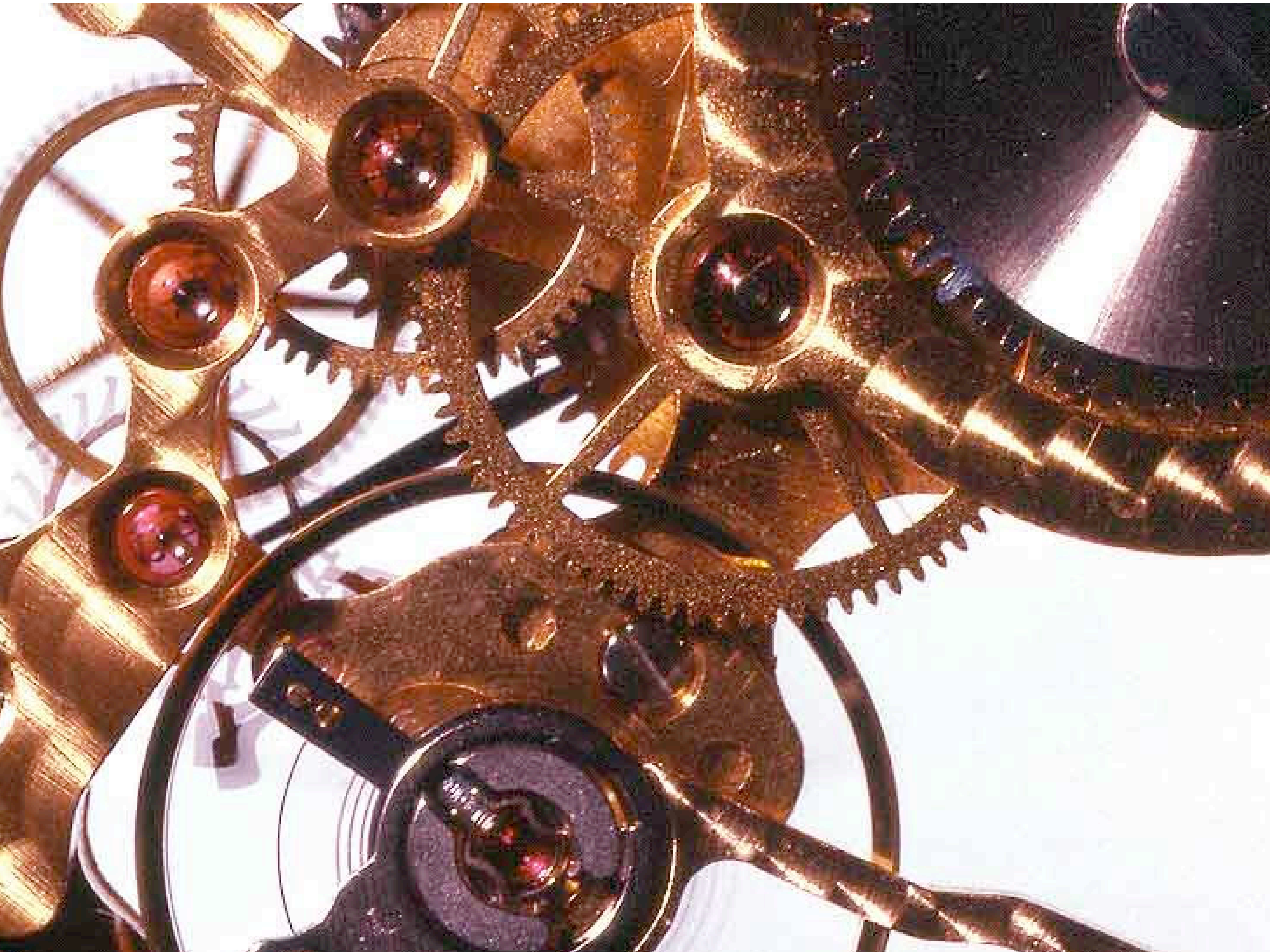
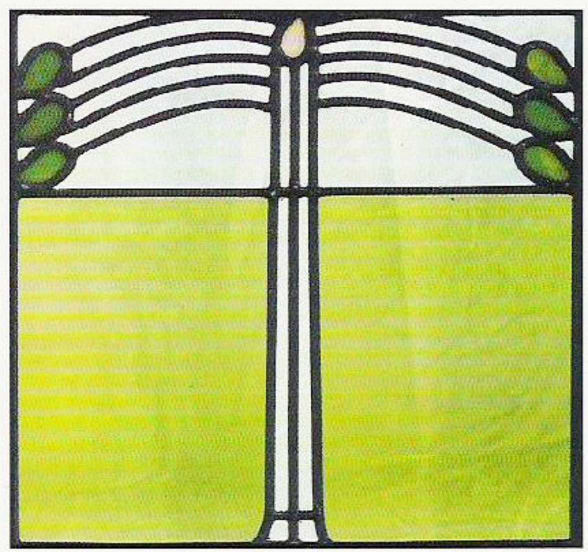


New Fixed Point Combinators from Old

Jan Willem Klop



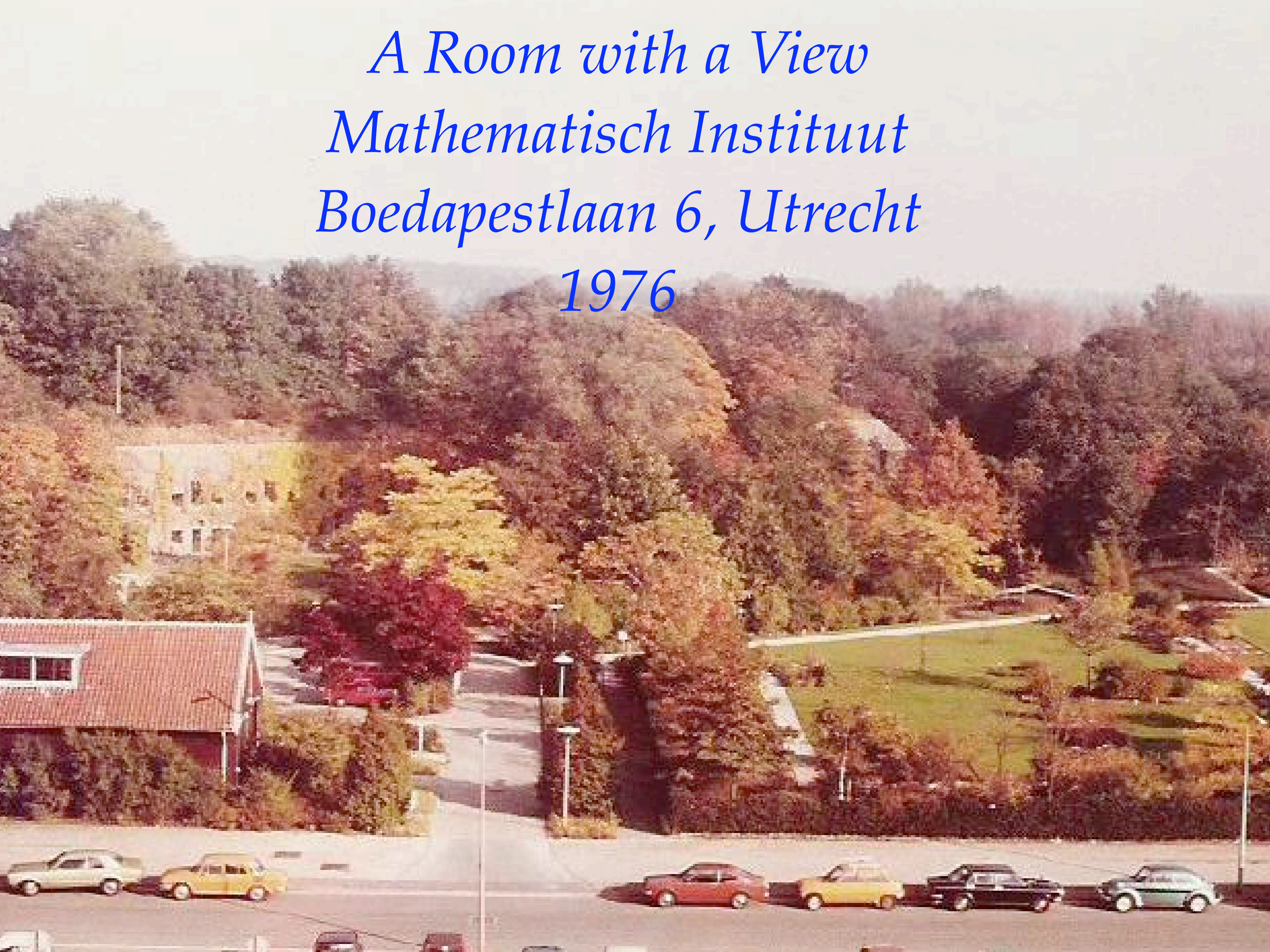
For Henk Barendregt

17 December 2007

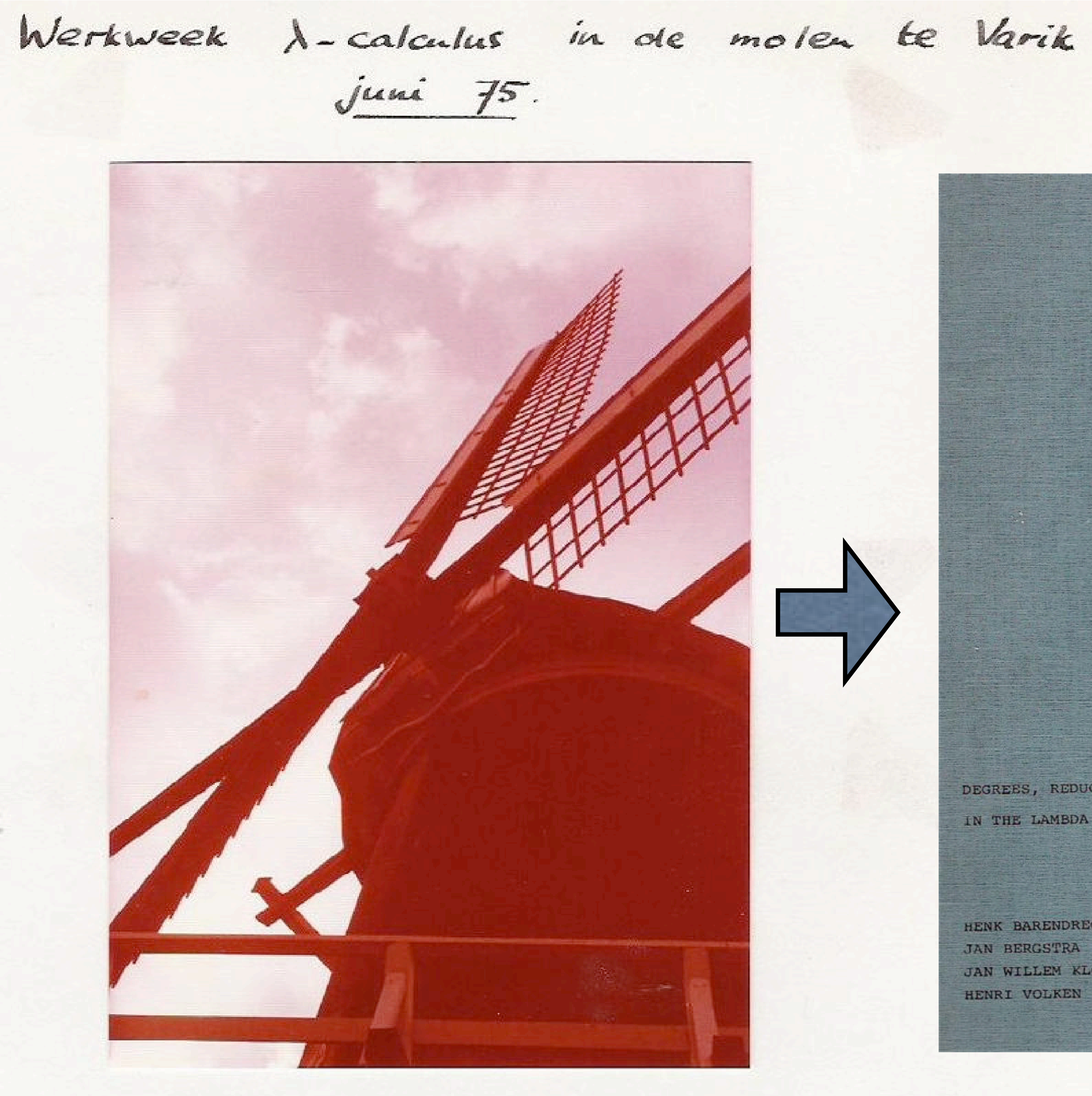
1. Reflections.
Henk as a fixed point

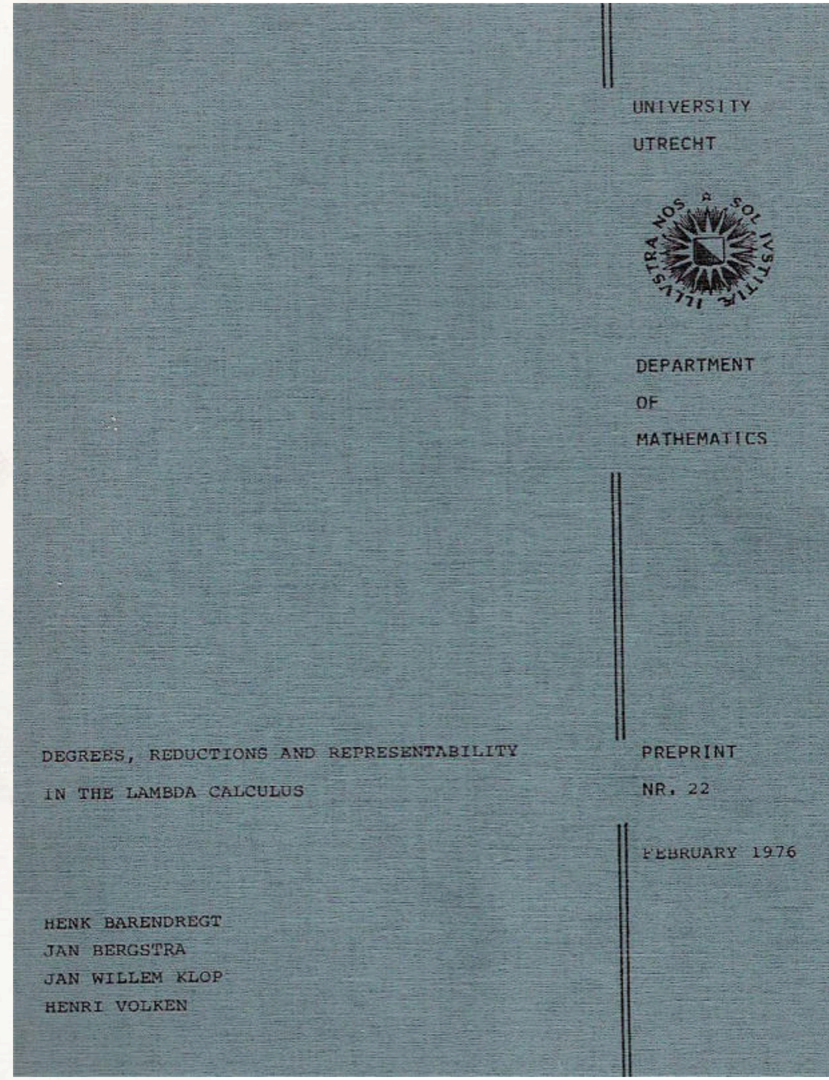
2. Constructing fixed point combinators from given ones

3. .... and proving that they are indeed new


A Room with a View
Mathematisch Instituut
Boedapestlaan 6, Utrecht
1976

Varik,
Betuwe




UNIVERSITY
UTRECHT

ILLVSTRA NOS A SOL IVSTITIÆ

DEPARTMENT
OF
MATHEMATICS

DEGREES, REDUCTIONS AND REPRESENTABILITY
IN THE LAMBDA CALCULUS

PREPRINT
NR. 22

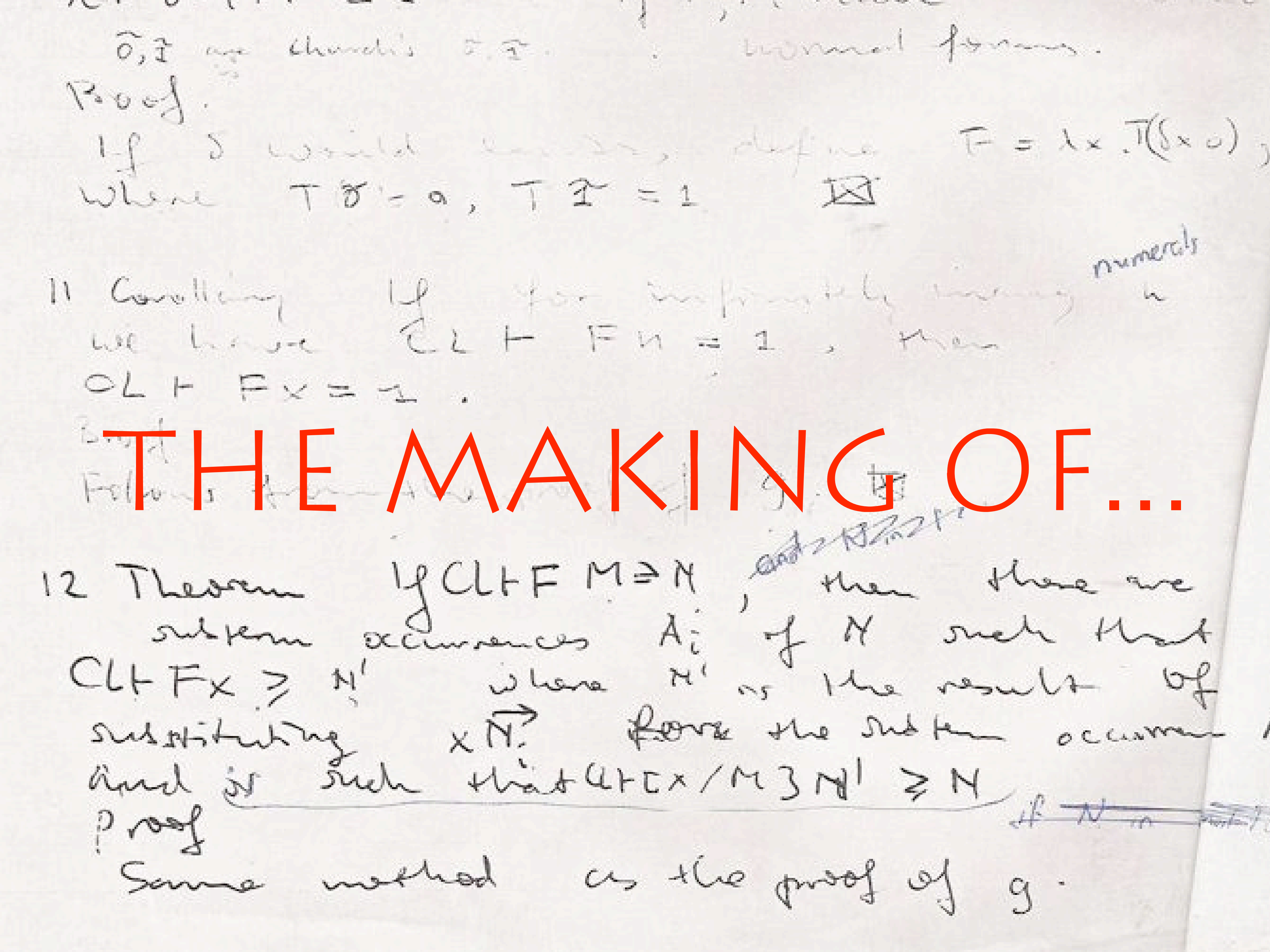
FEBRUARY 1976

HENK BARENDREGT
JAN BERGSTRA
JAN WILLEM KLOP
HENRI VOLKEN

The Blue Preprint

In the Citroën to the shops in Tiel

THE MAKING OF...



λ-演算的语法和语义

——计算机科学基础理论之一

[荷兰] H. P. 巴伦德莱赫特 著
朱一清 译

南 京 大 学 出 版 社
1992 · 南京


Henk
1975
Text

Dirkje Mariastraat Queen's Day 1976 or 1977

lambda calculus in Henk's kitchen

dish washing in Henk's kitchen

Ecole de Printemps La Chatre 1978, France,

Henk meets Corrado, Silvio, Mariangiola, Betty

Corrado,
Henk,
Böhm Tree

1985

To Ustica

Workshop on Reduction Machines

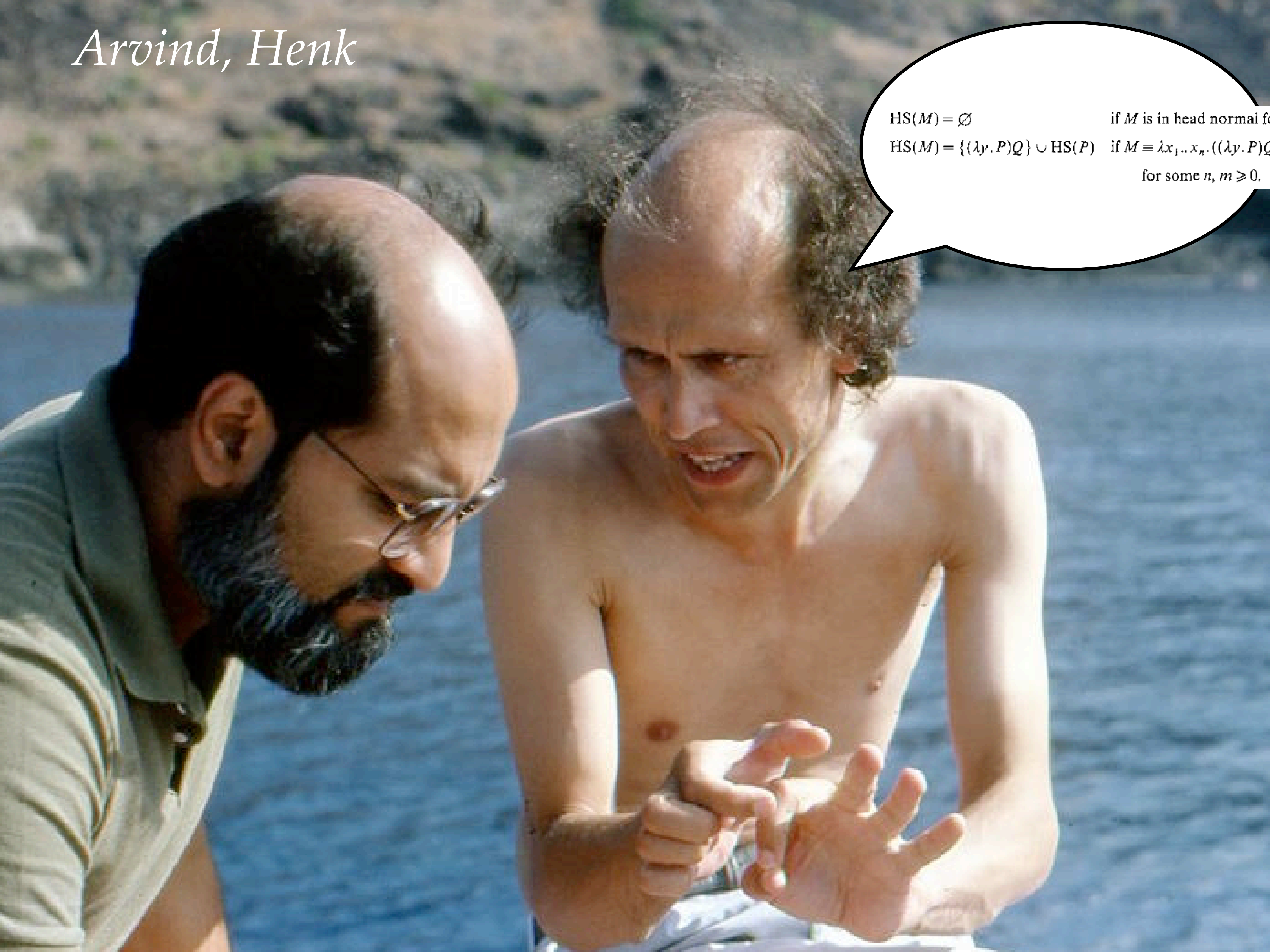
organized by ...

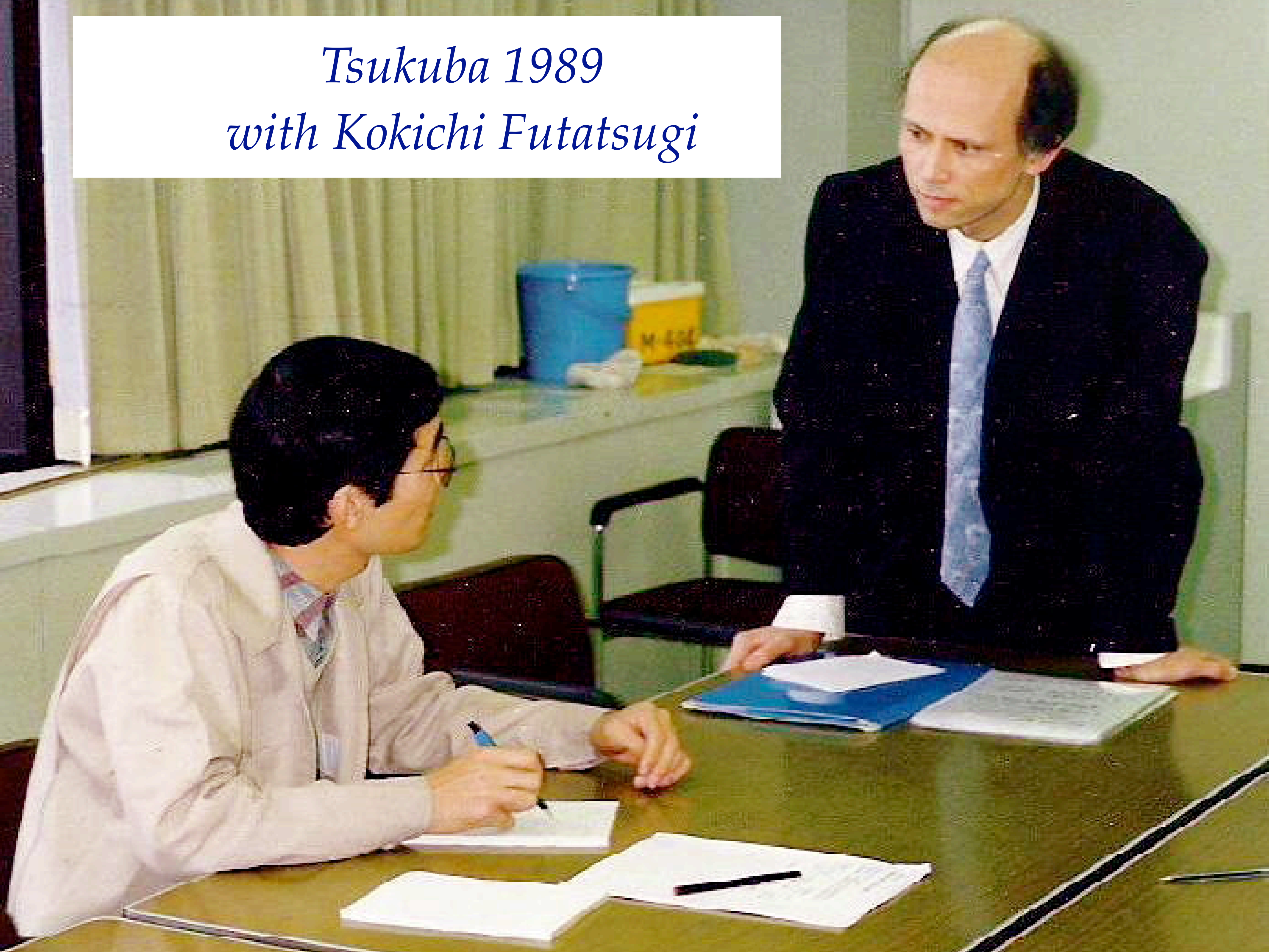
Ustica 1978, Corrado and staff

Henk,
Ronan Sleep

activities between theorems

Arvind, Henk


$\mathrm{HS}(M) = \varnothing$ if M is in head normal
$\mathrm{HS}(M) = \{(\lambda y.P)Q\} \cup \mathrm{HS}(P)$ if $M \equiv \lambda x_1 .. x_n.((\lambda y.P)$
for some n, $m \geqslant 0$.



Tsukuba 1989
with Kokichi Futatsugi

1. Reflections.
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The theory of sage birds (technically called fixed point combinators*) is a fascinating and basic part of combinatory logic; we have only scratched the surface.*

R. Smullyan, To Mock a Mockingbird, 1985.

fpc's:

recursion

=

reflection in itself

... x = x(Yx),

(Curry)

Y_1 ... η (T...)

Exercise. Find M such that Mx = Mx...

Answer: M ≡ Y(λmx.mxm)

LESSON 2

Y is a fixed point combinator if $Yx = x(Yx)$.

Böhm: AH! so Y is itself defined by a fixed point equation!

So, given a fpc Y’, we can construct Y:

$Y = Y' \, (\lambda yx.x(yx)) = Y'(SI) = Y' \, \delta$

$Y_0 \,,\, Y_0 \, \delta \,,\, Y_0 \, \delta \, \delta \,,\, Y_0 \, \delta \, \delta \, \delta \,,\, \ldots$ *Böhm sequence of fpc’s*

$Y_0 \,,\, Y_1 \,,\, Y_2 \,,\, Y_3 \,,\, \ldots$

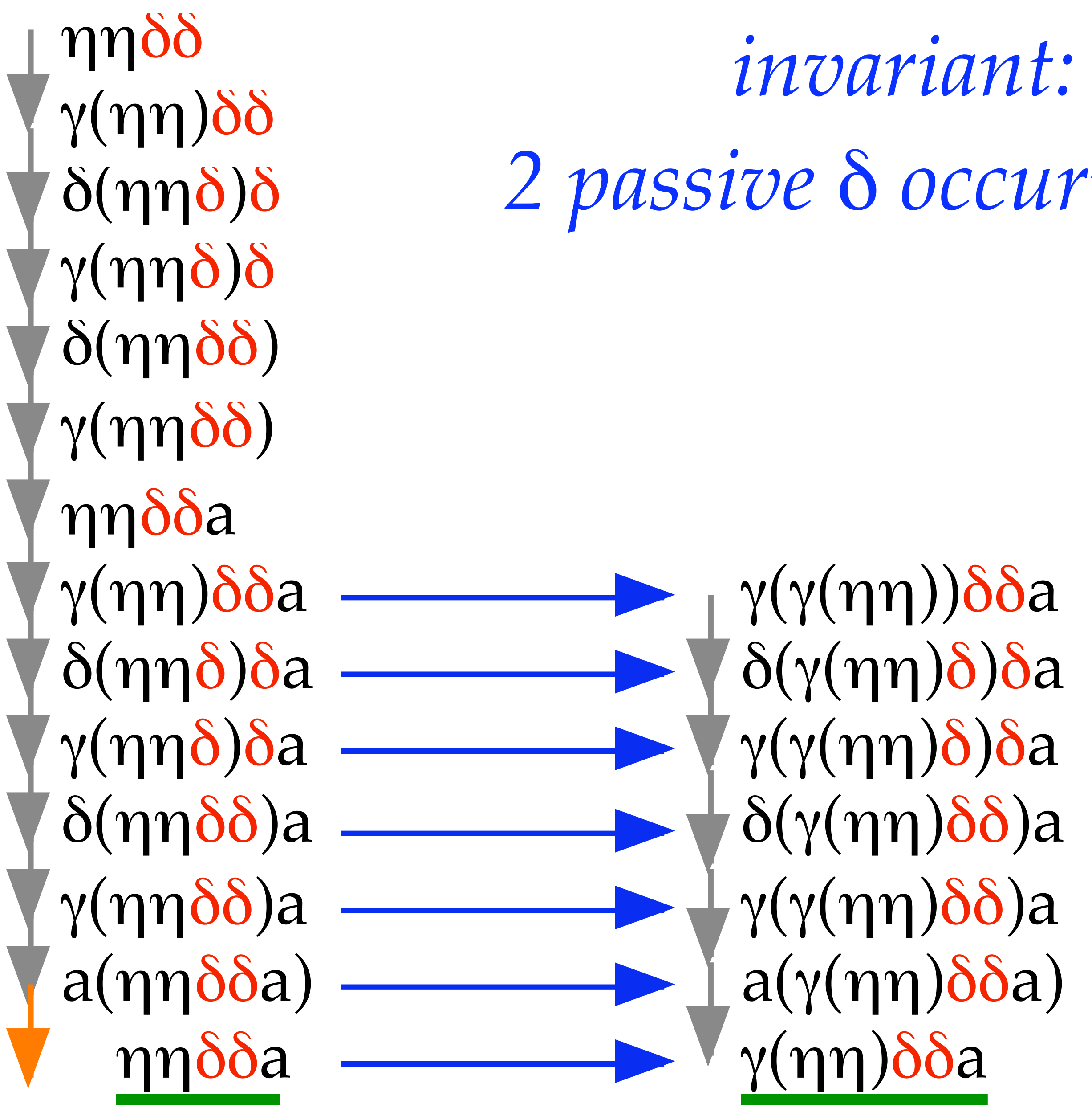
Easy exercise: first two are different

Hard exercise: they are all different

Y_3

invariant:

2 passive δ occurrences



super hard exercise (Statman hard):

If Y is a fpc, then $Y \neq Y\delta$

Theorem (Intrigila).

Statman's double fpc does not exist, indeed.

Question. Are there other postfixes that generate fpc's? Possibly with more arguments

INTERMEZZO: SMULLYAN'S OWL

$\delta xy \to y(xy)$

= δ = SI

δδδδδδ →
δ(δδ)δδδ →
δ(δδδ)δδ →
δ(δδδδ)δ →
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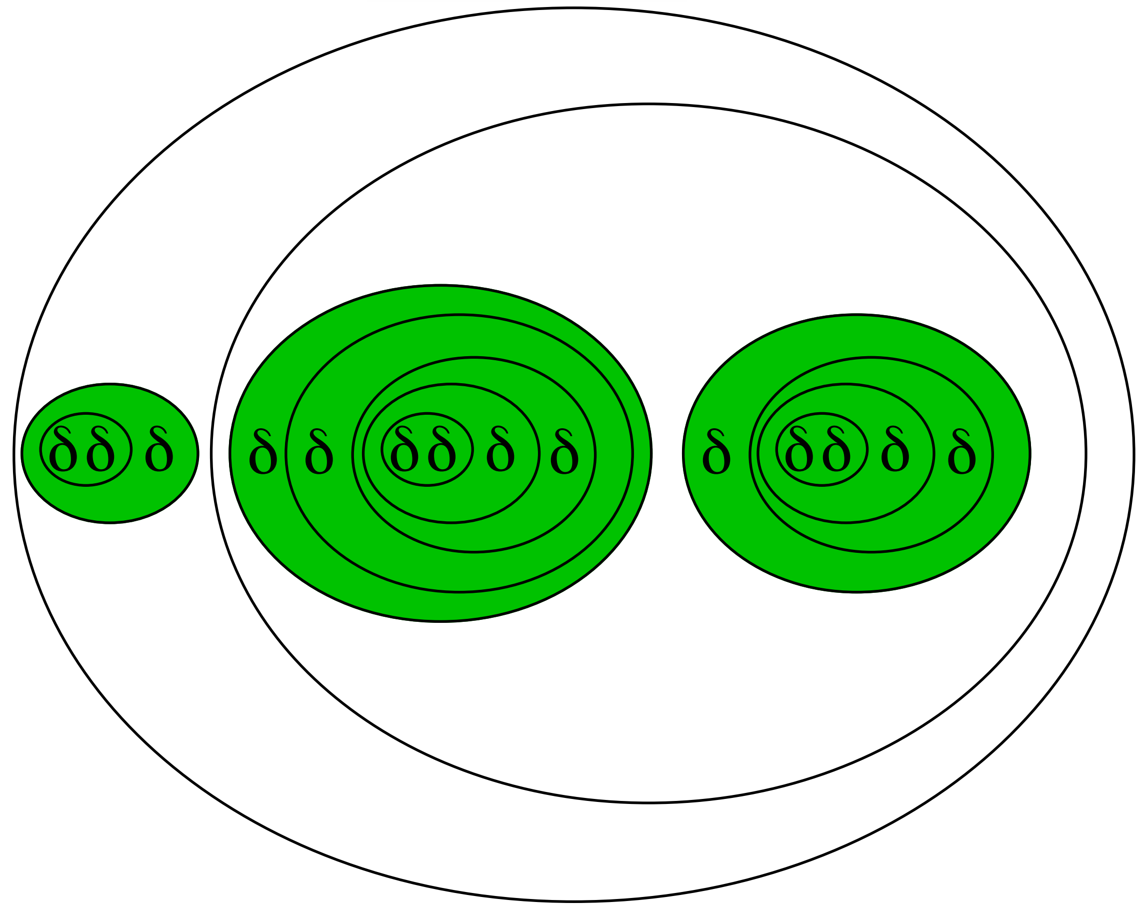

δδδ(δ(δ(δδδδ)))(δ(δδδδ))

$D = \lambda ab.b(ab)$

$A = DD$

AA

A(DA)

DA(D(DA))

D(DA)(A(D(DA)))

A(D(DA))(DA(A(D(DA))))

D(DA)(D(D(DA)))(DA(A(D(DA))))

D(D(DA))(DA(D(D(DA))))(DA(A(D(DA))))

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D(D(DA))(A(D(D(DA))))(D(DA)(DA(D(D(DA)))))(DA(A(D(DA))))

Word problem ?

SCOTT'S EQUATION BY = BYS

7. Show that the equation BYS = BY cannot be
λ-calculus without induction. Are there other
involving Y that can be proved by suitable
(For example Böhm's sequence of Y operators is

Easy exercise:

$Y_0 \neq Y_1$

$$BY_0I = (\lambda abc.\ a(bc))\ Y_0I = \lambda c.\ Y_0(Ic) = Y_0$$

$$BY_0SI = (\lambda abc.\ a(bc))Y_0SI = Y_0(SI) = Y_0\ \delta = Y_1$$

QED

For general Y, apply Intrigila's theorem: $Y \neq Y\delta$

BY = BYS CONTINUED (2)

$BY = (\lambda abc.\ a(bc))Y = (\lambda bc.\ Y(bc)) = \lambda bc.\ (bc)^{\omega}$

$BYS = (\lambda abc.\ a(bc))YS = (\lambda c.\ Y(Sc)) = \lambda c.\ (Sc)^{\omega}$

$= \lambda c.\ Sc(Sc)^{\omega} = \lambda cz.cz((Sc)^{\omega}z) = \lambda cz.cz(cz((Sc)^{\omega}z)) =$

$\cdots\ \lambda cz.\ (cz)^{\omega} = \lambda bc.\ (bc)^{\omega}$

Note that $(\lambda bc.\ (bc)^{\omega})I = \lambda c.\ (Ic)^{\omega} = \lambda c.\ c^{\omega} = Y$ in *infinitary lambda calculus* λ^{∞}.

And note that in λ^{∞} $BY = BYS = BYSS = BYSSS = \ldots = \lambda bc.\ (bc)^{\omega}$

BY = BYS CONTINUED (3)

$BY = BYS = BYSS = BYSSS = \ldots = \lambda bc.\ (bc)^{\omega}$

$BYI = BYSI = BYSSI = BYSSSI = \ldots = \lambda c.\ c^{\omega} = Y$ in λ^{∞}

Now return to finitary λ-calculus:

- every $BYS^{\sim n}I$ is a fpc. The first two are as in the Böhm fpc sequence, but the subsequent ones deviate. The sequence contains no duplications.

- If Y is a fpc, then $Y(SS)S^{\sim n}I$ is a fpc.

BY = BYS CONCLUDED

Similar: the equation BBBY = BBBY(BS) = λabc. (abc)$^{\omega}$ yields fpc generating schemes (with A = BS):

$$Y \Rightarrow Y(S(AI))I$$

$$Y \Rightarrow Y(AAA)II$$

$$Y \Rightarrow Y(AII)$$

$$Y \Rightarrow Y(AAI)I$$

$$Y \Rightarrow Y(AAA)A^{\sim n}II$$

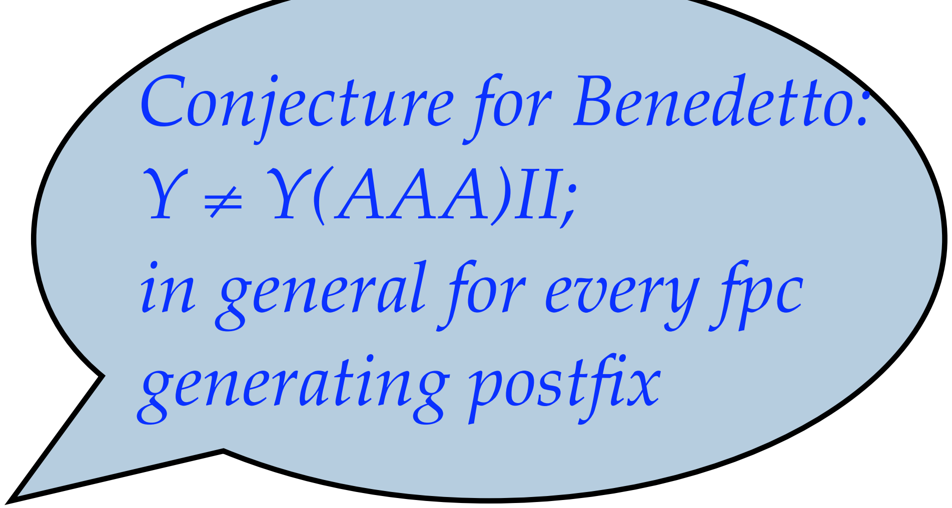
AII = δ !

bachelor thesis: compute the schemes obtainable from λabcd. (abcd)$^{\omega}$ etc.

I=\x.x
K=\xy.x
S=\xyz.xz(yz)
B=\xyz.x(yz)
C=\xyz.xzy
1=\xy.xy
Y=\f.(\x.f(xx))\x.f(xx)
T=\xy.x
F=\xy.y
J=\abcd.ab(adc)

.li leftmost innermost
.lo leftmost outermost [default]
.po parallel outermost
.gk gross knuth
.l lambda reduction [default]
.c combinator reduction
.ex eta reduction
.in no eta reduction [default]
.+ fold combinators
.- don't fold combinators [default]
.tau translate to CL
.tau' translate economically to CL
.. normalize
.<< previous input term
.< previous term
.> next term
.>> next input term
.? help
. exit

D=\ab.b(ab)
A=BS



Y(AAA)IIx
Y(AAA)IIx .c
Y(AAA)IIx
(\x.AAA(xx))(\x.AAA(xx))IIx
AAA((\x.AAA(xx))\x.AAA(xx))IIx
BSAA((\x.AAA(xx))\x.AAA(xx))IIx
S(AA)((\x.AAA(xx))\x.AAA(xx))IIx
AAI((\x.AAA(xx))(\x.AAA(xx))I)Ix
BSAI((\x.AAA(xx))(\x.AAA(xx))I)Ix
S(AI)((\x.AAA(xx))(\x.AAA(xx))I)Ix
AII((\x.AAA(xx))(\x.AAA(xx))II)x
BSII((\x.AAA(xx))(\x.AAA(xx))II)x
S(II)((\x.AAA(xx))(\x.AAA(xx))II)x
IIx((\x.AAA(xx))(\x.AAA(xx))IIx)
Ix((\x.AAA(xx))(\x.AAA(xx))IIx)
x((\x.AAA(xx))(\x.AAA(xx))IIx)

GENERAL CONJECTURE

- $Y\delta^{\sim n} \neq Y$ for every fpc Y
- $Y \neq Y' \Rightarrow Y\delta \neq Y'\delta$ for al fpc's Y, Y'
- Every fpc Y can be factorized uniquely in a prime fpc followed by a string of (prime) fpc postfixes
- There are no non-trivial equations between fpc's, no postfix derivation cycles, no intersecting derivation trails
- Fpc's form a 'free structure'

Caveat: $Y(KY') = Y'$

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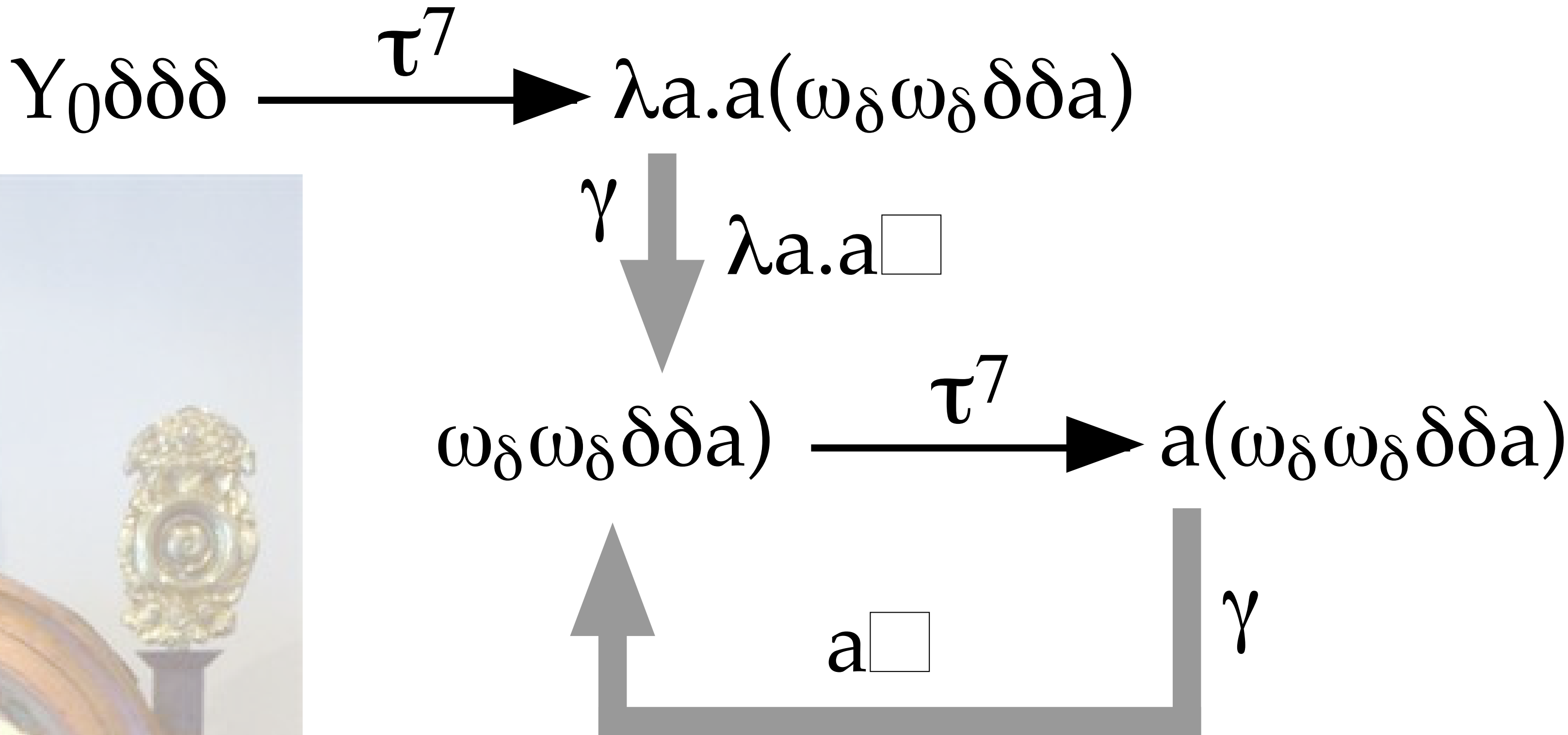


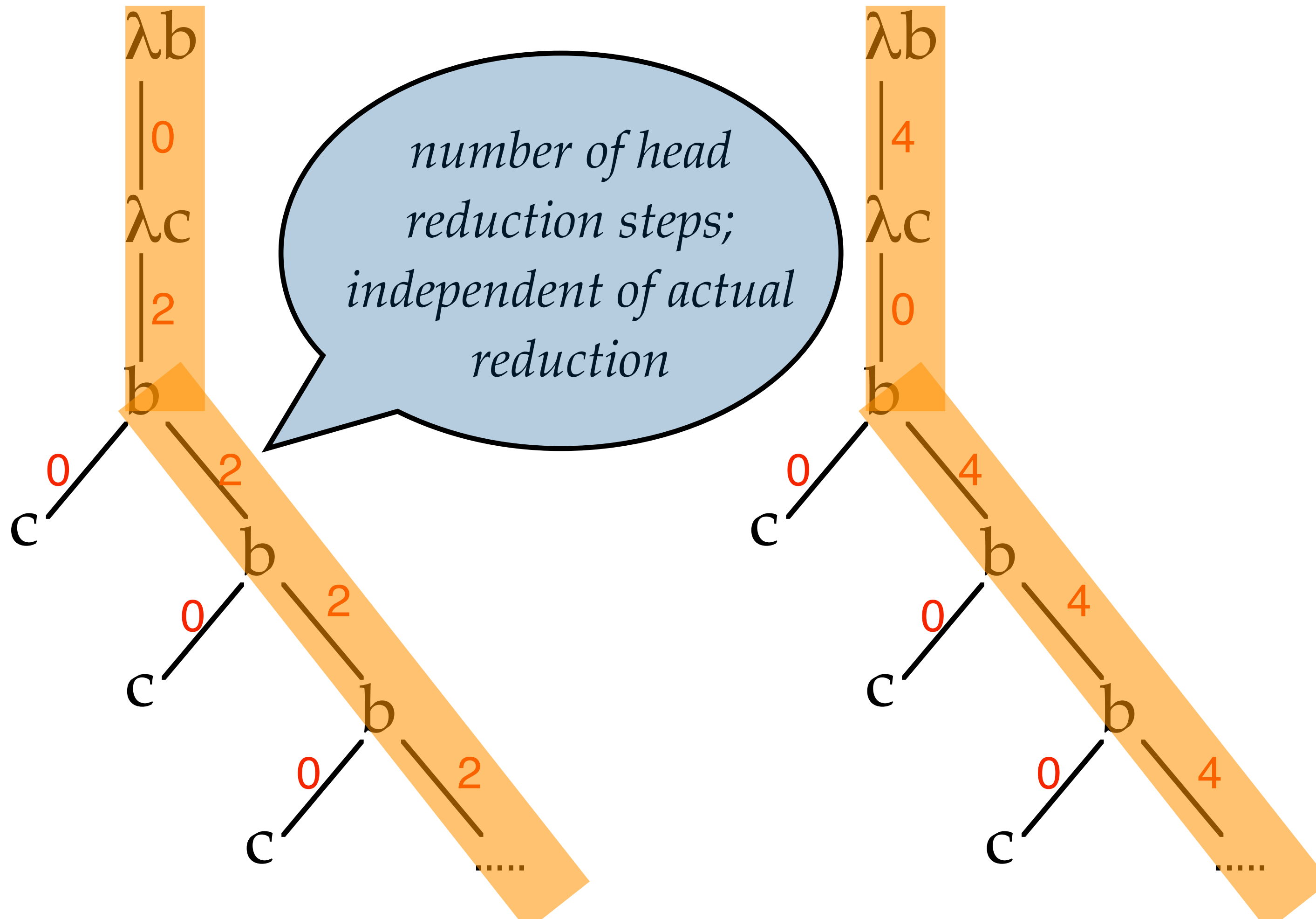
fixed point combinator

Y_3 as a clock $\tau^7\gamma(\tau^7\gamma)^\omega$

Y_n has clock $\tau^{2n+1}\gamma\,(\tau^{2n+1}\gamma)^\omega$

CLOCKED BOHM TREES

$\lambda bc.\ (bc)^{\omega}$



clocked BT of BY

clocked BT of BYS

Clock invariance for *simple* *terms; reduct of simple term has same clock*

For simple terms clock behaviour is discriminating feature

Alternative proof that Böhm sequence of fpc's is free of duplicates