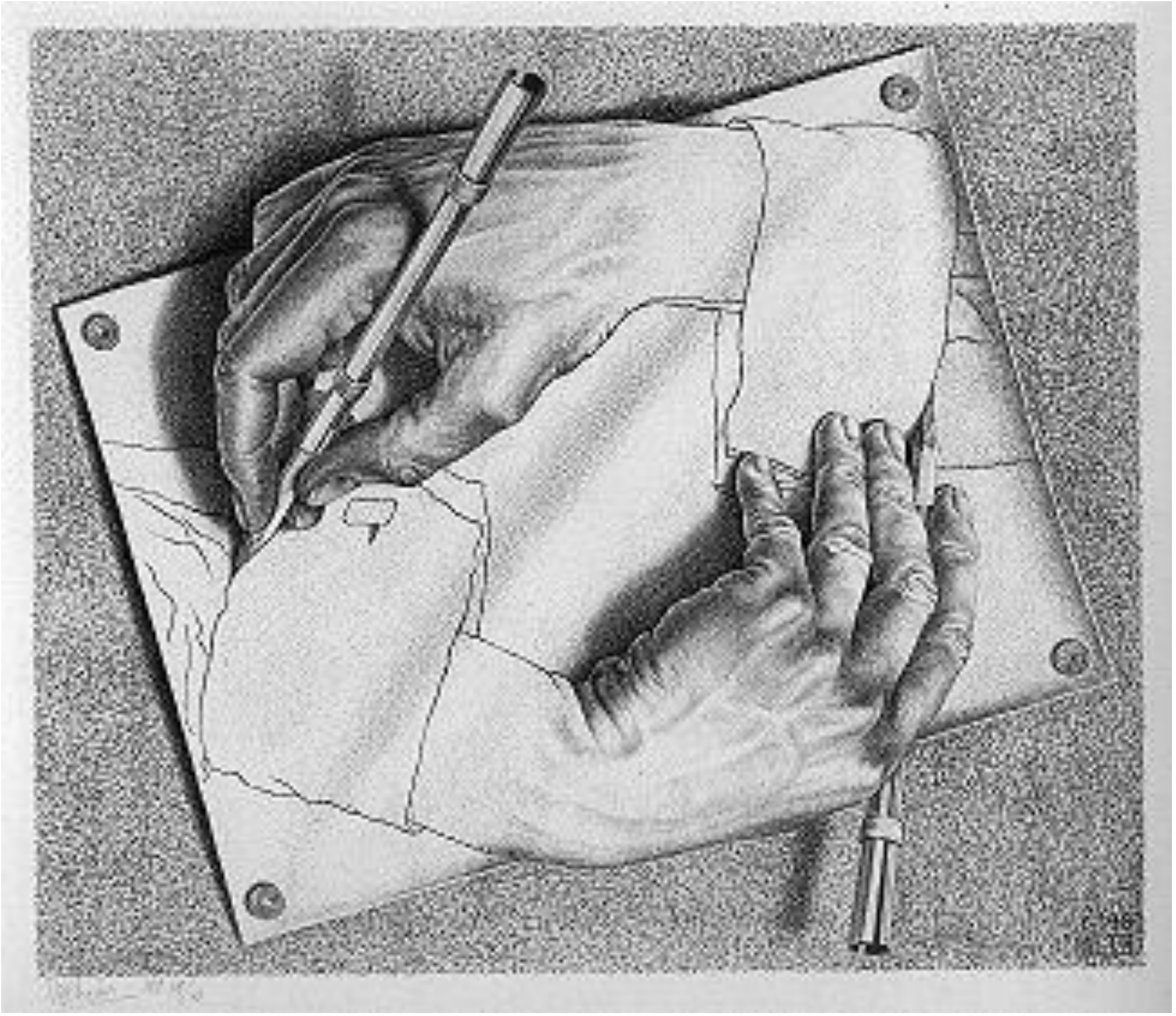
BIJVOORBEELD, EEN VERZAMELING KAN ANDERE VERZAMELINGEN BEVATTEN ... EN ZELFS ZICHZELF!
HOE KAN ZIJ ZICHZELF BEVATTEN?
DE VERZAMELING VAN ALLE IDEEËN IS EEN IDEE...
... DUS BEVAT ZIJ ZICHZELF ALS ELEMENT.
MAAR NIET ALLE VERZAMELINGEN BEVATTEN ZICHZELF?
NEE! DE VERZAMELING VOGELS IS GEEN VOGEL!
HMM... EEN INTERESSANTE TWEEDELING: DE VERZAMELING VERZAMELINGEN DIE ZICHZELF BEVATTEN...
... EN DE VERZAMELING VERZAMELINGEN DIE DAT NIET DOEN.
EN DAARVAN KUNNEN WE ONS AFVRAGEN...
BEVAT DIE –
!?
WACHT EENS EVEN!

3.6. Circularity in alchemy

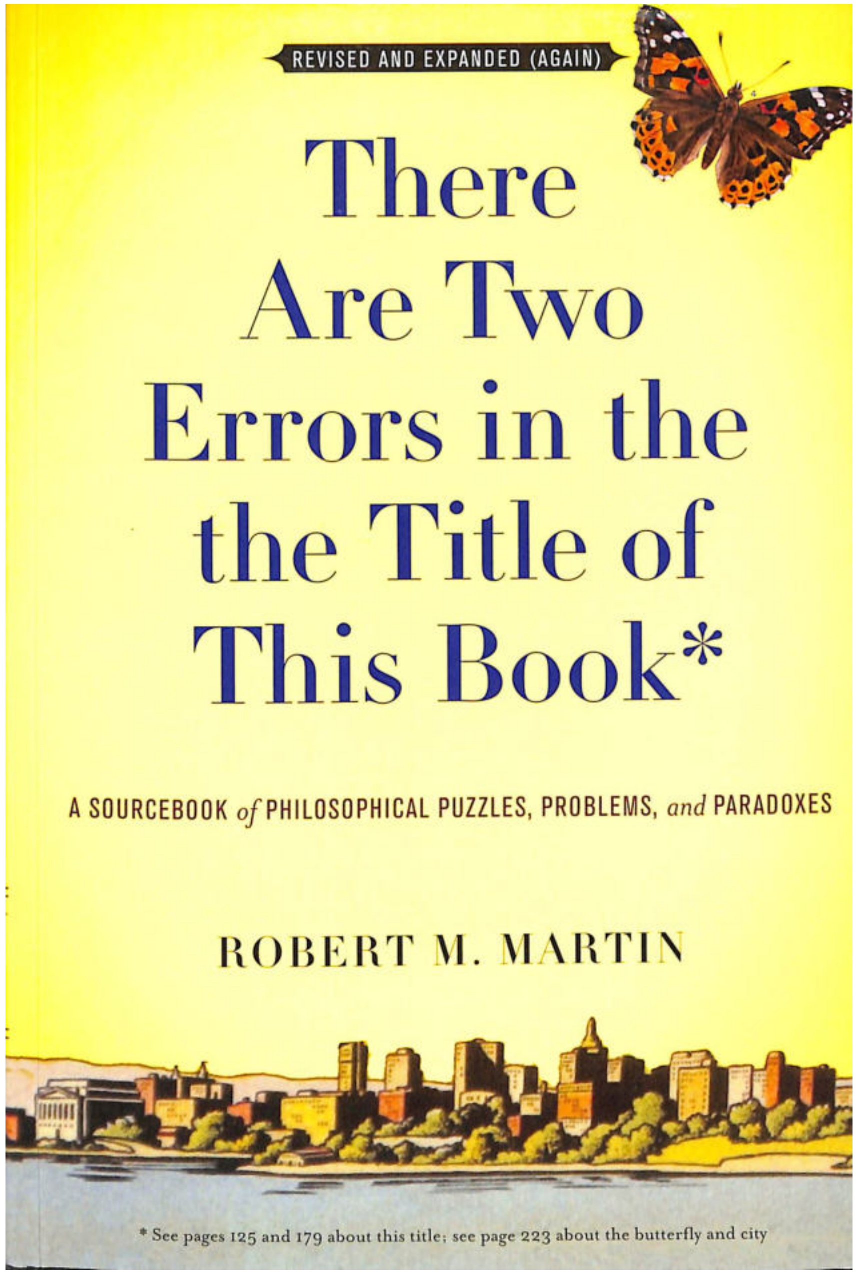
Engraving by Lucas Jennis, in alchemical tract titled *De Lapide Philosophico*.. Around 1600.

3.7. Circularity on the dining table

3.8. Circularity in Escher's art



3.9. Self-referentiality pun in my library



smullyan:
What is the name of this book?

3.10. Self-referentiality in streams (infinite sequences of natural numbers)

Naming sequence:

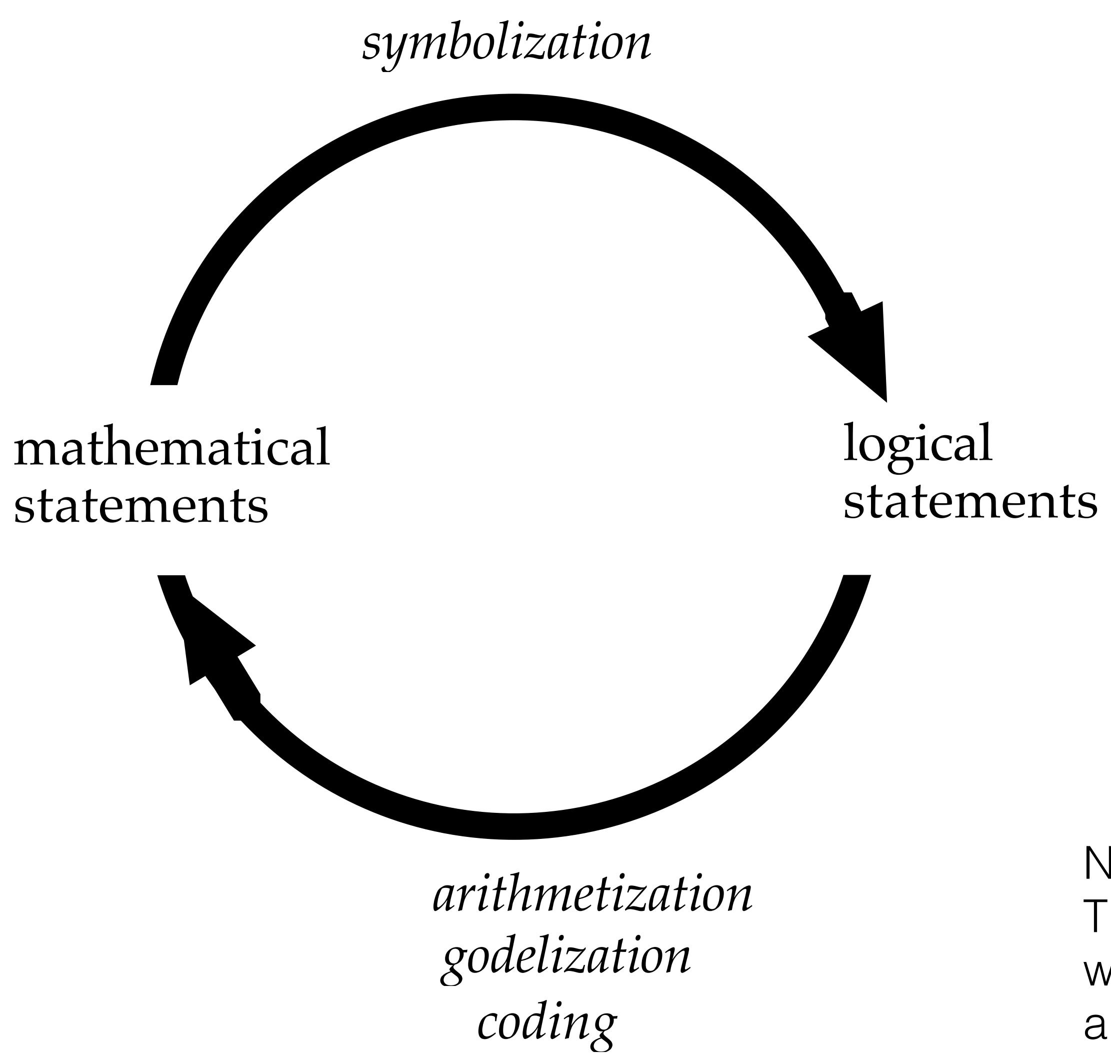
Kolakoski sequence, equal to the sequence of its own run lengths

Thue-Morse sequence; morphic sequences

Toeplitz sequence

self-similarity;
fixed point construction,
fractals

3.9. Circularity, self-referentiality in logic



Noson Yanowsky 2013:
The outer limits of reason - what science, mathematics, and logic *cannot* tell us

Emil Post (1897-1954):

Symbolic Logic may be said to be Mathematics become self-conscious.

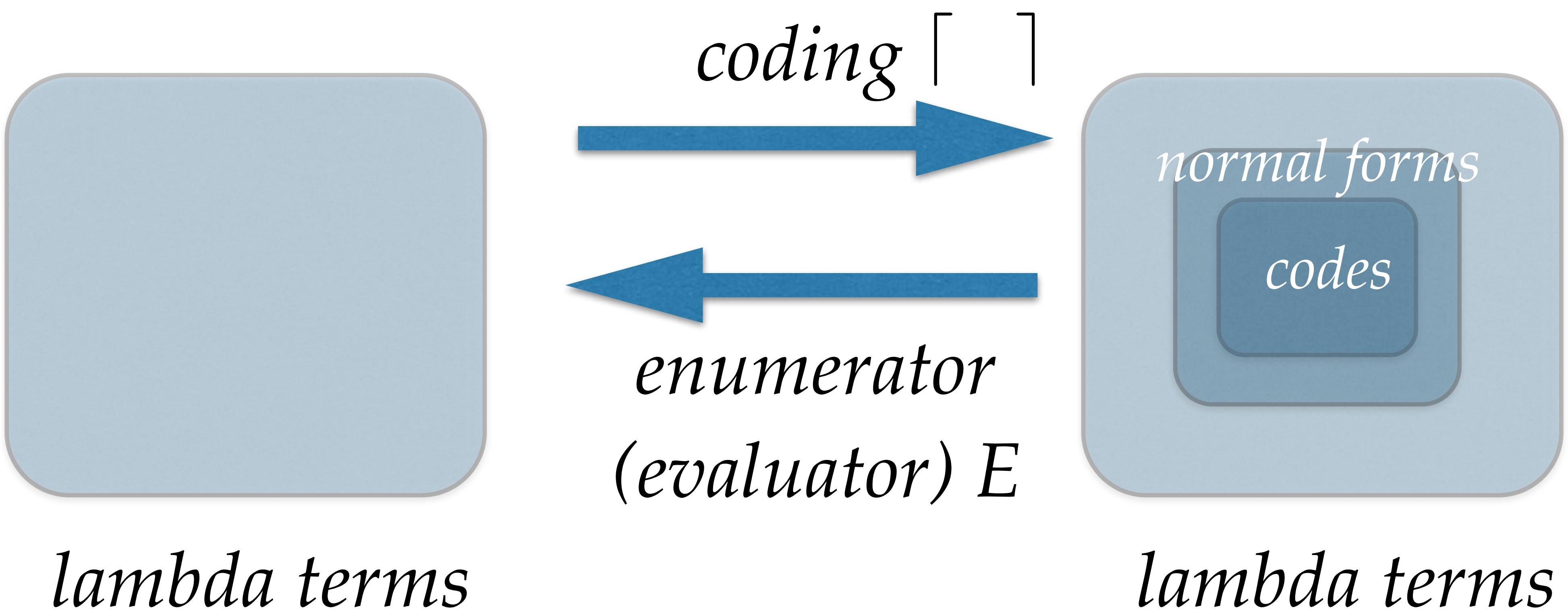
PA, Peano's Arithmetic

Peano's nine axioms, rephrased in contemporary notation, are:

1 1 is a natural number.
2 Every natural number is equal to itself (equality is reflexive).
3 For all natural numbers *a* and *b*, *a*=*b* if and only if *b*=*a* (equality is symmetric).
4 For all natural numbers *a*, *b*, and *c*, if *a*=*b* and *b*=*c* then *a*=*c* (equality is transitive).
5 If *a* = *b* and *b* is a natural number then *a* is a natural number.
6 If *a* is a natural number then S*a* is a natural number.
7 If *a* and *b* are natural numbers then *a*=*b* if and only if S*a* =S*b*.
8 If *a* is a natural number then S*a* is not equal to 1.
9 For every set *K*, if *1* is in *K* and for every natural number *x* in *K*, S*x* is also in *K*, then every natural number is in *K*. (It makes no difference here whether all elements of *K* are natural numbers.)

Gödel's inzicht:
reflectie - het systeem (PA)
over zichzelf laten praten

3.10. Reflection in the lambda calculus: an inner model of the lambda calculus




Stephen Cole Kleene
1909-1994

$$E\lceil M \rceil =_\beta M$$

$$E\lceil M \rceil \twoheadrightarrow_\beta M$$

Conjecture by Barendregt,
proved by Statman for enumerators.
Also for this evaluator?

$$E \equiv \langle\langle K, S, C \rangle\rangle$$

$$K \equiv \lambda ab.a$$
$$S \equiv \lambda abc.ac(bc)$$
$$C \equiv \lambda abc.acb$$

DEFINITION. (Mogensen) Define for a lambda term M its code $\ulcorner M \urcorner$ as follows.

$$\begin{array}{rcl c l} \ulcorner x \urcorner & \equiv & \lambda e.eU^3_1 xe & = & F_L x; \\ \ulcorner MN \urcorner & \equiv & \lambda e.eU^3_2 \ulcorner M \urcorner \ulcorner N \urcorner e & = & F_P \ulcorner M \urcorner \ulcorner N \urcorner; \\ \ulcorner \lambda x.M \urcorner & \equiv & \lambda e.eU^3_3 (\lambda x. \ulcorner M \urcorner) e & = & F_! (\lambda x. \ulcorner M \urcorner). \end{array}$$

The trick here is to code the lambda with lambda itself, one may speak of an inner model of the lambda calculus in itself. Putting the ideas of Mogensen [1992] and Böhm et al. [1994] together, as done by Berarducci and Böhm [1993], one obtains a very smooth way to create the mechanism of reflection in the lambda calculus. The result was already proved in Kleene [1936][3].

THEOREM. *There is a lambda term* E *(evaluator or self-interpreter) such that*

$$\begin{array}{rcl} \mathsf{E} \ulcorner x \urcorner & = & x; \\ \mathsf{E} \ulcorner MN \urcorner & = & \mathsf{E} \ulcorner M \urcorner (\mathsf{E} \ulcorner N \urcorner); \\ \mathsf{E} \ulcorner \lambda x.M \urcorner & = & \lambda x.(\mathsf{E} \ulcorner M \urcorner). \end{array}$$

It follows that for all lambda terms M *one has*

$$\mathsf{E} \ulcorner M \urcorner = M.$$

COROLLARY. *The term* $\langle\langle \mathsf{K}, \mathsf{S}, \mathsf{C} \rangle\rangle$ *is a self-interpreter for the lambda calculus with the coding as above.*

3.11. Lego blocks for fixed point combinators

$Y \Rightarrow Y\delta \quad (\delta = SI)$ *(Böhm)*

$Y \Rightarrow Y(SS)S\sim^{n}I$ *(Scott)*

$Y \Rightarrow Y(AAA)A\sim^{n}II \quad (A = BS)$

$Y_0\ \delta\ (SS)SI\ \delta\ (AAA)AAAAII\ (SS)SI$ *is a fpc*

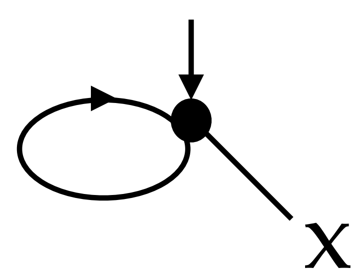
Are all these composite fpc's really different?

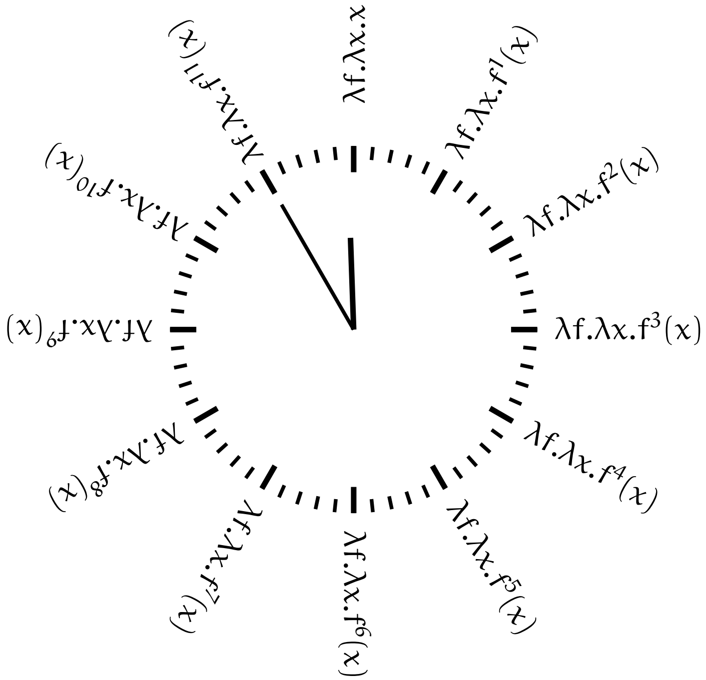
3.12. *Weak fixed point combinators, aka looping combinators*

An example of a weak fpc is the term $\mathsf{A}(\mathsf{BAB})$ where $\mathsf{A} \equiv \mathsf{BM}$ and $\mathsf{M} \equiv \lambda x.xx$. This example was found by Statman, in his study of terms composed only of symbols B and M. Here the generator changes in each 'production cycle'. We have the following reduction:

$$
\begin{array}{ll}
 & \mathsf{A}(\mathsf{BAB})x \\
\to^3 & \mathsf{M}(\mathsf{BAB}x) \\
\to & \mathsf{BAB}x(\mathsf{BAB}x) \\
\to^3 & \mathsf{A}(\mathsf{B}x)(\mathsf{BAB}x) \\
\to^3 & \mathsf{M}(\mathsf{B}x(\mathsf{BAB}x)) \\
\to & \mathsf{B}x(\mathsf{BAB}x)(\mathsf{B}x(\mathsf{BAB}x)) \\
\to^3 & x(\mathsf{BAB}x(\mathsf{B}x(\mathsf{BAB}x))) \\
\to^3 & x(\mathsf{A}(\mathsf{B}x)(\mathsf{B}x(\mathsf{BAB}x))) \\
\to^3 & x(\mathsf{M}(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x)))) \\
\to & x(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x))(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x)))) \\
\to^3 & x(x(\mathsf{B}x(\mathsf{BAB}x)(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x))))) \\
\to^3 & x(x(x(\mathsf{BAB}x(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x)))))) \\
\to^3 & x(x(x(\mathsf{A}(\mathsf{B}x)(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x)))))) \\
\to^3 & x(x(x(\mathsf{M}(\mathsf{B}x(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x))))))) \\
\to & x(x(x(\mathsf{B}x(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x)))(\mathsf{B}x(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x))))))) \\
\to^3 & x(x(x(x(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x))(\mathsf{B}x(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x)))))))) \\
\to^3 & x(x(x(x(x(\mathsf{B}x(\mathsf{BAB}x)(\mathsf{B}x(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x))))))))) \\
\to^3 & x(x(x(x(x(x(\mathsf{BAB}x(\mathsf{B}x(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x)))))))))) \\
\to^3 & x(x(x(x(x(x(\mathsf{A}(\mathsf{B}x)(\mathsf{B}x(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x)))))))))) \\
\to^3 & x(x(x(x(x(x(\mathsf{M}(\mathsf{B}x(\mathsf{B}x(\mathsf{B}x(\mathsf{B}x(\mathsf{BAB}x)))))))))))
\end{array}
$$

in functional languages the Y-combinator is represented by a term graph with a cycle: self-reference!

$Yx =$ 



4. Clocks in the lambda calculus

The evaluation of lambda terms M to their infinite expansion BT(M) turns out to possess an underlying clock mechanism, a rhythm, that can be used to discriminate lambda terms wrt beta-conversion.

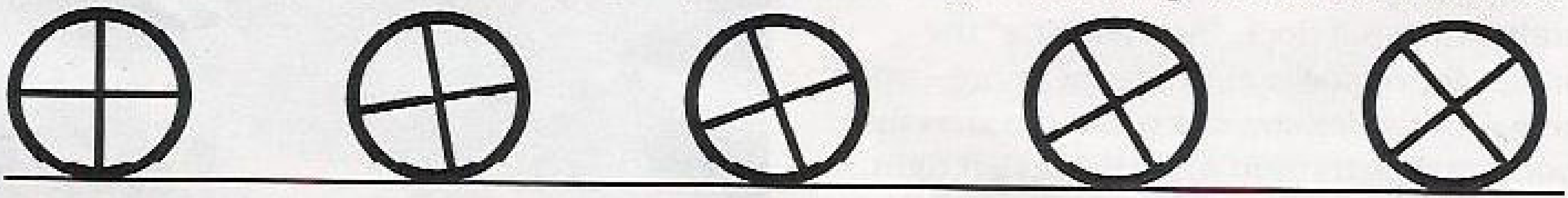
But first we pay attention to the curious phenomenon of a clock mechanism in human consciousness.

4.1. The stream of consciousness …

is not a continuous stream but has a rhythm, a heart beat, a clock

The wagon wheel illusion

If these frames were played in succession, which way would the wheel appear to be rolling – clockwise or anticlockwise?



No camera required

In a movie this wheel would appear to be moving anticlockwise, when in fact it is rolling clockwise

Each frame captured by the camera shows the wheel after just under a quarter of a revolution. The brain assumes the wheel has moved the smaller distance – a slight angle anticlockwise with each frame, rather than the bigger rotation clockwise

The illusion, often seen in old westerns can be created in real life too, with no camera present, suggesting that the brain creates the perception of continuity from a series of discrete frames, just like a film reel

whether you are in control of your own body," says William Hetrick, who studies the brain's timekeeping and schizophrenia at Indiana University in Bloomington. "The ability to attribute actions to oneself versus others, to perceive one's own thoughts against thoughts generated from external sources, perhaps requires a tight coupling in time [within the brain]."

The idea could explain many of the experiences reported by people with schizophrenia. By muddling the order of thoughts and perceptions within your brain, for example, you might move your hand before you are conscious of the decision, making it feel as if someone else is controlling your movements. And when an advert appears on TV, your brain might picture the product before it consciously registers seeing it on screen – creating the disturbing illusion that your thoughts are being broadcast on television.

"By upsetting the brain's clock, you can recreate some of the delusions seen in schizophrenia"

If poor time-processing really does underlie many psychotic delusions, it could point to a single culprit in the brain: the cerebellum. For decades, the cerebellum has been seen as a centre for timing the movement of muscles, but some neuroscientists now reckon that it might coordinate thoughts and the processing of sensory perceptions too.

That would fit with the neurological evidence. "During a broad range of mental tasks, people with schizophrenia have lower rates of cerebellar blood flow than healthy people do," says Nancy Andreasen, a schizophrenia researcher at the University of Iowa in Iowa City.

The idea has sparked plenty of interest. David Eagleman at Baylor College of Medicine in Houston, Texas, has studied people with schizophrenia using a video game similar to the aircraft game, which lets him manipulate delays between volunteers' actions and their outcomes.

When he alters time delays, people with schizophrenia find it more difficult to compensate than healthy controls. "Schizophrenic brains seem to be temporally inflexible," he says. "They don't recalibrate." Eagleman hopes such games might be useful in the future to measure the severity of schizophrenia, or patients' responses to treatment and drugs.

2014

Discrete states of consciousness

An inquiry into the dynamics of conscious cognition based on neurophysiological evidence.

Name : Anthony Mahabir
1st supervisor : Henk Barendregt
2nd supervisor : Menno Lievers

Credits : 60 ECTS
Date : 19-09-2014

clock behaviour of fpc in Böhm sequence of fpc's

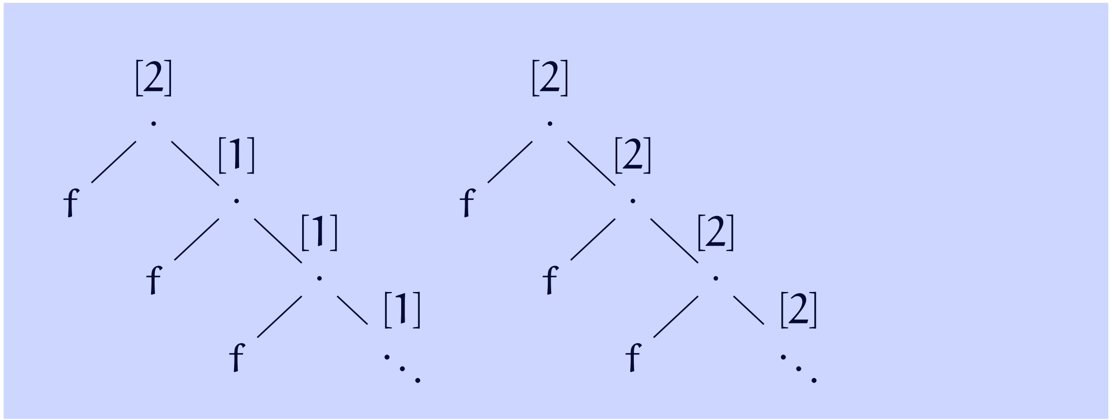
$Y_0,\ Y_0\,\delta,\ Y_0\,\delta\delta,\ Y_0\,\delta\delta\delta,\ Y_0\,\delta\delta\delta\delta,\ \ldots$

$$\mathsf{Y}_3 \equiv \mathsf{Y}_0\delta\delta\delta \ \xrightarrow[h]{7}\ \lambda a.a(\omega_\delta\omega_\delta\delta\delta a)$$

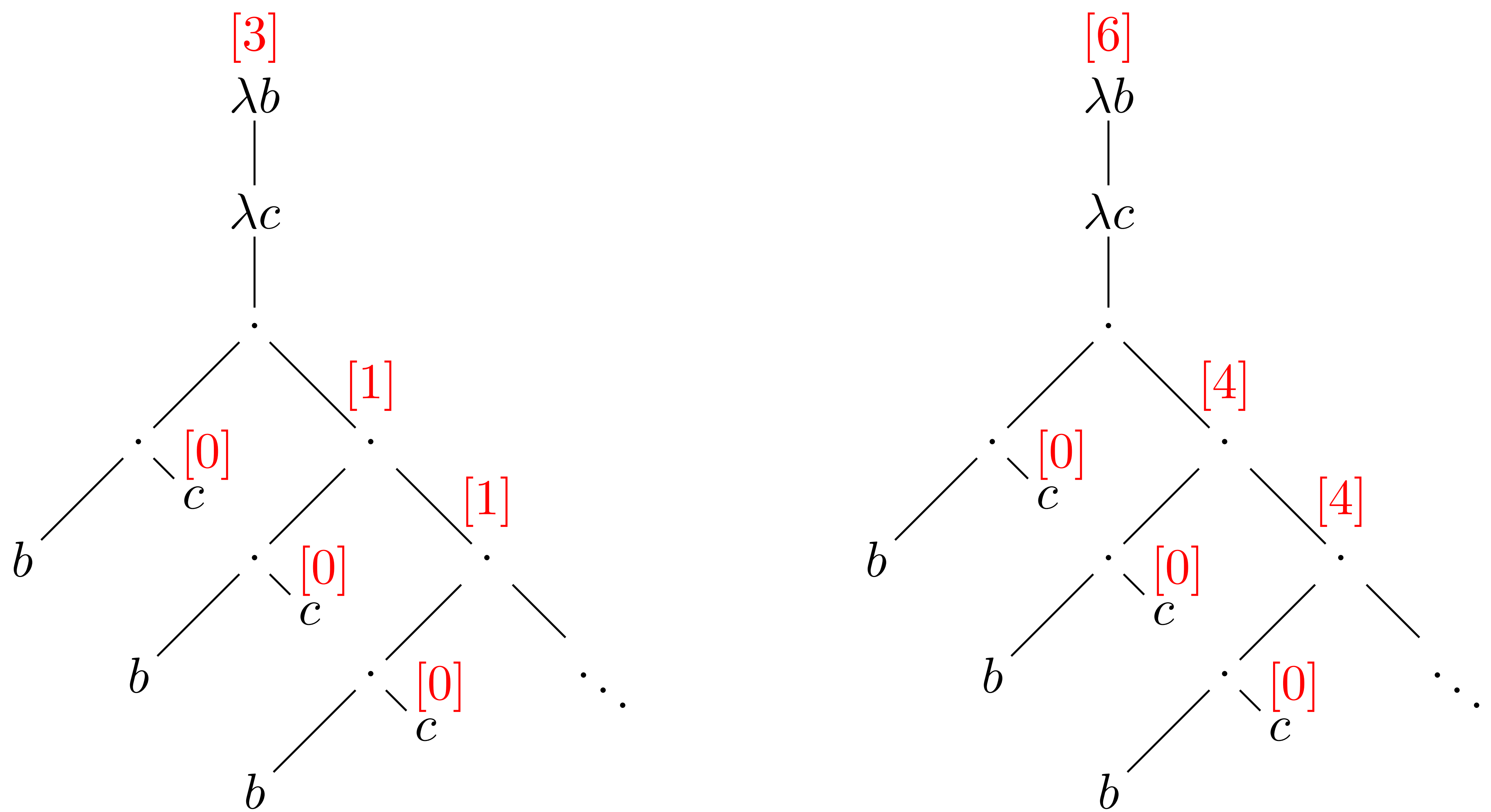
$$\downarrow\ \lambda a.a\Box$$

$$\omega_\delta\omega_\delta\delta\delta a \ \xrightarrow[h]{7}\ a(\omega_\delta\omega_\delta\delta\delta a)$$

$$a\Box$$



Clocked BT's of $\mathrm{Y_0f}$ *and* $\mathrm{Y_1f}$



Clocked Böhm trees of $\mathsf{BY_0}$ *and* $\mathsf{BY_0S}$.

Atomic clocks

Record not only the number steps, but also the positions of the steps.

Let $M \in \Lambda$. The atomic clock Böhm tree $\mathrm{BT}_{☢}(M)$ of M is defined by $\mathrm{BT}_{☢}(M) = \bot$ if M has no hnf. Otherwise, there is a head reduction

$$M \rightarrow_{hd,p_1} \cdots \rightarrow_{hd,p_k} \lambda x_1 \ldots . \lambda x_n . y N_1 \ldots N_m$$

and then put $\mathrm{BT}_{☢}(M) = [\langle p_1, \ldots, p_k \rangle] \lambda \vec{x}. y \mathrm{BT}_{☢}(N_1) \ldots \mathrm{BT}_{☢}(N_m)$.

The theory for clocked BT's generalizes to atomic trees: Here we use the subsequence relation for comparing annotations (= lists of positions):

$$\langle a_1, \ldots, a_n \rangle \leq \langle b_1, \ldots, b_m \rangle \overset{\text{def}}{\iff} \langle a_1, \ldots, a_n \rangle = \langle b_{i_1}, \ldots, b_{i_n} \rangle$$

for some indices $i_1 < i_2 < \ldots < i_n$.

Example: $\mathbf{Y}_2 \neq_\beta \mathbf{U}_2$.

Lévy–Longo and Berarducci trees

- The theory easily generalizes to Lévy–Longo and Berarducci trees.
- LLT-semantics is based on head reduction to weak hnf (whnf = abstraction, or application with leading var).
- Clocked LLT's can distinguish e.g. $PP \neq_\beta QQ$ where

$$P = \lambda xy.xx \qquad\qquad Q = \lambda xyz.xx$$

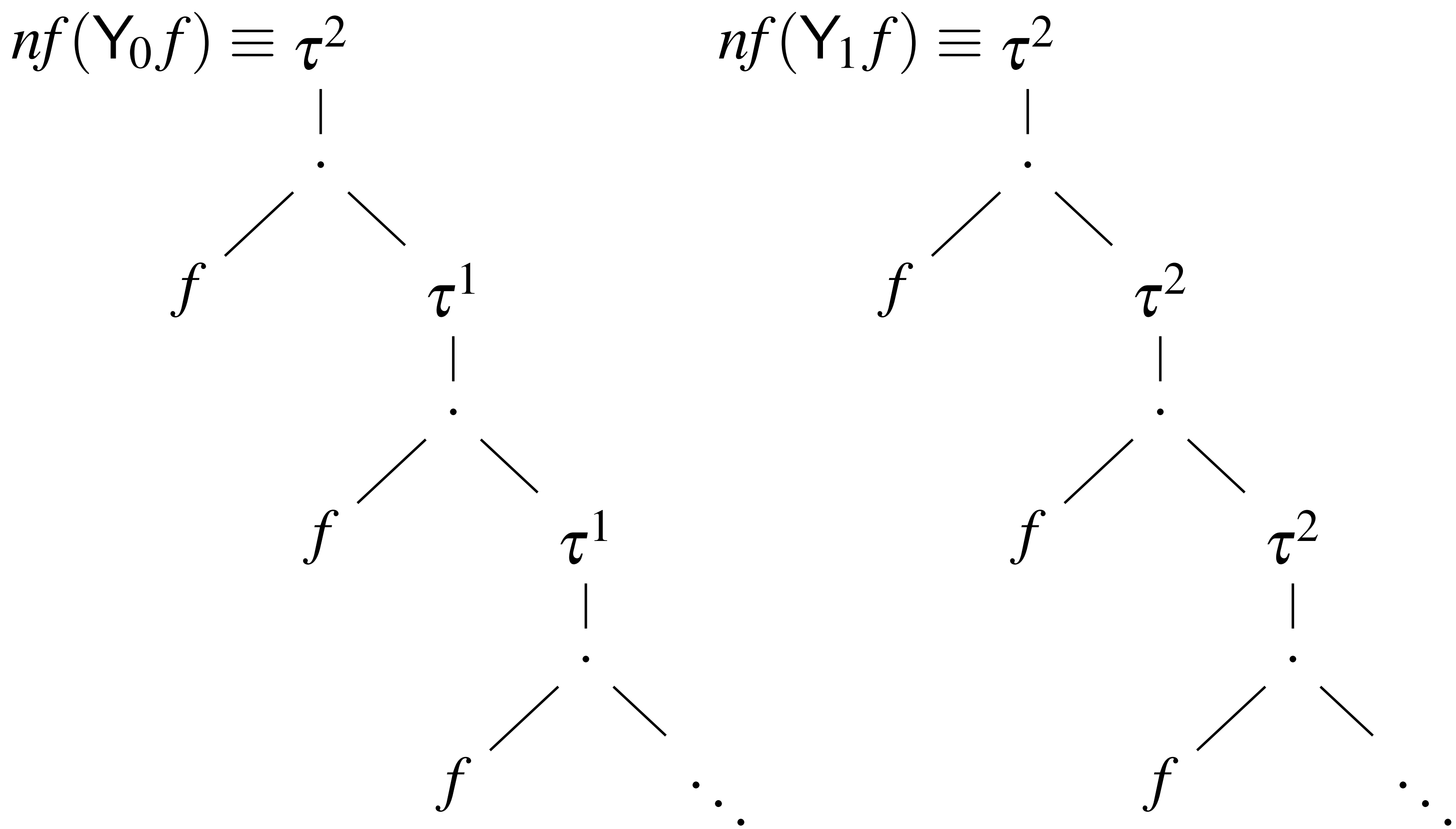
Clocked Lambda Calculus

$$(\lambda x.M)N \to \tau(M[x:=N])$$
$$\tau(M)N \to \tau(MN)$$

The τ's are ticks of the clock (measure of efficiency).

Properties: orthogonal, SN^∞, CR^∞, UN^∞

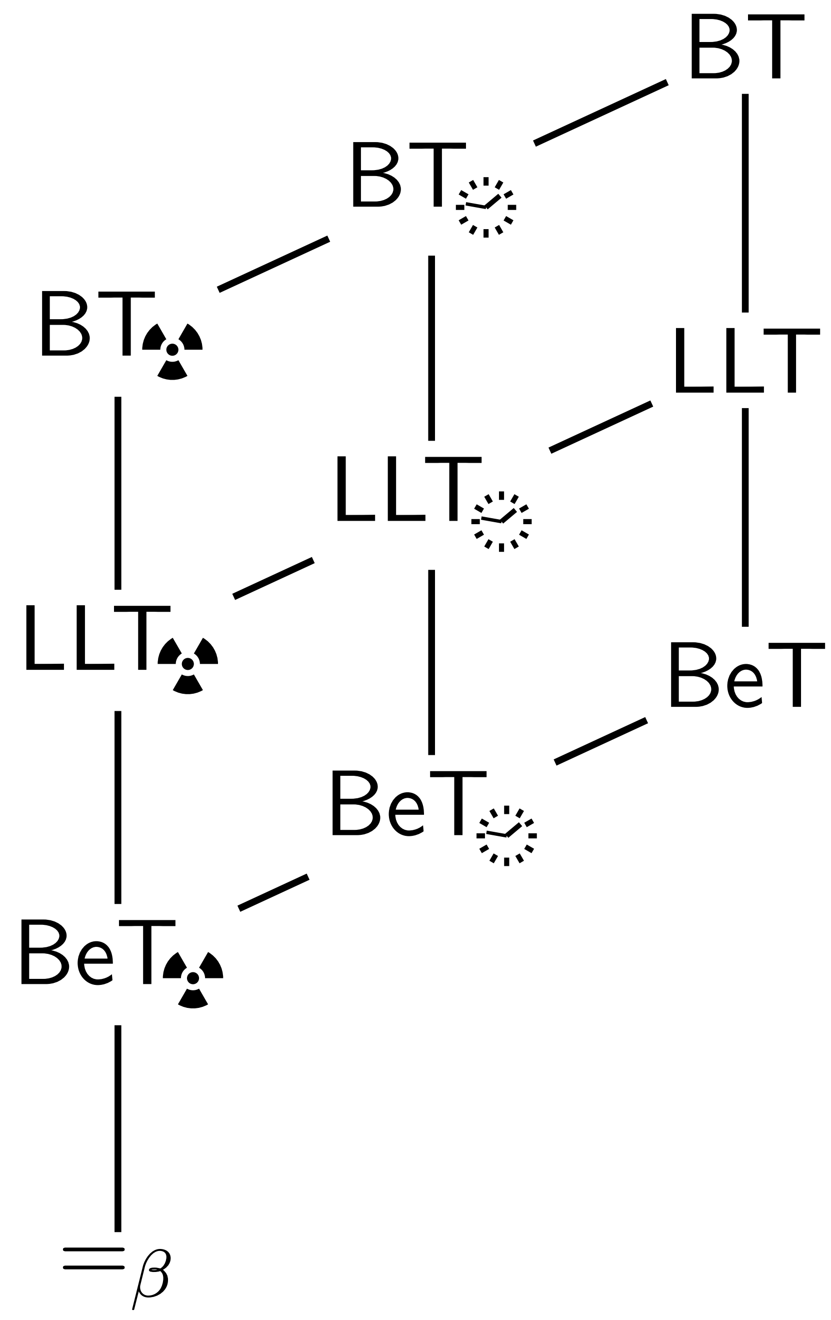
Normal forms are clocked Lévy–Longo trees:



different clock $\Rightarrow \mathsf{Y}_0 \neq \mathsf{Y}_1$

Making lambda calculus see sharper:

we find a spectrum of intermediate equalities between β-equality and BT-equality, each a discrimination tool for β-equality

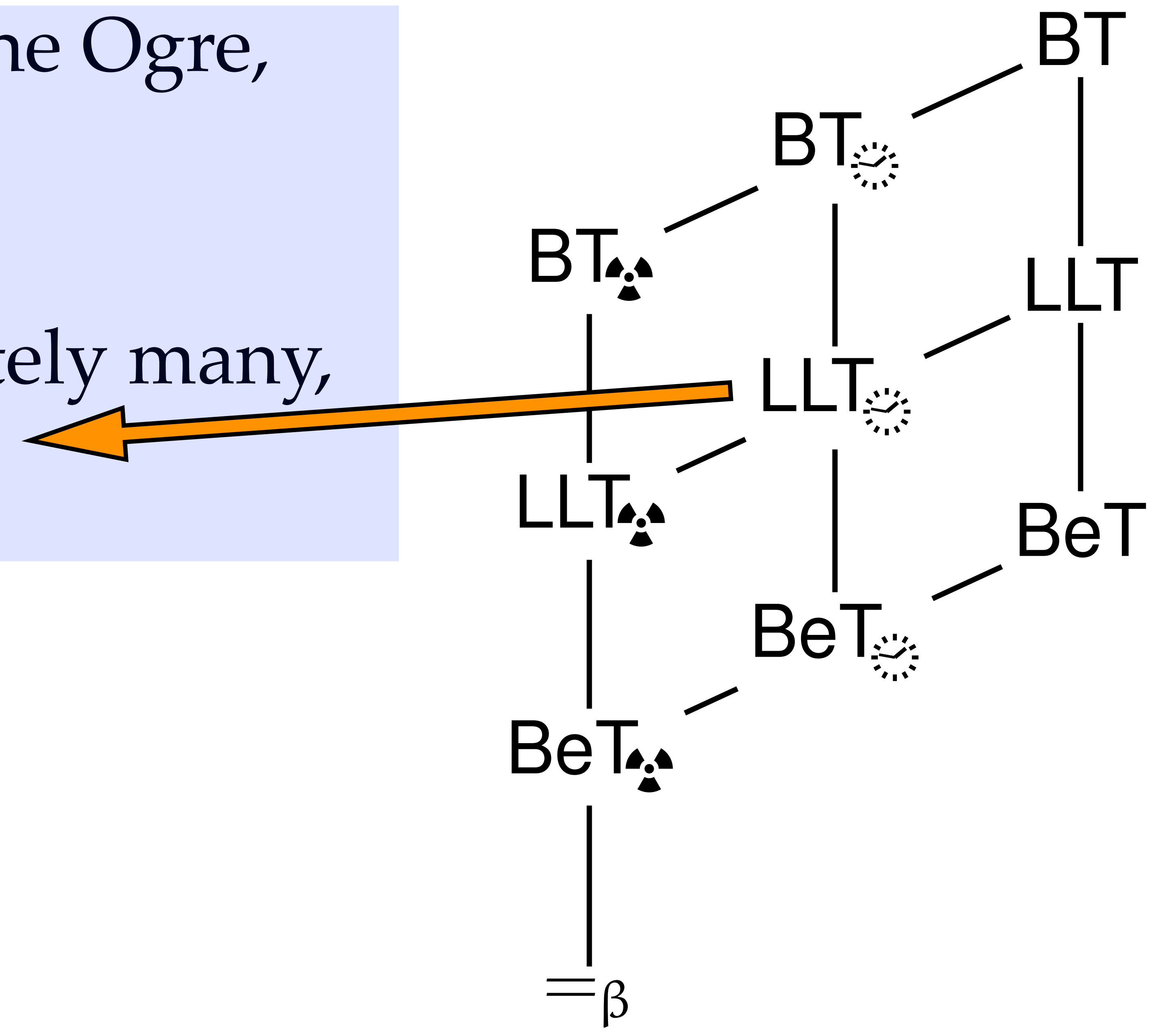


clocked lambda theories

Exercise.

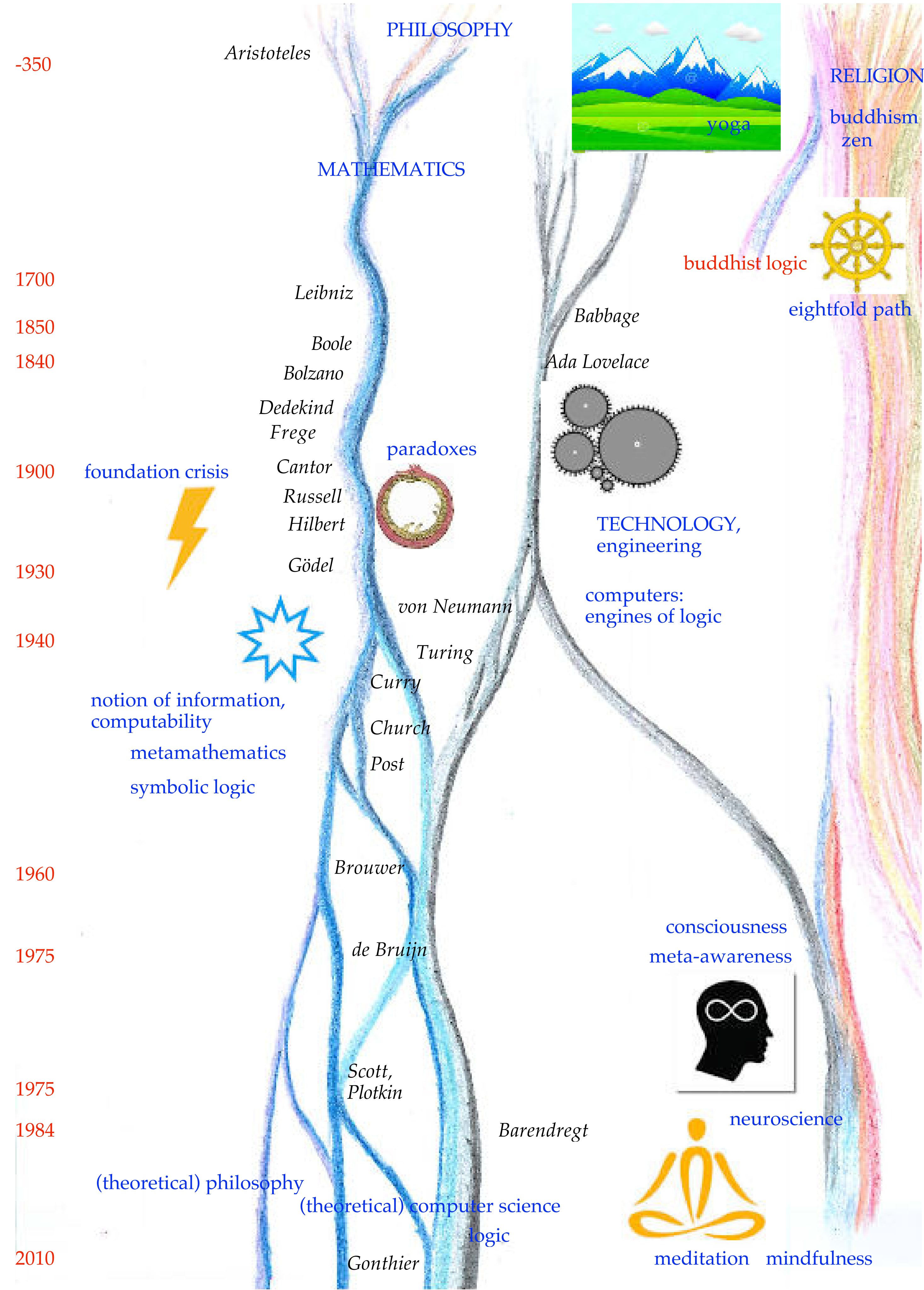
(i) in $\lambda^{\infty}\beta$ there is only one Ogre, Omnivore;

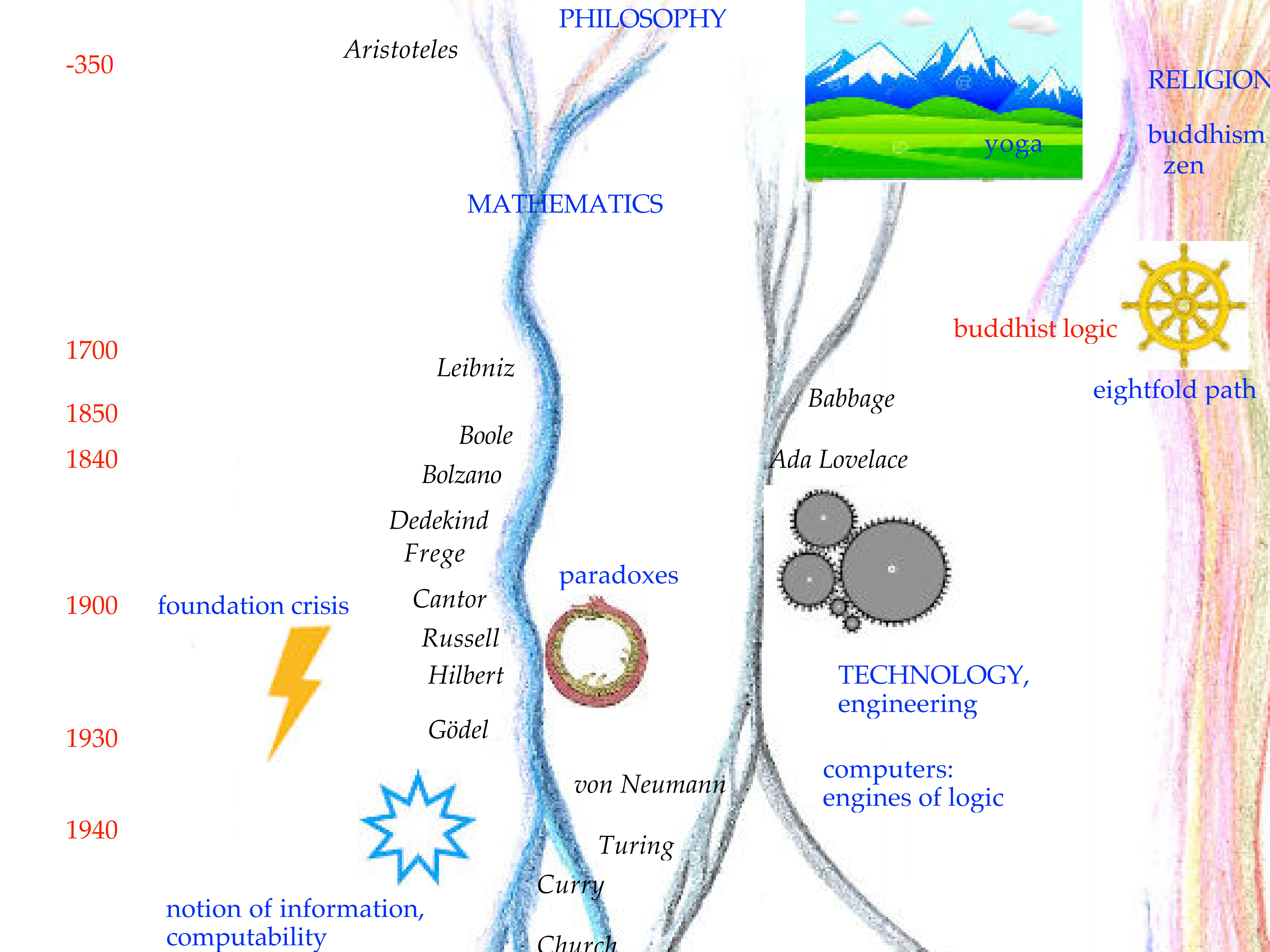
(ii) in $\lambda\beta$ there are infinitely many, i.p. all Y_nK are different

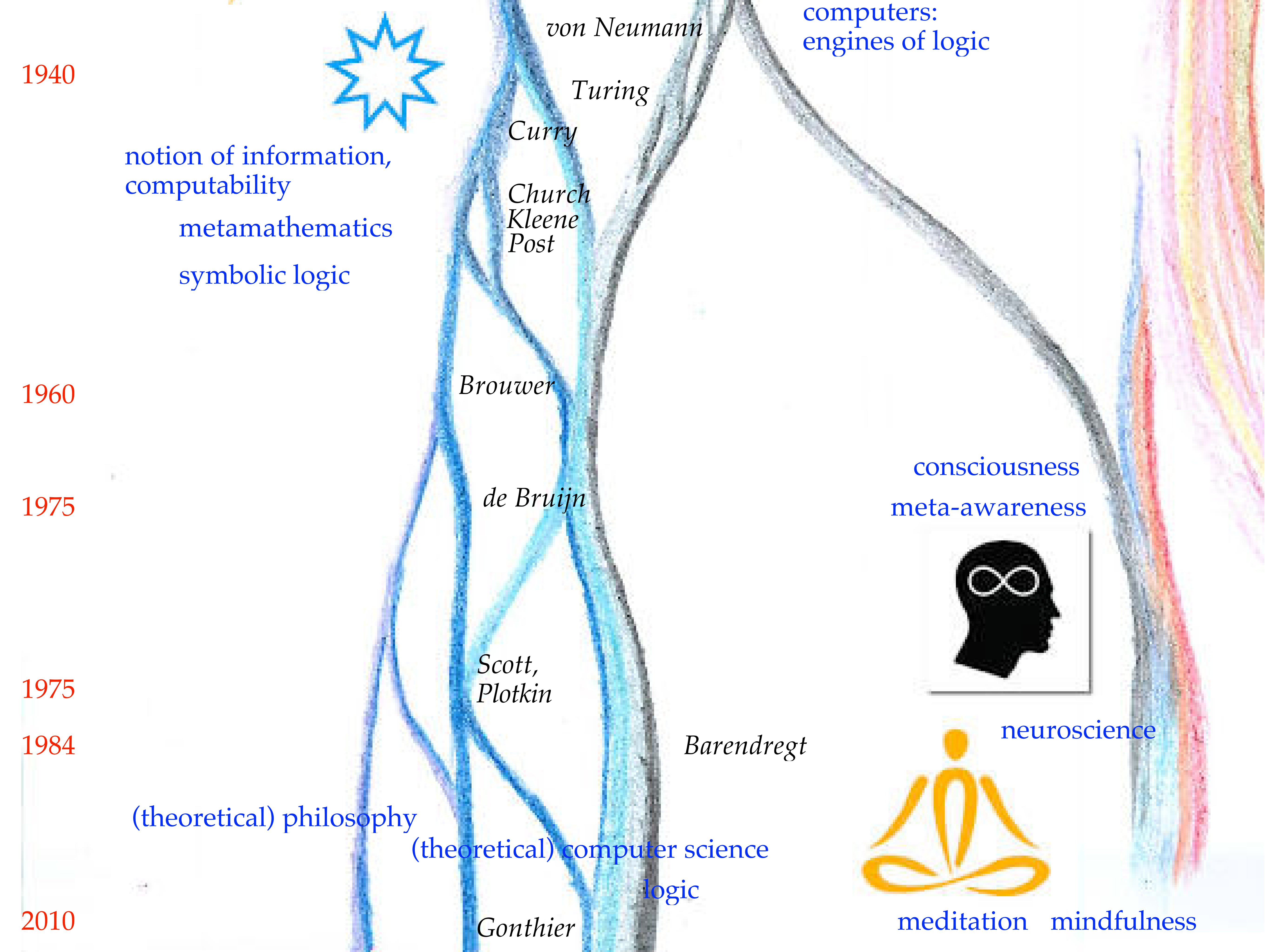


5. Rivers of knowledge

philosophy
religion
mathematics
symbolic logic
technology
neuroscience




PHILOSOPHY
-350
Aristoteles
RELIGION
buddhism
zen
yoga
MATHEMATICS
buddhist logic
1700
Leibniz
eightfold path
Babbage
1850
Boole
1840
Ada Lovelace
Bolzano
Dedekind
Frege
paradoxes
1900
foundation crisis
Cantor
Russell
Hilbert
TECHNOLOGY,
engineering
1930
Gödel
computers:
engines of logic
von Neumann
1940
Turing
Curry
notion of information,
computability
Church



von Neumann
computers:
engines of logic
1940
Turing
Curry
notion of information,
computability
Church
Kleene
Post
metamathematics
symbolic logic
1960
Brouwer
consciousness
meta-awareness
1975
de Bruijn
Scott,
Plotkin
1975
1984
Barendregt
neuroscience
(theoretical) philosophy
(theoretical) computer science
logic
2010
Gonthier
meditation
mindfulness

To Henk: much success in the coming chapter of your life,
bridging the two streams, and much मुदिता *(mudita) in doing so!*