

IWC 2013  
2nd International

Workshop

on Confluence



Workshop on  
Infinitary Rewriting  
2013

# CONFLUENCE & INFINITARY REWRITING

*A bus tour with twelve stops*

BUS DRIVER: JAN WILLEM KLOP

*VU University Amsterdam  
CWI Amsterdam*

*June 28, 2013 Eindhoven*



1. Introduction

2. Lambda and CL: basic confluence

3. Surjective Pairing: confluence lost

4. Confluence of a higher order

5. Lambda with black holes: confluence

6. Confluence lost in infinity

7. The threefold path

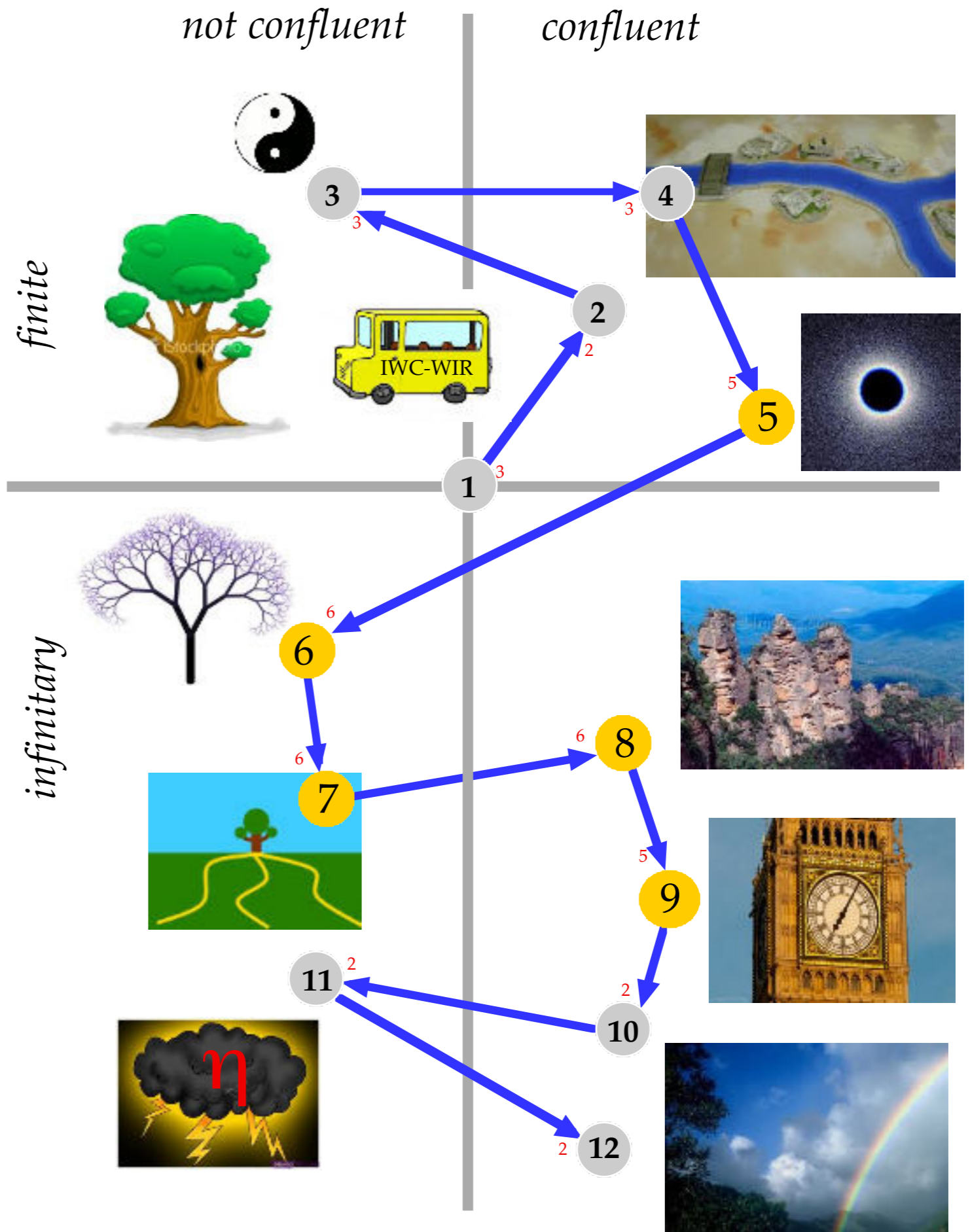
8. Black holes to the rescue

9. The rhythm of lambda terms

10. Getting rid of ordinals

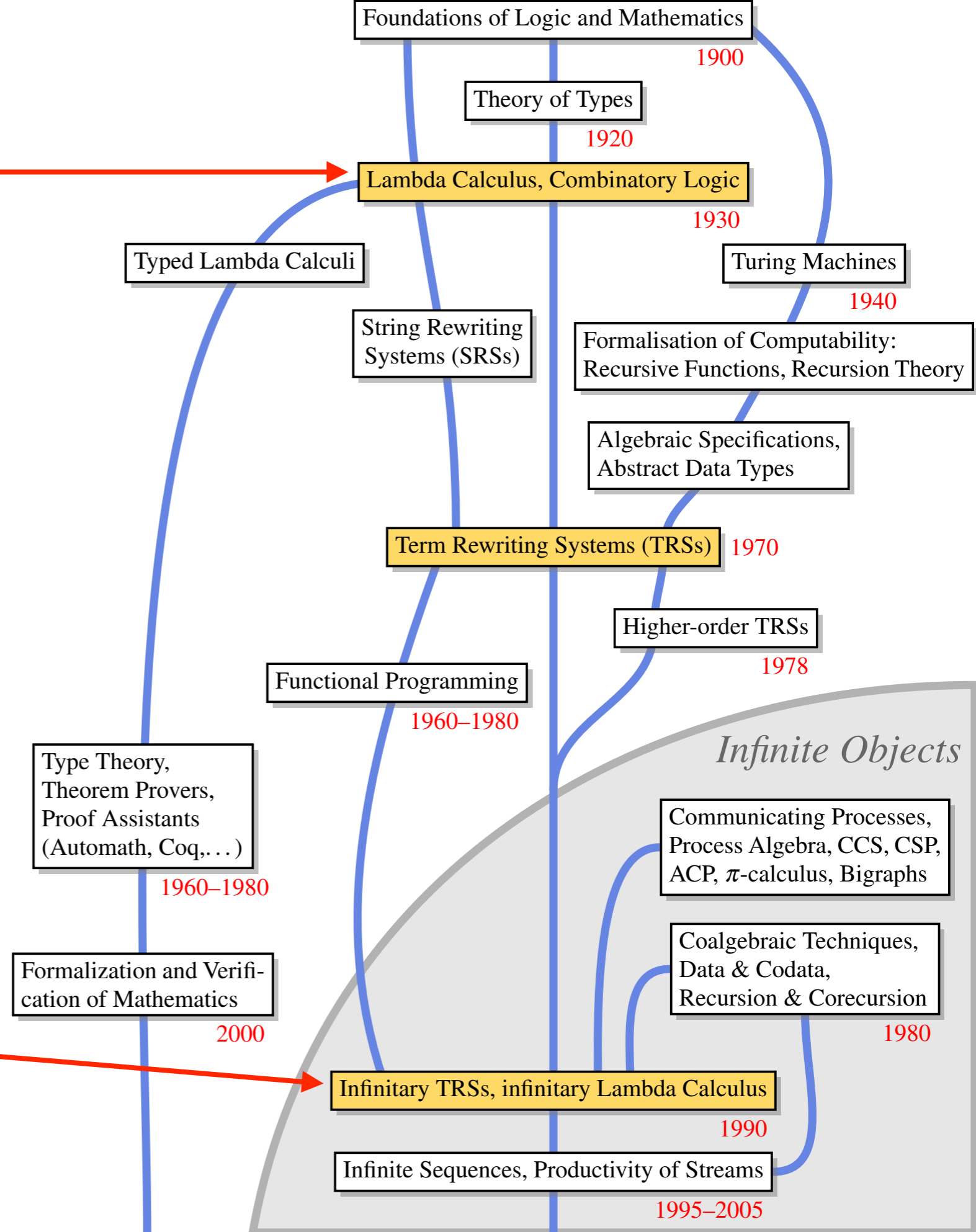
11. Infinity and eta: total breakdown

12. A lambda universe



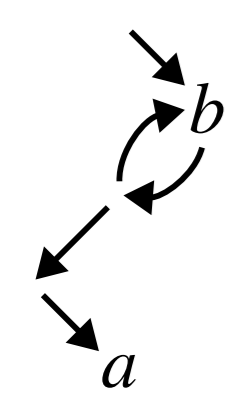
*historical time-flow*

*confluence arose here*

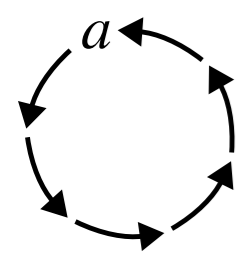


*infinitary rewriting arose here*

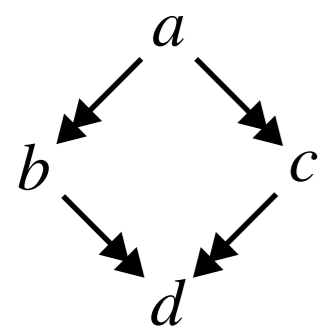
# 1. REWRITING DICTIONARY



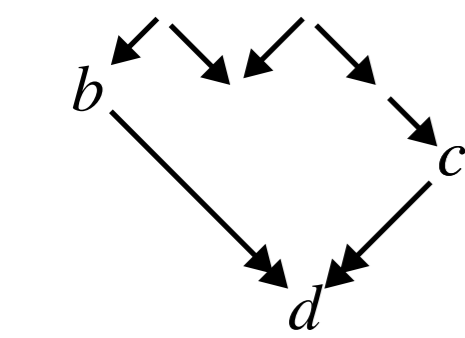
*normal form*



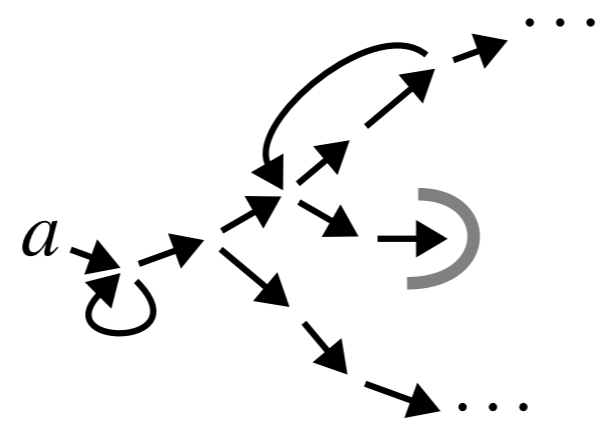
*reduction cycle;  
loop if one step*



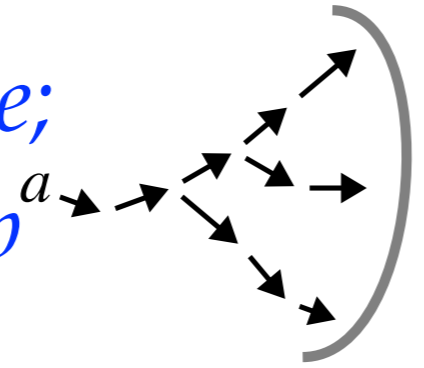
*CR, Church-Rosser*



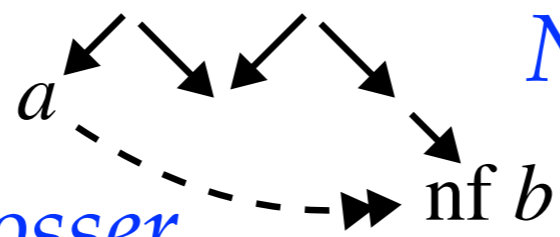
*equivalent: CR,  
Church-Rosser*



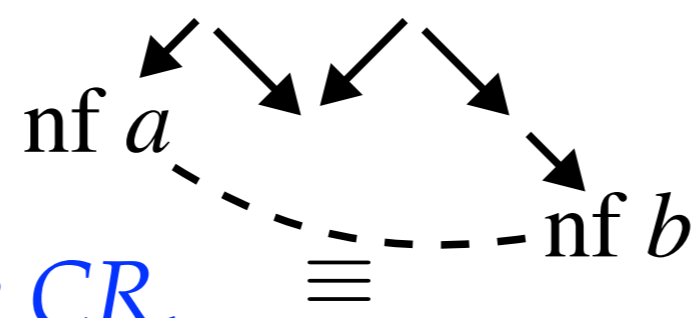
*WN, weakly normalizing*



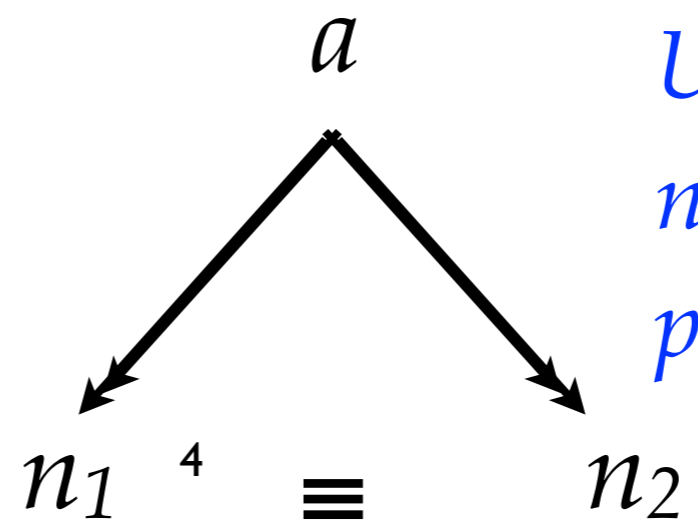
*SN, strongly  
normalizing; terminating; noetherian*



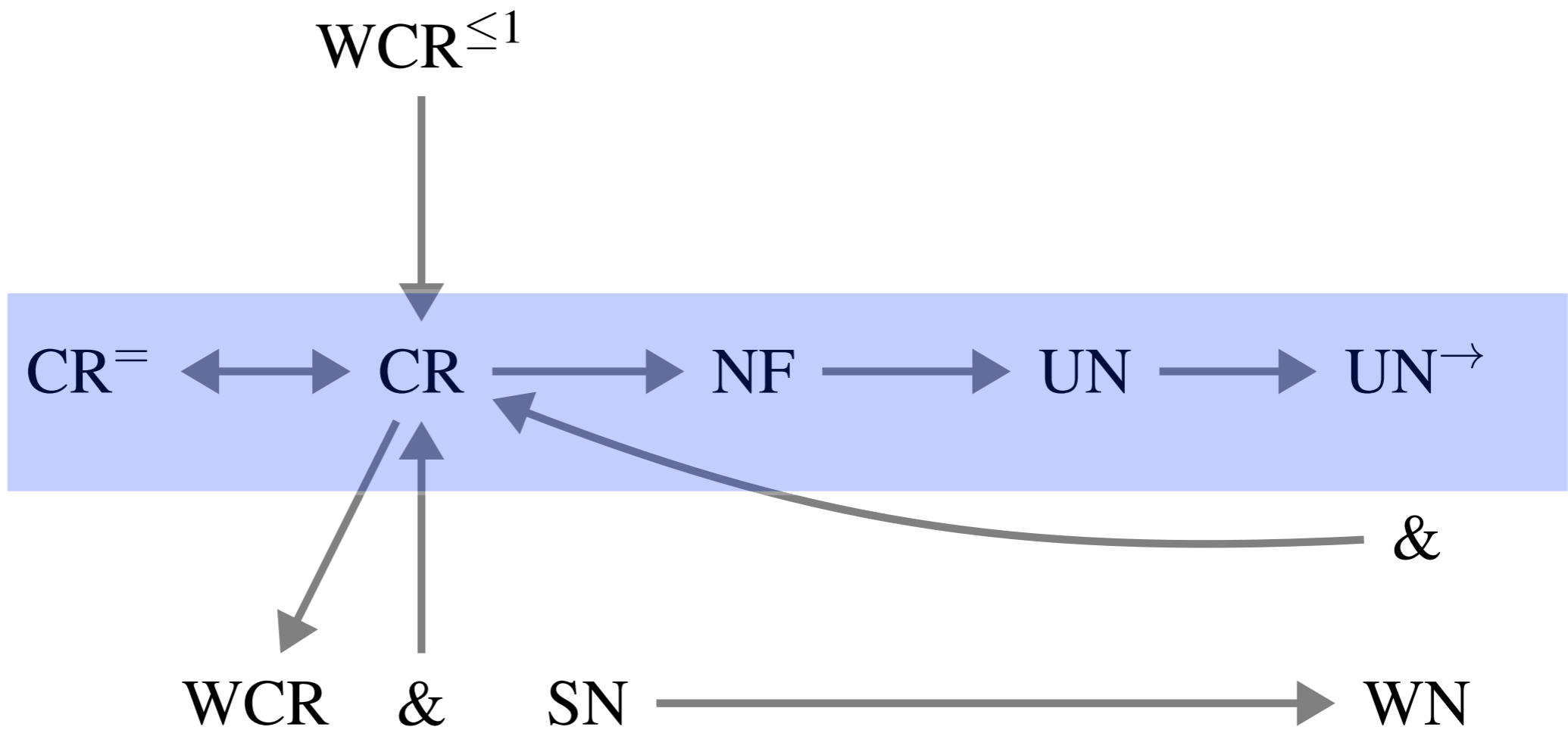
*NF, normal form property*



*UN=, unique normal  
form property wrt =*



*UN->, unique  
normal form  
property wrt →*

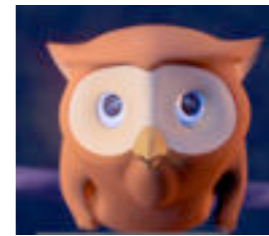


I, K, S, B

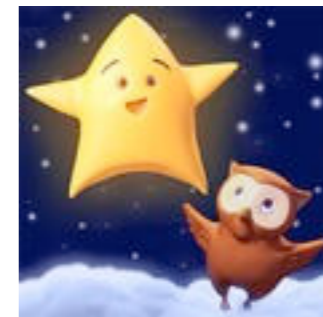
$$\omega = \lambda x. xx$$

$$\Omega = \omega \omega$$

$$\delta = \lambda xy. y(xy) = \text{SI, Smullyan's Owl}$$



$$\Delta = \delta^\omega = \delta(\delta(\delta(\delta \dots = Y\delta$$



$$Y_0 = \lambda f. (\lambda x. f(xx)) (\lambda x. f(xx)) \text{ Curry's fpc}$$

$$Y_1 = (\lambda ab. b(aab)) (\lambda ab. b(aab)) \text{ Turing's fpc}$$

$$Y_0 \delta = Y_1$$



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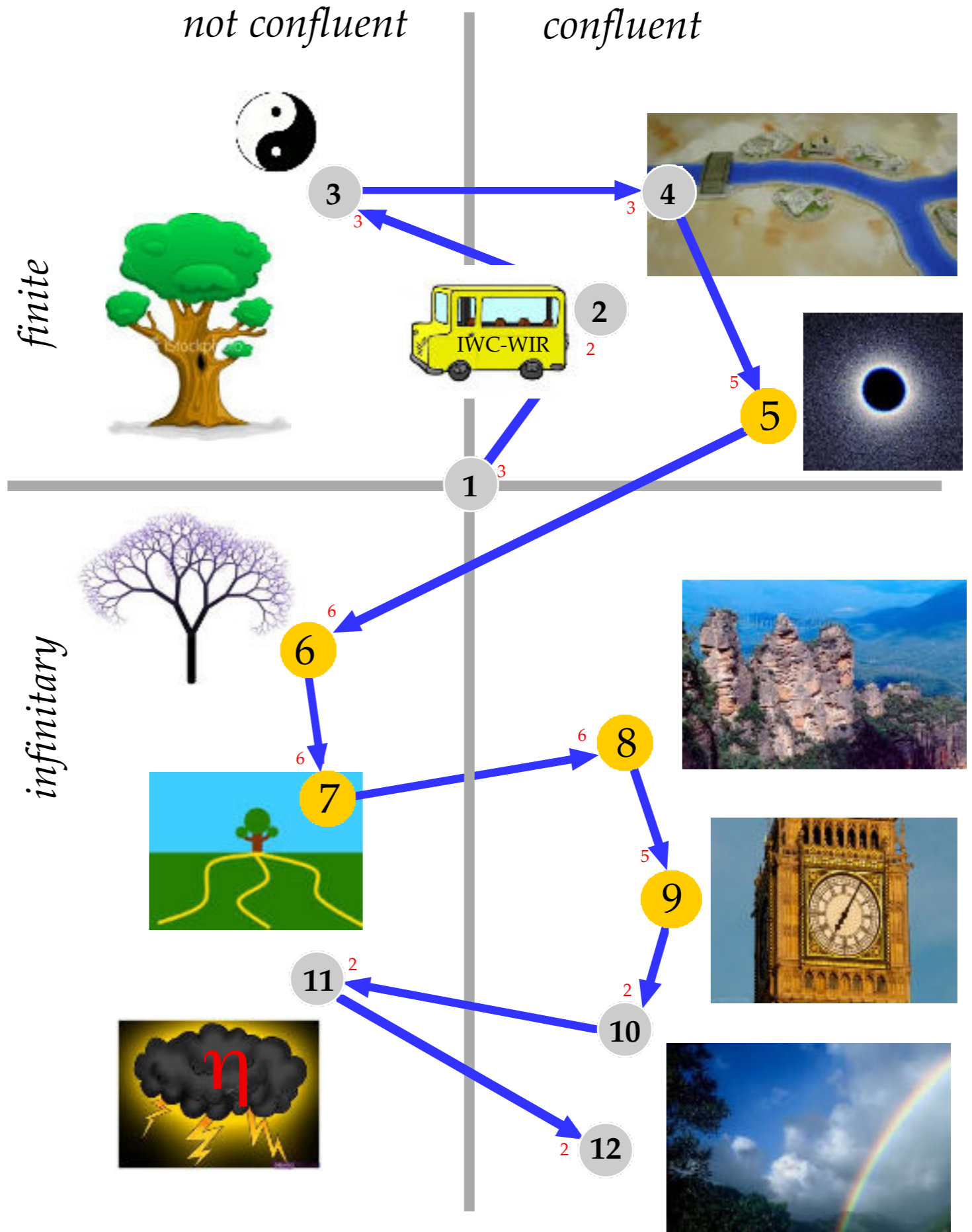
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Henk Barendregt, spring 1971

Parallel Reduction à la Tait and Martin-Löf

$$\begin{array}{c}
 M \twoheadrightarrow M \\
 \\
 \frac{M \twoheadrightarrow M'}{\lambda x. M \twoheadrightarrow \lambda x. M'} \\
 \\
 \frac{M \twoheadrightarrow M' \quad N \twoheadrightarrow N'}{MN \twoheadrightarrow M'N'} \\
 \\
 \frac{M \twoheadrightarrow M' \quad N \twoheadrightarrow N'}{(\lambda x. M)N \twoheadrightarrow M'[x := N']}
 \end{array}$$

We use the notation  $\twoheadrightarrow$  for parallel reduction. In the style of Tait and Martin-Löf, it is defined by the inductive clauses in Table 10. It characterizes complete developments, in the sense that  $M \twoheadrightarrow N$  if and only if there is a complete development from  $M$  to  $N$ .

In Aczel [Acz78] the last clause is replaced by:

$$\frac{M \twoheadrightarrow \lambda x. M' \quad N \twoheadrightarrow N'}{MN \twoheadrightarrow M'[x := N']}$$

Now there is a complete  $\beta$ -superdevelopment from  $M$  to  $N$  if and only if  $M \twoheadrightarrow N$  according to Aczel's definition.

EXAMPLE 12.1. In the first definition, due to Tait and Martin-Löf, we do not have  $III \twoheadrightarrow I$  (with  $I \equiv \lambda x. x$ ); in Aczel's definition we do.

Likewise  $(\lambda xyz. xyz) abc \twoheadrightarrow abc$  and even  $II(\lambda xyz. xyz) abc \twoheadrightarrow abc$ .

Dear

I would like to mention you a strikingly simple proof of the Church-Rosser theorem for the  $\lambda$ -calculus, due to Martin-Löf in his unpublished paper 'A theory of types', Stockholm 1971. In this paper an extension of the  $\lambda$ -calculus is considered. However the proof of the Church-Rosser theorem immediately carries over to the  $\lambda$ -calculus itself. The idea of the proof arised from cut-elimination properties of certain formal systems. In fact the Church-Rosser theorem is a kind of cut-elimination theorem, the transitivity of  $=$  in the  $\lambda$ -calculus corresponds to the cut.

The trick is to define a relation  $\geq_1$  between terms in such a way that

- 1) The transitive closure of  $\geq_1$  is the (classical) reduction relation ( $\geq$ ).
- 2) If  $M_1 \geq_1 M_2$ ,  $M_1 \geq_1 M_3$ , then there exists a term  $M_4$ , such that  $M_2 \geq_1 M_4$  and  $M_3 \geq_1 M_4$ .

From 1) and 2) the analogue of 2) for  $\geq$  can be derived.

From this the Church-Rosser theorem easily follows.

Now  $\geq_1$  is defined as follows:

$M \geq_1 M$ .

If  $M \geq_1 M'$ , then  $\lambda x M \geq_1 \lambda x M'$ .

If  $M \geq_1 M'$ ,  $N \geq_1 N'$  then  $MN \geq_1 M'N'$ .

If  $M \geq_1 M'$ , then  $\lambda x M \geq \lambda y [x/y] M'$ , where  $y \notin FV(M')$ .

If  $M \geq_1 M'$ ,  $N \geq_1 N'$  then  $(\lambda x M)N \geq_1 [x/N'] M'$  if

$$FV(N') \cap BV(M') = \emptyset$$

( $[x/N]M$  stands for the result of substituting  $N$  in the free occurrences of  $x$  in  $M$ ;  $FV(M)$  resp.  $BV(M)$  is the set of free resp. bound variables of  $M$ ).

It is clear that  $\geq_1$  satisfies 1). A simple inductive proof shows that  $\geq_1$  also satisfies 2).

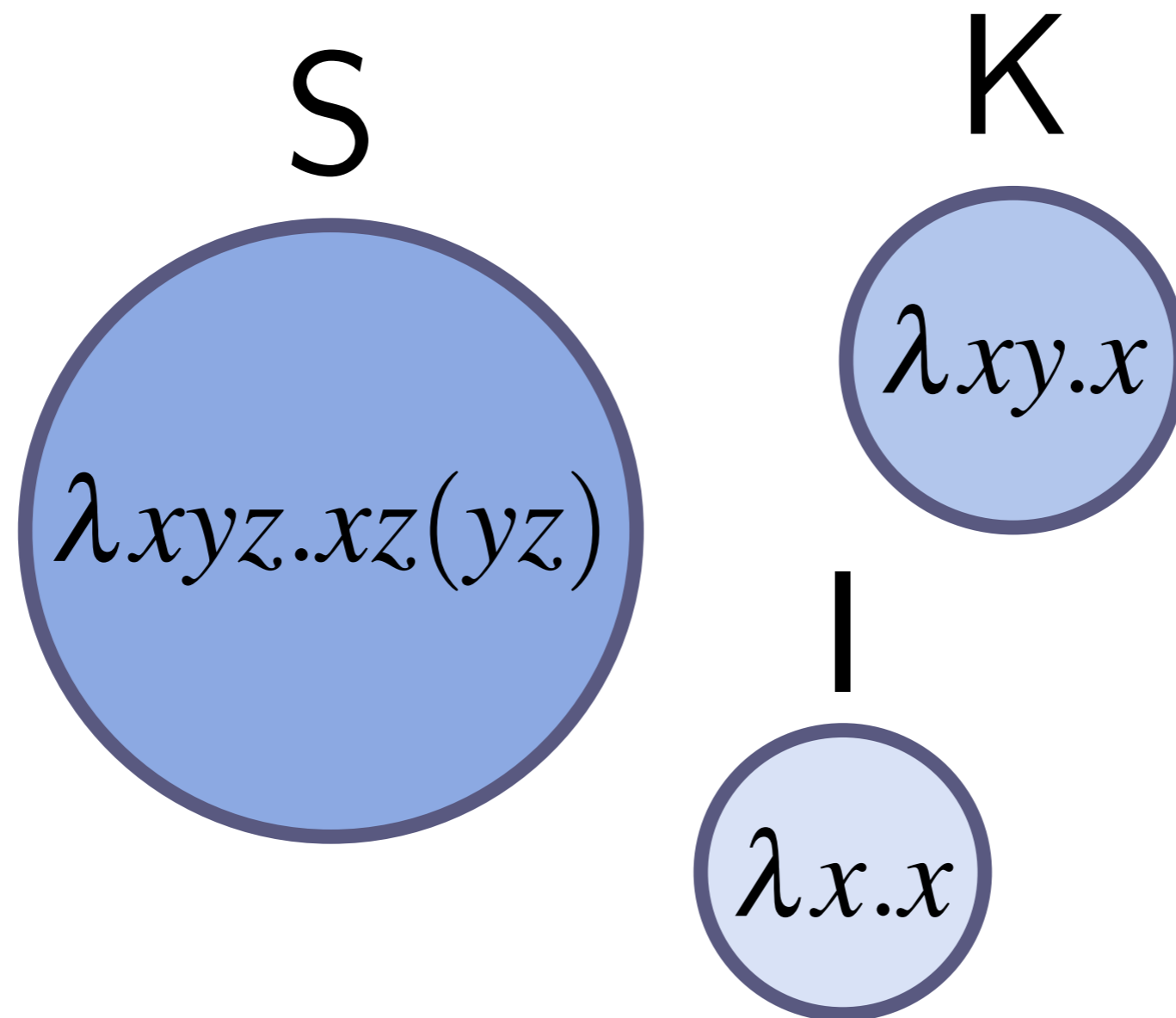
In the same way the Church-Rosser theorem can be proved when  $\eta$ -reduction is included.

Sincerely yours,

Henk Barendregt.

Mathematisch Instituut  
Budapestlaan 6  
Utrecht- De Uithof  
Holland



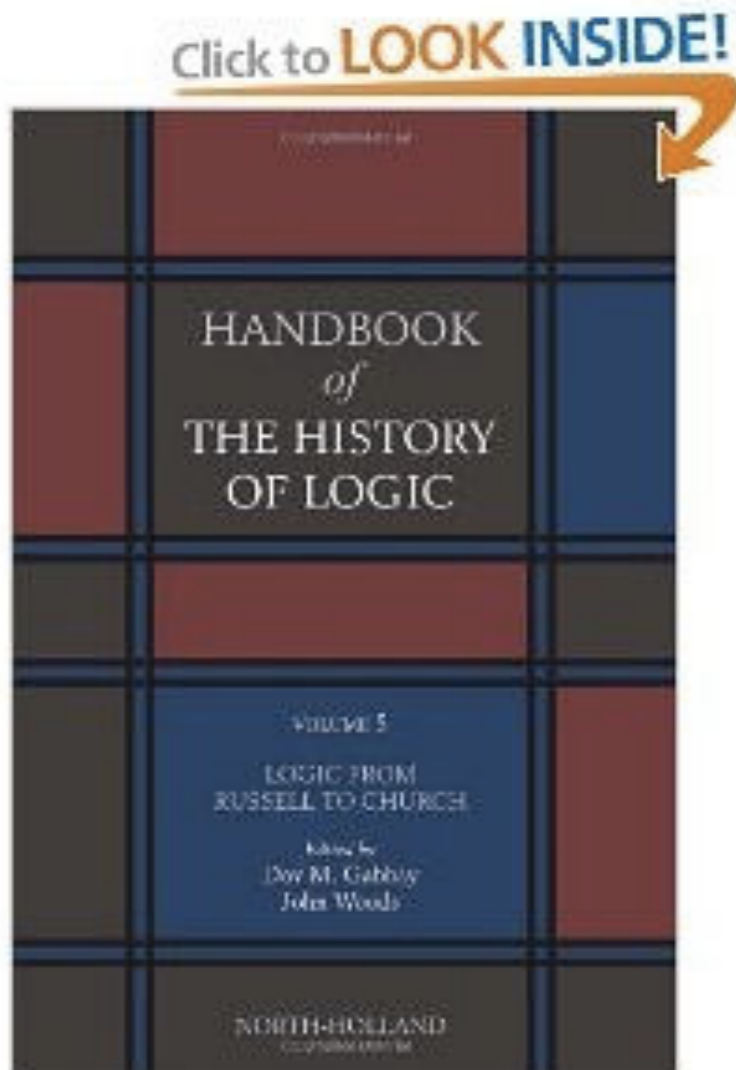


1924. "Über die Bausteine der mathematischen Logik"

Moses Schönfinkel

# History of Lambda-calculus and Combinatory Logic

Felice Cardone \* J. Roger Hindley †  
2006,



Ever since the original proof of the confluence of  $\lambda\beta$ -reduction in [Church and Rosser, 1936], a general feeling had persisted in the logic community that a shorter proof ought to exist. The work on abstract confluence proofs described in §5.2 did not help, as it was aimed mainly at generality, not at a short proof for  $\lambda\beta$  in particular.

For CL, in contrast, the first confluence proof was accepted as reasonably simple; its key idea was to count the simultaneous contraction of a set of non-overlapping redexes as a single unit step, and confluence of sequences of these unit steps was easy to prove, [Rosser, 1935, p.144, Thm. T12].

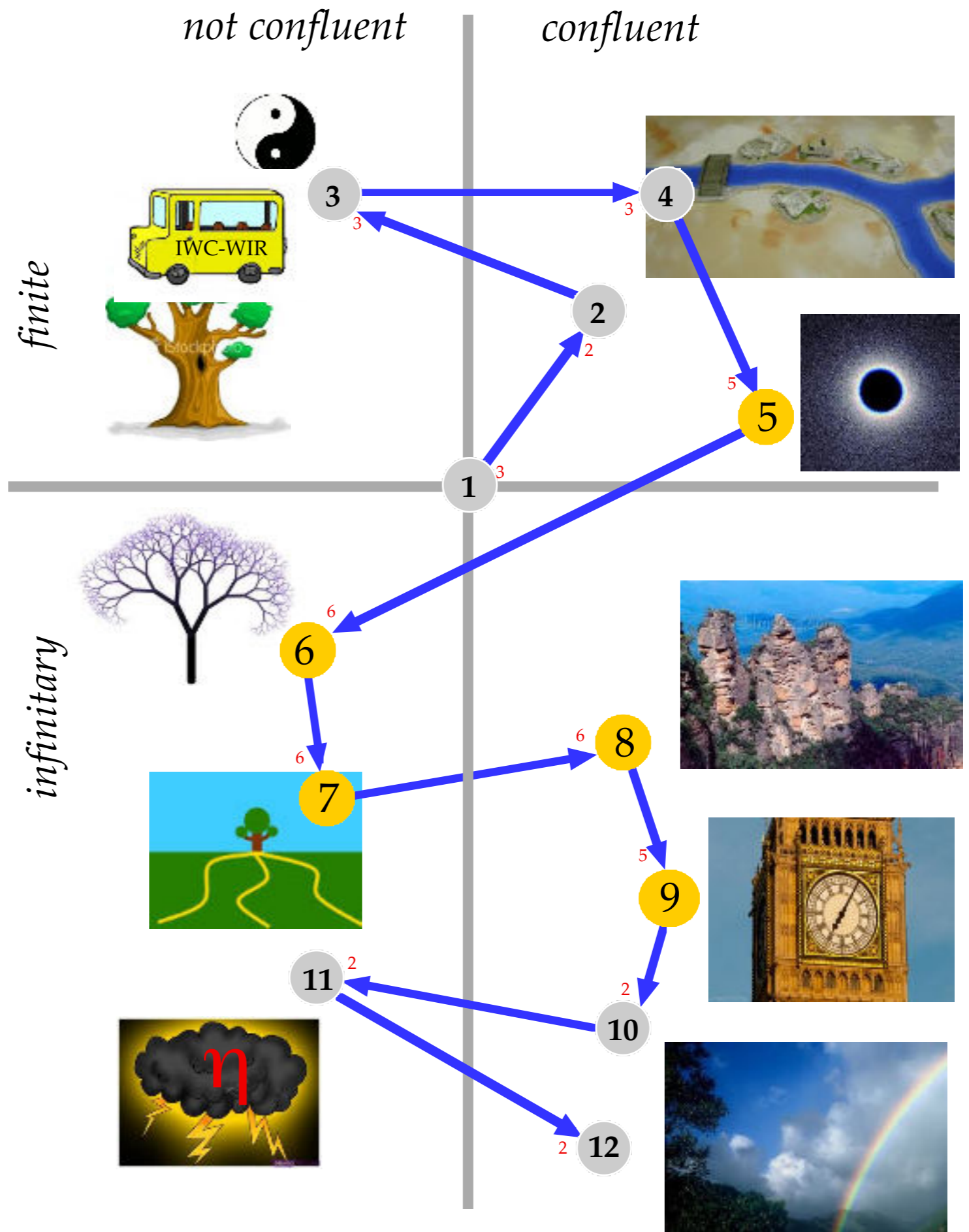
Then in 1965 William Tait presented a short confluence proof for CL to a seminar on  $\lambda$  organized by Scott and McCarthy at Stanford. Its key was a very neat definition of a unit-step reduction by induction on term-structure. Tait's units were later seen to be essentially the same as Rosser's, but his inductive definition was much more direct. Further, it could be adapted to  $\lambda\beta$ . (This possibility was noted at the seminar in 1965, see [Tait, 2003, p.755 footnote]). Tait did not publish his method directly, but in the autumn of 1968 he showed his CL proof to Per Martin-Löf, who then adapted it to  $\lambda\beta$  in the course of his work on type theory and included the  $\lambda\beta$  proof in his manuscript [Martin-Löf, 1971b, pp.8–11, §2.5].

Martin-Löf's  $\lambda\beta$ -adaptation of Tait's proof was quickly appreciated by other workers in the subject, and appeared in [Barendregt, 1971, Appendix II], [Stenlund, 1972, Ch. 2] and [Hindley *et al.*, 1972, Appendix 1], as well as in a report by Martin-Löf himself, [Martin-Löf, 1972b, §2.4.3].<sup>42</sup>

In  $\lambda$ , each unit step defined by Tait's structural-induction method turned out to be a minimal-first development of a set of redexes (not necessarily disjoint). Curry had introduced such developments in [Curry and Feys, 1958, p.126], but had used them only indirectly; Hindley had used them extensively in his thesis, [Hindley, 1969a, p.547, "MCD"], but only in a very abstract setting. They are now usually called *parallel reductions*, following Masako Takahashi. In [Takahashi, 1989] the Tait-Martin-Löf proof was further refined, and the method of dividing reductions into these unit steps was also applied to simplify proofs of other main theorems on reductions in  $\lambda$ .

Tait's structural-induction method is now the standard way to prove confluence in  $\lambda$  and CL. However, some other proofs give extra insights into reductions that this method does not, see for example the analysis in [Barendregt, 1981, Chs. 3, 11–12].

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# *De Vrijer 1989*

## **Extending the lambda calculus with surjective pairing is conservative**

Consideration is given to the equational theory  $\lambda\pi$  of lambda calculus extended with constants  $\pi$ ,  $\pi_0$ ,  $\pi_1$  and axioms for surjective pairing:

$$\pi_0(\pi XY) = X, \pi_1(\pi XY) = Y, \pi(\pi_0 X)(\pi_1 X) = X.$$

*nasty overlap*

The reduction system that one obtains by reading the equations as reductions (from left to right) is not Church-Rosser.

Despite this failure, the author obtains a syntactic consistency proof of  $\lambda\pi$  and shows that it is a conservative extension of the pure  $\lambda$  calculus

*Klop, de Vrijer 1989: but UN holds*



# A Question of Balance (The Moody Blues 1970)

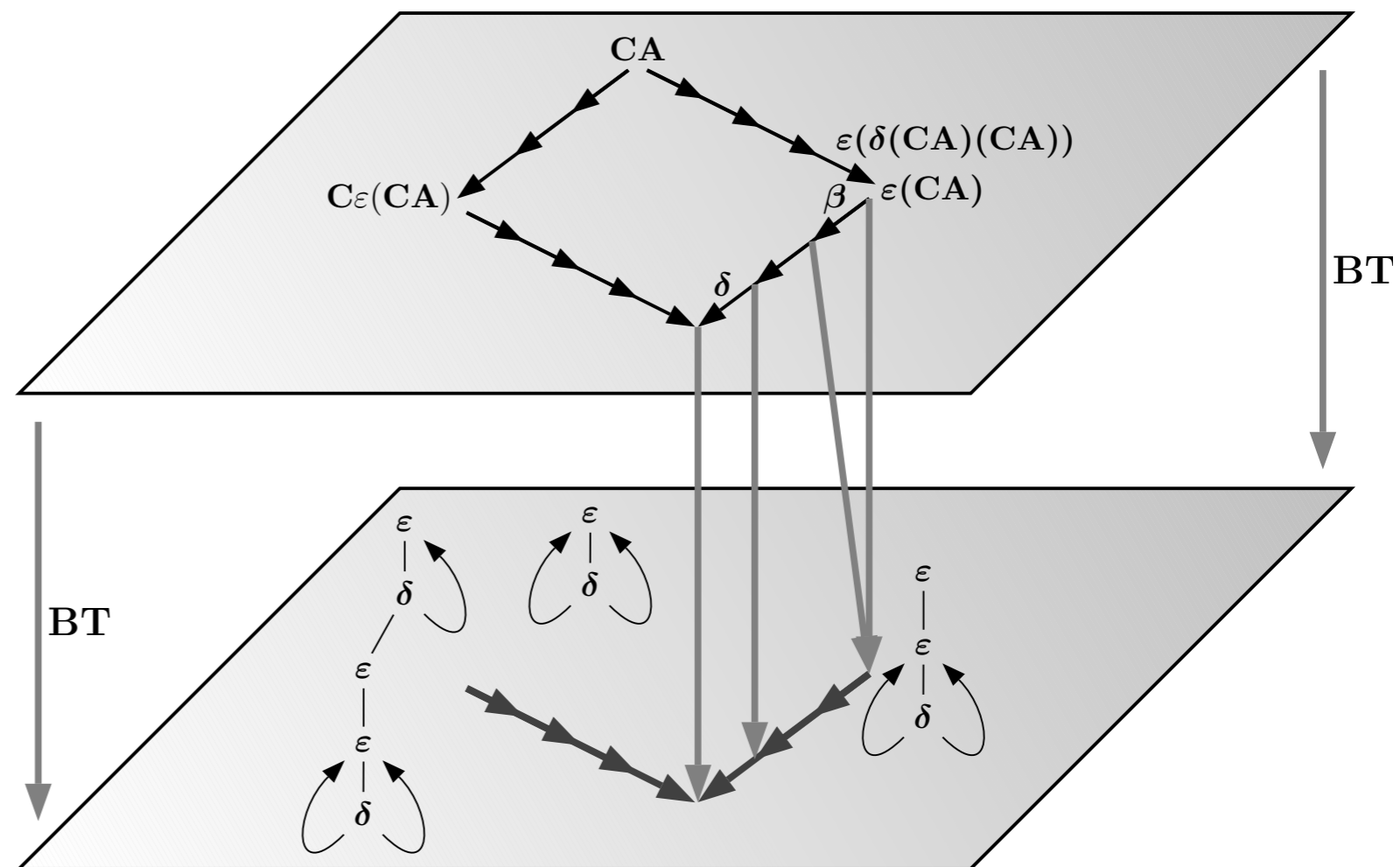
$$\delta x x \rightarrow \delta_H x$$

$$Cx \rightarrow \varepsilon(\delta x(Cx))$$

$$A \rightarrow CA$$

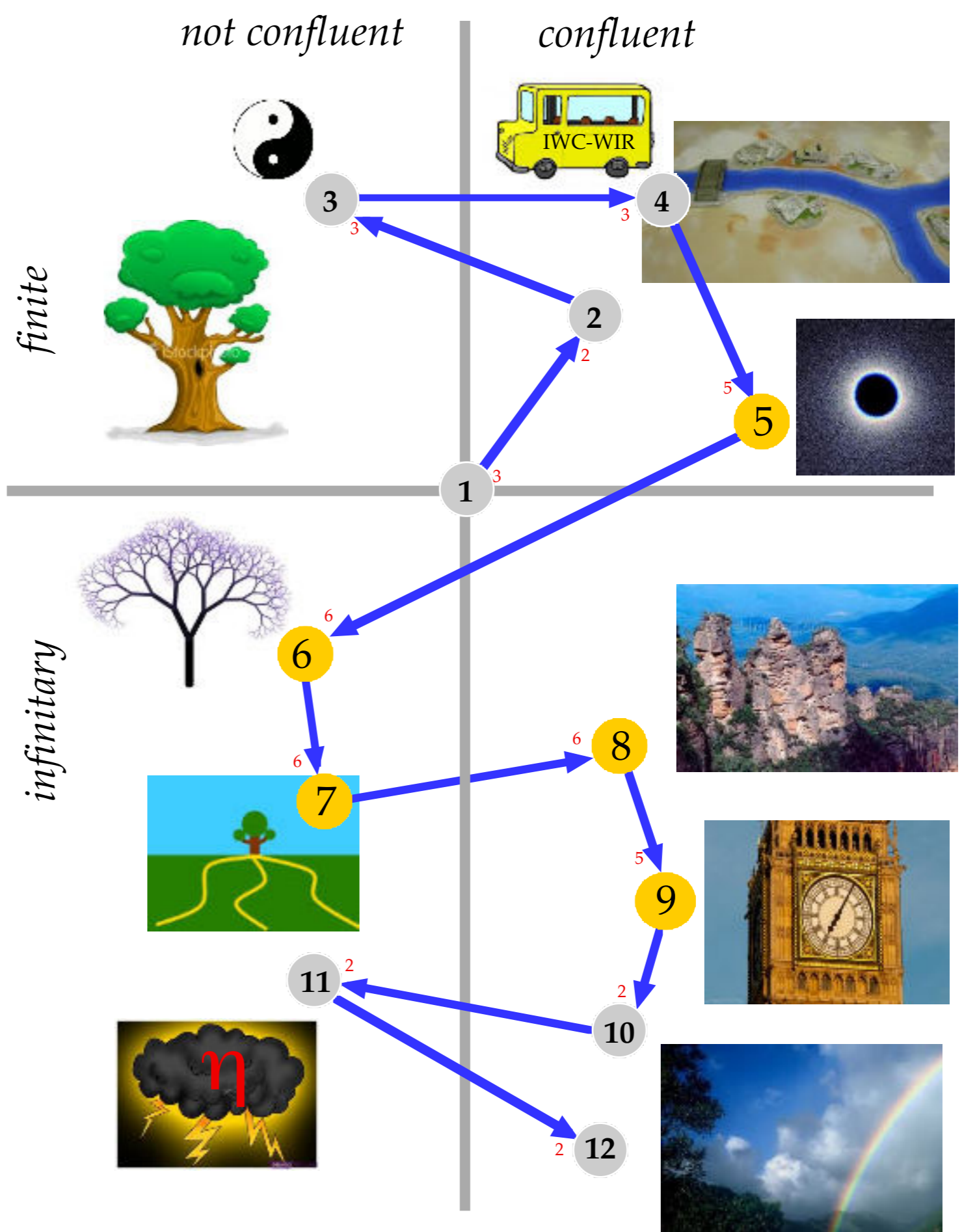
$$A \longrightarrow CA \longrightarrow \varepsilon(\delta A(CA)) \longrightarrow \varepsilon(\delta(CA)(CA)) \longrightarrow \varepsilon(CA)$$

$$C(\varepsilon(CA))$$



Question: what about  $\lambda^\infty \beta \delta$  and  $\lambda^\infty \beta \pi$  ?

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Peter Aczel

## coherent contraction schemes

We prove the Church-Rosser theorem in a general framework. Our result easily yields the standard result for the lambda calculus but also has wider application.

### §1. The Main Theorem

1.1 An expression system consists of an infinite set of variables and a set of forms. Each form has an arity i.e. a finite sequence  $k_1, \dots, k_m$  ( $m \geq 0$ ) of natural numbers. If  $m = 0$  the form is a constant. If  $k_1 = \dots = k_m = 0$  the form is simple.

Expressions are inductively generated using the two rules:-

- 1) Every variable is an expression.
- 2) If  $F$  is a form with arity  $k_1, \dots, k_m$  ( $m \geq 0$ ) and  $a_1, \dots, a_m$  are expressions then  $F((\vec{x}_1)a_1, \dots, (\vec{x}_m)a_m)$  is an expression, where for  $i = 1, \dots, m$   $\vec{x}_i$  is a list of  $k_i$  variables.

The expression generated by 2) is said to have form  $F$  and parts  $a_1, \dots, a_m$ . Free and bound occurrences of variables are defined in the usual way, so that occurrences of a variable in the list  $\vec{x}_i$  that are free in  $a_i$  become bound in  $F((\vec{x}_1)a_1, \dots, (\vec{x}_m)a_m)$ . Alphabetic variants of expressions are identified in the standard way.

Below we shall usually write  $F(a_1, \dots, a_m)$  instead of  $F((\vec{x}_1)a_1, \dots, (\vec{x}_m)a_m)$ . It must be kept in mind that with this abuse of notation a variable that is free in  $a_i$  can become bound in  $F(a_1, \dots, a_m)$ . Also, an expression  $F(a_1, \dots, a_m)$  may be the same as an expression  $F(b_1, \dots, b_m)$  while  $a_i$  is not the same as  $b_i$  for  $i = 1, \dots, m$ .

1.2 We shall be concerned with a partial function on the expressions which we shall call a contraction operation. An expression in the domain of the operation will be called a redex and its value under the operation will be called the contraction<sup>u/m</sup> of the redex. We shall insist that no variable is a redex. Each contraction operation generates a relation of definitional equality.

The methods to prove confluence of orthogonal higher-order rewriting systems both can be adapted to the case where critical pairs are allowed, but only if they are of the form  $(s, s)$ . Such a critical pair is said to be *trivial*. The notion of trivial critical pair is used to define the class of weakly orthogonal higher-order rewriting systems; the definition is analogous to the one for the first-order case.

**Definition 3.** *A higher-order rewriting system is weakly orthogonal if it is left-linear and all its critical pairs are trivial.*

Examples of weakly orthogonal rewriting systems that are not orthogonal are  $\{a \rightarrow b, f(a) \rightarrow f(b)\}$  and  $\{f(x) \rightarrow f(b), f(a) \rightarrow f(b)\}$ . Moreover, lambda-calculus with both  $\beta$ - and  $\eta$ -reduction is a weakly orthogonal rewriting system.

Typed Lambda Calculi and Applications

Lecture Notes in Computer Science Volume 664, 1993, pp 306-317

## Orthogonal higher-order rewrite systems are confluent

### Abstract

The results about higher-order critical pairs and the confluence of OHRs provide a firm foundation for the further study of higher-order rewrite systems. It should now be interesting to lift more results and techniques both from term-rewriting and  $\lambda$ -calculus to the level of HRSs. For example termination proof techniques are much studied for TRSs and are urgently needed for HRSs; similarly the extension of our result to weakly orthogonal HRSs or even to Huet's “parallel closed” systems is highly desirable.

Conversely, a large body of  $\lambda$ -calculus reduction theory has been lifted to CRSs [10] already and should be easy to carry over to HRSs.

Finally there is the need to extend the notion of an HRS to more general left-hand sides. For example the *eta*-rule for the *case*-construct on disjoint unions [15]  $case(U, \lambda x.F(inl(x)), \lambda y.G(inr(y))) \rightarrow F(U)$  is outside our framework, whichever way it is oriented.



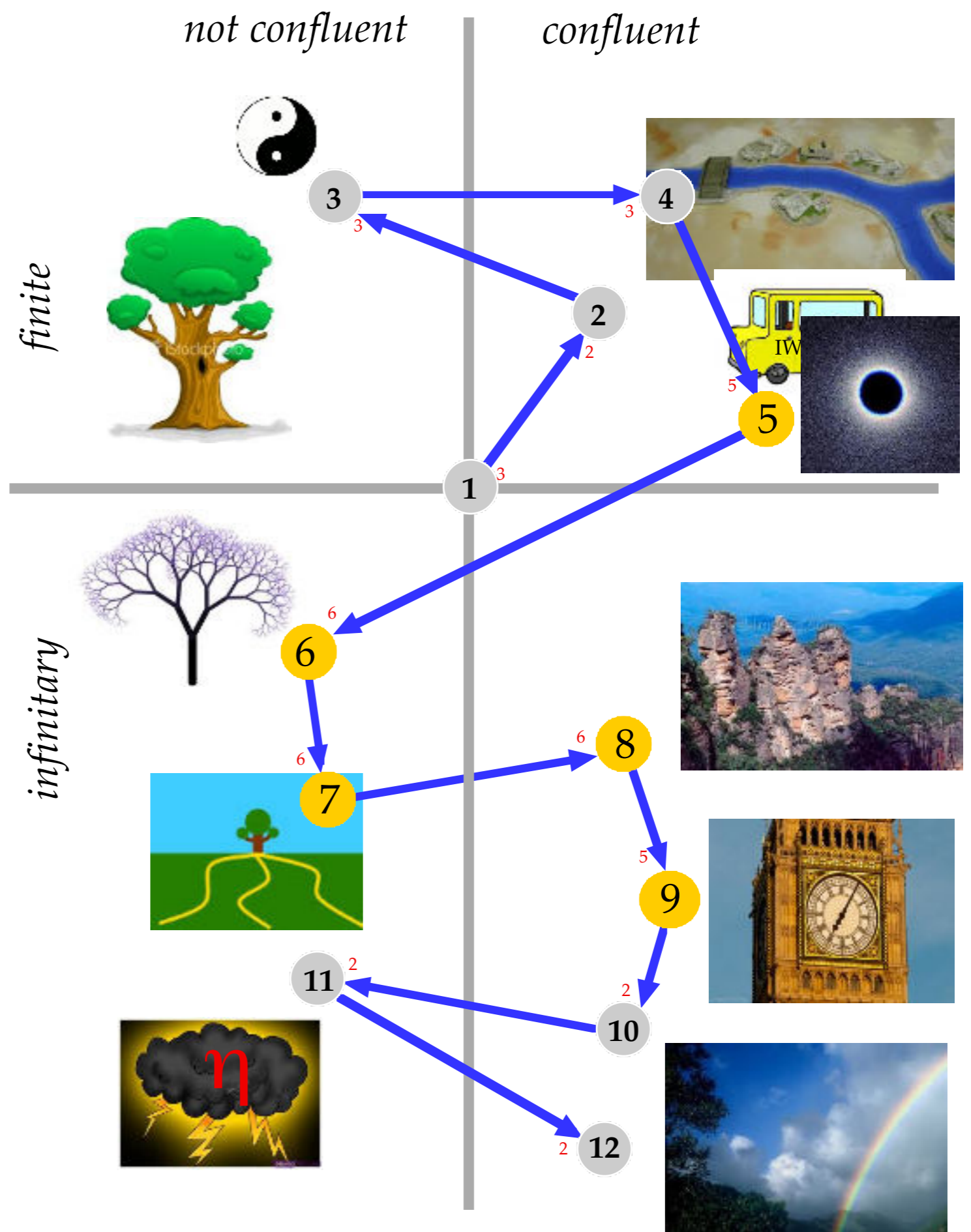
$$\frac{\frac{\vdots}{A} \quad [A] \quad [B]}{A \vee B} \quad \frac{\vdots}{C} \quad \frac{\vdots}{C}}{C} \rightarrow \frac{\vdots}{A} \quad \frac{\vdots}{C} \quad \frac{\vdots}{C} \rightarrow \frac{\vdots}{B} \quad \frac{\vdots}{C} \quad \frac{\vdots}{C} \rightarrow \frac{\vdots}{B} \quad \frac{\vdots}{C} \quad \frac{\vdots}{C}}{C}$$

These rules can be written in the formalism of Combinatory Reduction Systems. They then take the following form:

$$\begin{aligned} \text{el}(\text{inl}(Z), [x]Z_0(x), [y]Z_1(y)) &\rightarrow Z_0(Z) \\ \text{el}(\text{inr}(Z), [x]Z_0(x), [y]Z_1(y)) &\rightarrow Z_1(Z) \end{aligned}$$

$\lambda\beta\eta \models \text{CR}$

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*Modulo unsolvables:*

$$(\lambda x.Z(x))Z' \rightarrow Z(Z')$$

$(\beta)$

$$M \rightarrow \Omega$$

if  $M \neq \Omega$  is unsolvable (uns)

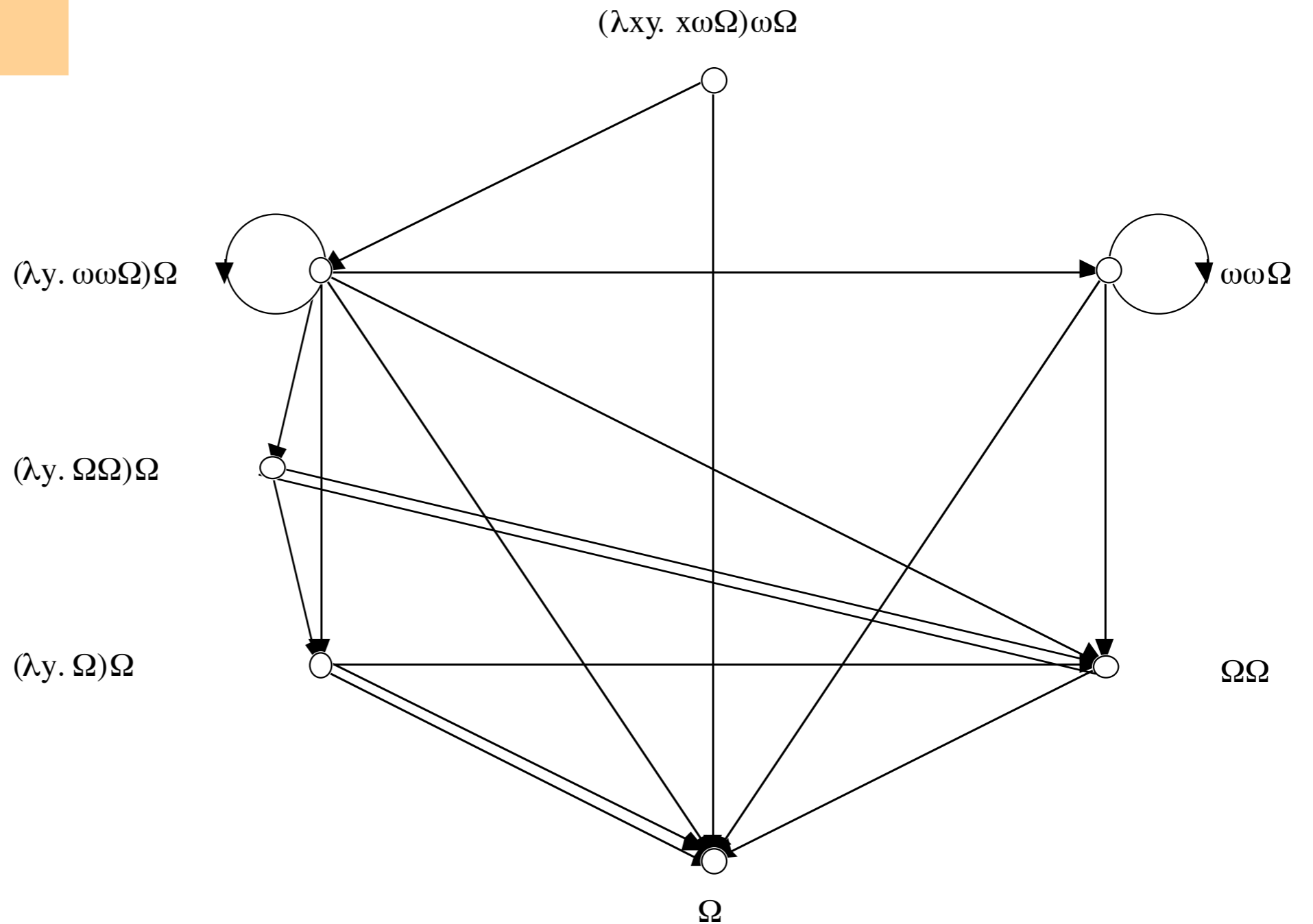
$$\Omega M \rightarrow \Omega$$

$(\Omega_l)$

$$\lambda x.\Omega \rightarrow \Omega$$

$(\Omega_d)$

$\lambda\beta\Omega \models \text{CR}$



Blue preprint 1976, Barendregt, Bergstra, Klop, Volken:  
youth sentiment and contortuous casuistics

$$\lambda\beta\eta\Omega \models CR$$



*later question:*

$$\lambda^\infty\beta\eta\Omega \models CR^\infty$$

Scheme of relative positions of redices.

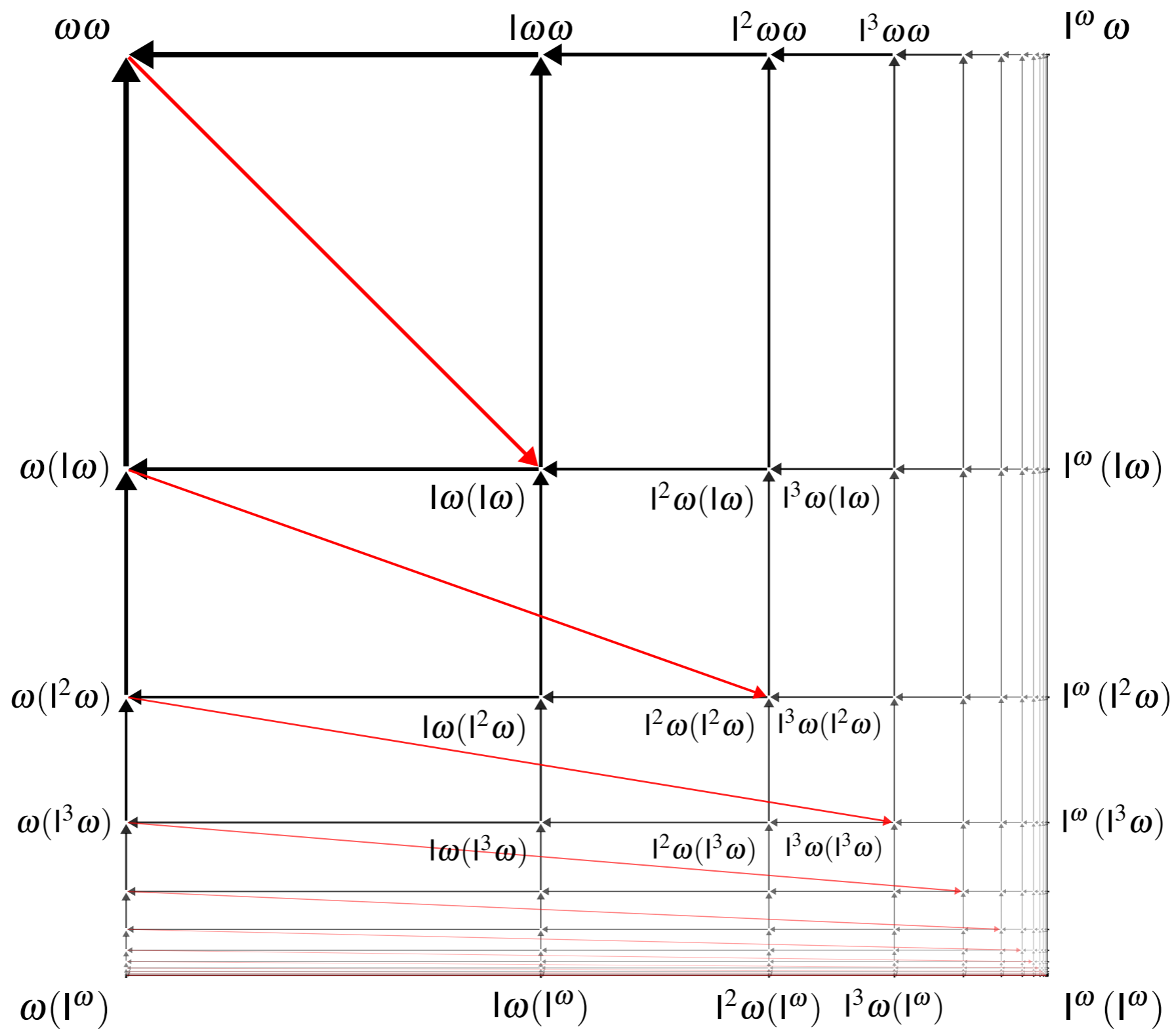
	1 $R = (\lambda x.P)Q$	2 $E = \lambda y.Dy$	3 $H$ ...
1	111 $R' \cap R = \emptyset$	121 $R' \cap E = \emptyset$	131 $R' \cap H = \emptyset$
$R' = (\lambda x'.P')Q'$	112 $R' \subset R$	122 $R' \subset E$	132 $R' \subset H$
	1121 $R' \subset P$	1221 $R' \subset D$	1321 $R' \subset H$
	1122 $R' \subset Q$	1222 $R' \in Dy$	1322 $R' \in H$
	113 $R' \equiv R$	123 $R' \supset E$	133 $R' \equiv H$
114 $R' \supset R$	1231 $E \subset P'$	134 $R' \supset H$	
	1141 $R \subset P'$	1232 $E = \lambda x'.P'$	1341 $H \subset P'$
	1142 $R \subset Q'$	1233 $E \subset Q'$	1342 $H = \lambda x'.P'$
			1343 $H \subset Q'$
2		221 $E' \cap E = \emptyset$	231 $E' \cap H = \emptyset$
$E' = \lambda y'.D'y'$		222 $E' \subset E \ (\Rightarrow E' \subset D)$	232 $E' \subset H$
		223 $E' \equiv E$	233 $E' \equiv H$
		224 $E' \supset E \ (\Rightarrow E \subset D')$	234 $E' \supset H$
			2341 $H \subset D'$
			2342 $H = D'y'$
3			331 $H' \cap H = \emptyset$
...			332 $H' \subset H$
			333 $H' \equiv H$
			334 $H' \supset H$

In Barendregt 84: section 15.2, 8 pages



Werkweek  $\lambda$ -calculus in de molen te Varik  
juni 75.



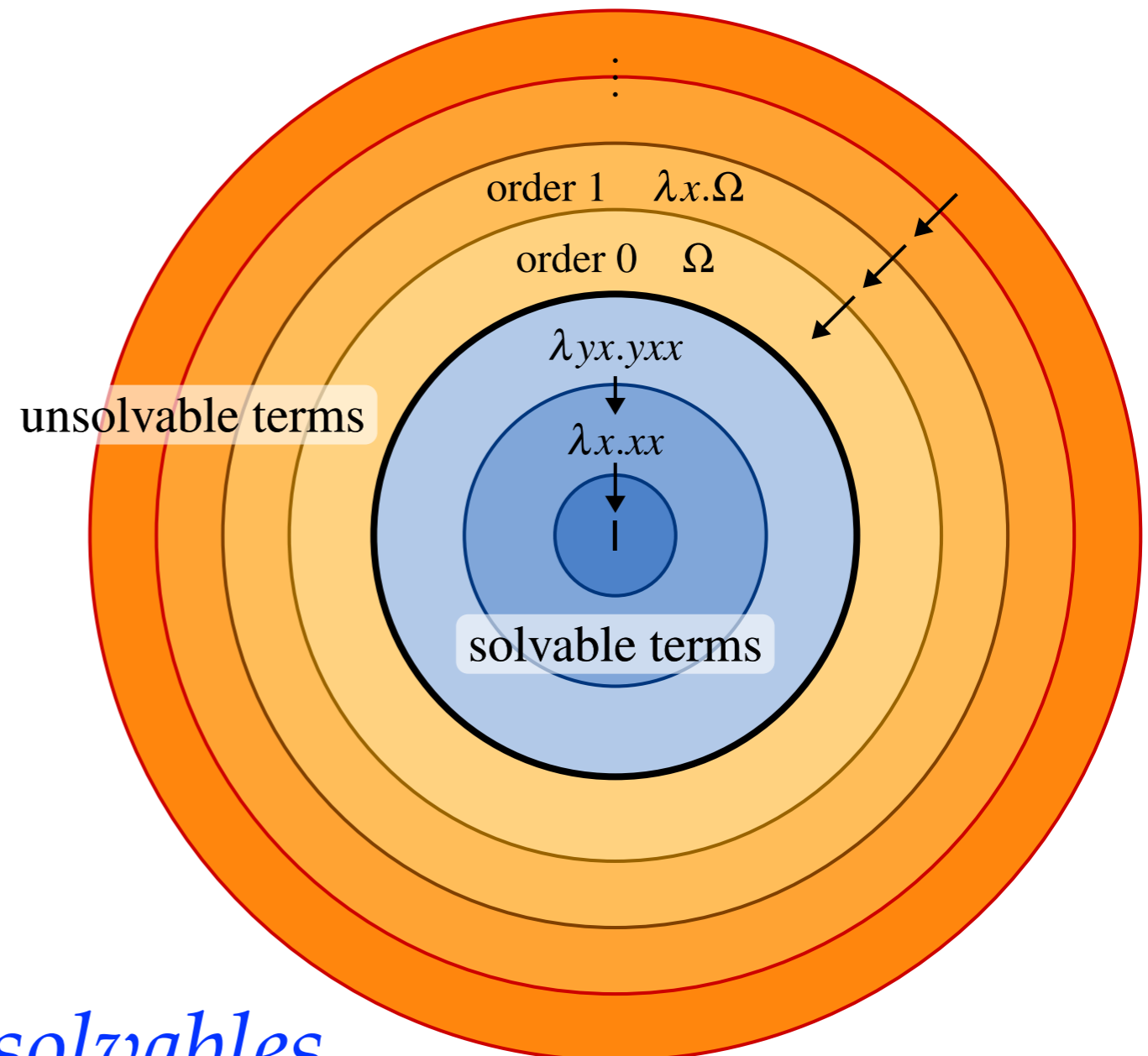


# Statman 1978

instead of  $MA \twoheadrightarrow_{\beta} N$ , write  $M \overset{A}{\rightarrow} N$

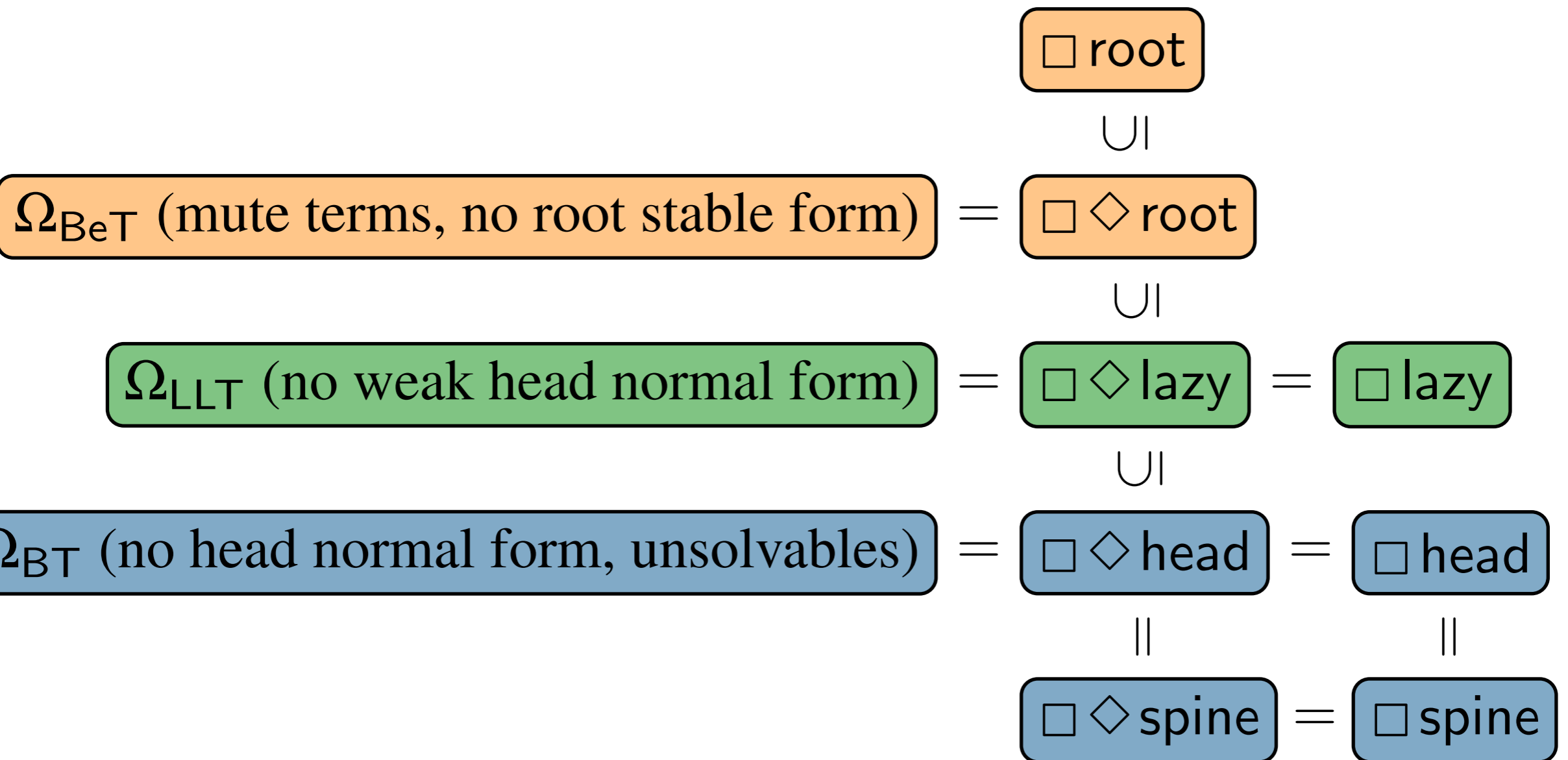
$M \rightarrow N$ :  $M$  is more solvable than  $N$ .

order  $\infty$   $YK \equiv \lambda x_1 x_2 x_4 \dots$



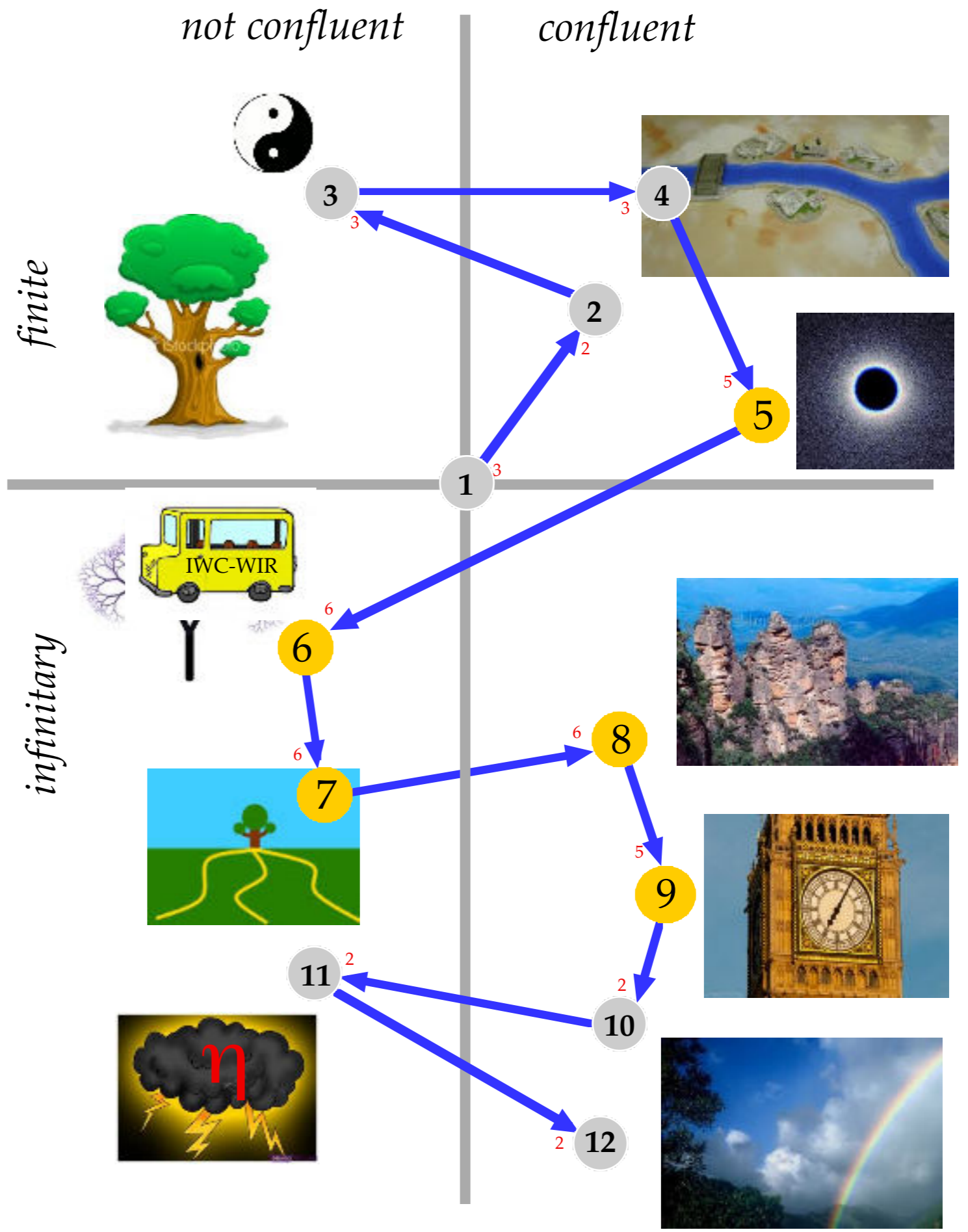
*Every countable poset is embeddable in poset of unsolvables*

# head normalization theorems





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# REWRITE, REWRITE, REWRITE, REWRITE, REWRITE, ...\*

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David A. PLAISTED<sup>§</sup>

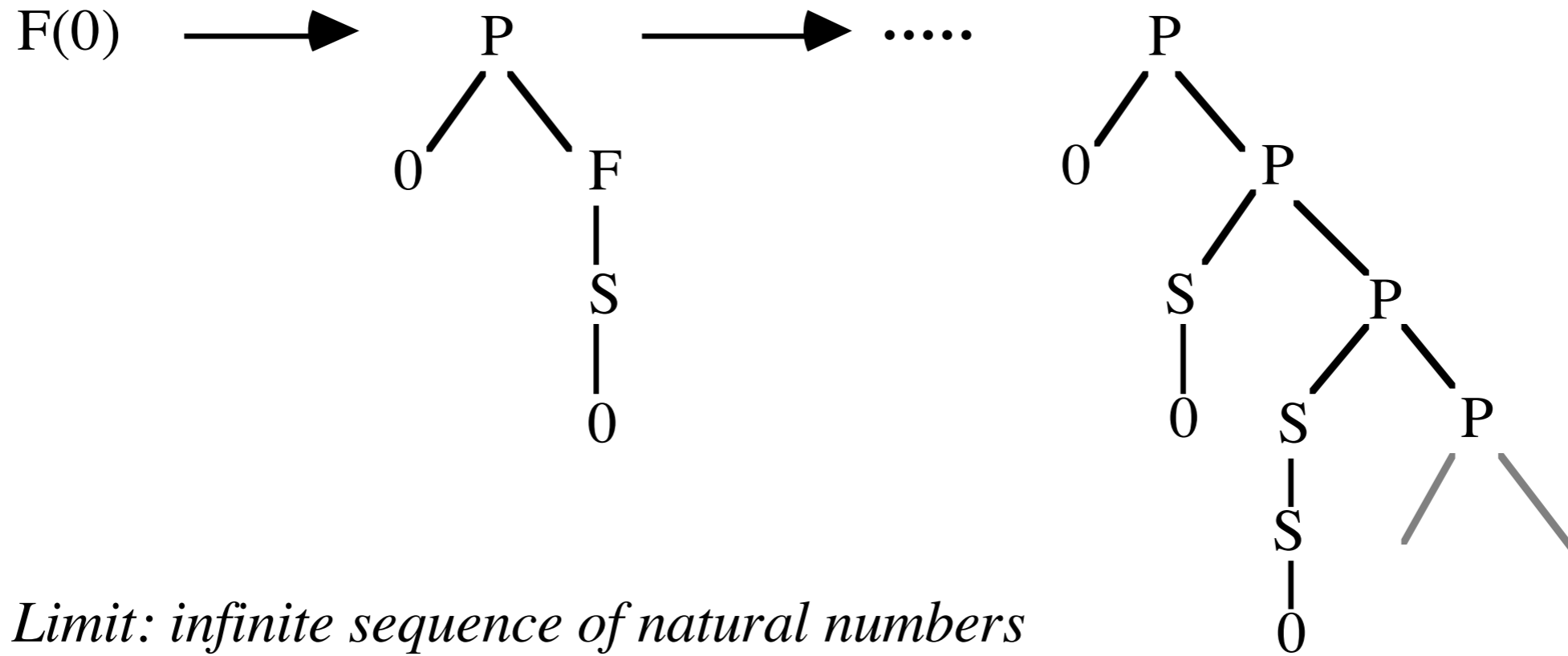
*Department of Computer Science, University of North Carolina at Chapel Hill,  
Chapel Hill, NC 27514-3175, U.S.A.*

Communicated by

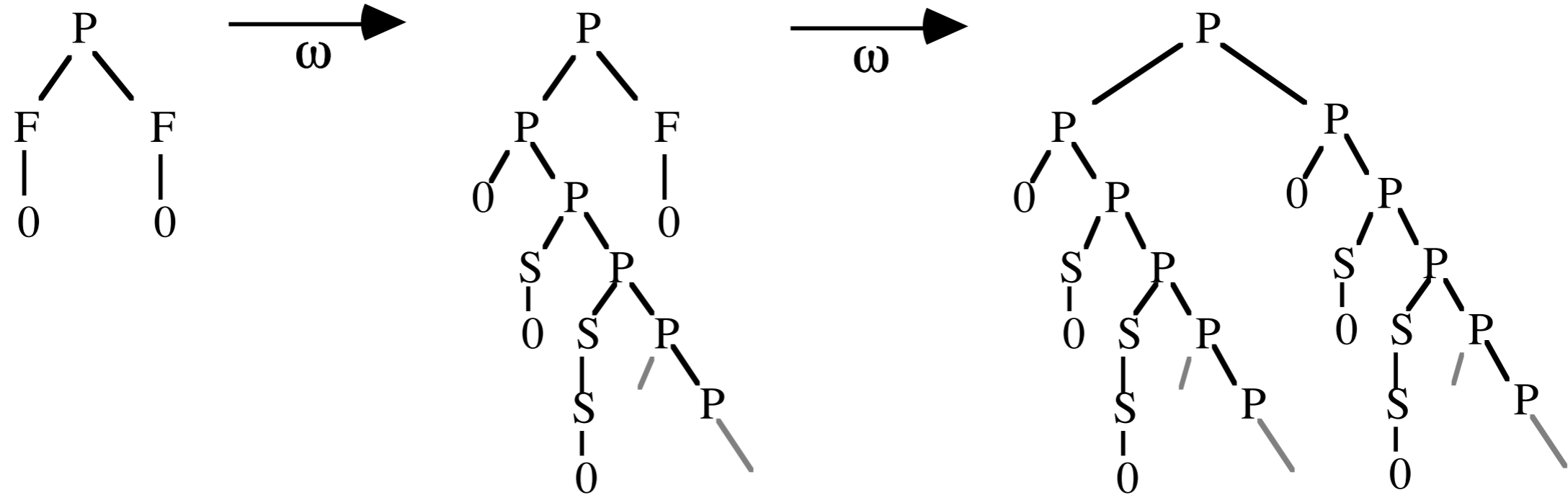
Received

Revised

**Abstract.** We study properties of rewrite systems that are not necessarily terminating, but allow instead for transfinite derivations that have a limit. In particular, we give conditions for the existence of a limit and for its uniqueness and relate the operational and algebraic semantics of infinitary theories. We also consider sufficient completeness of hierarchical systems.

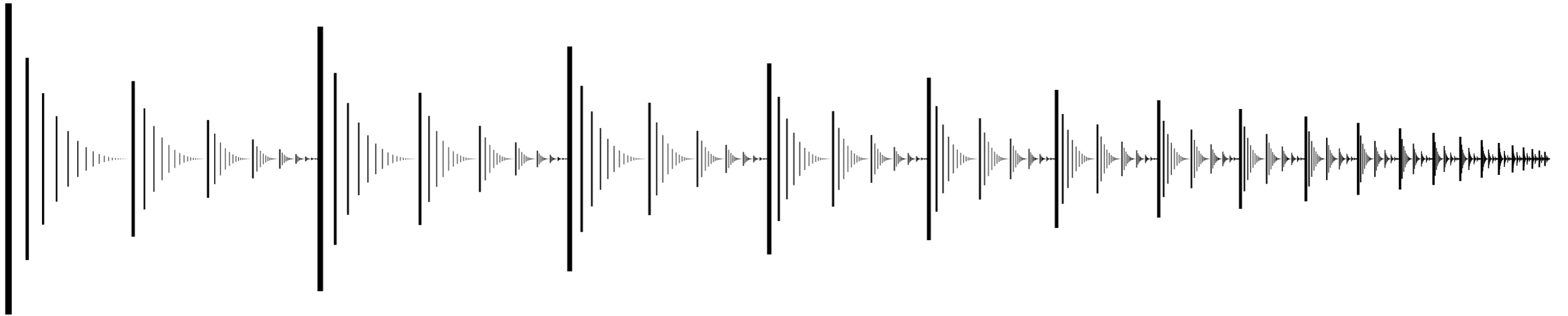
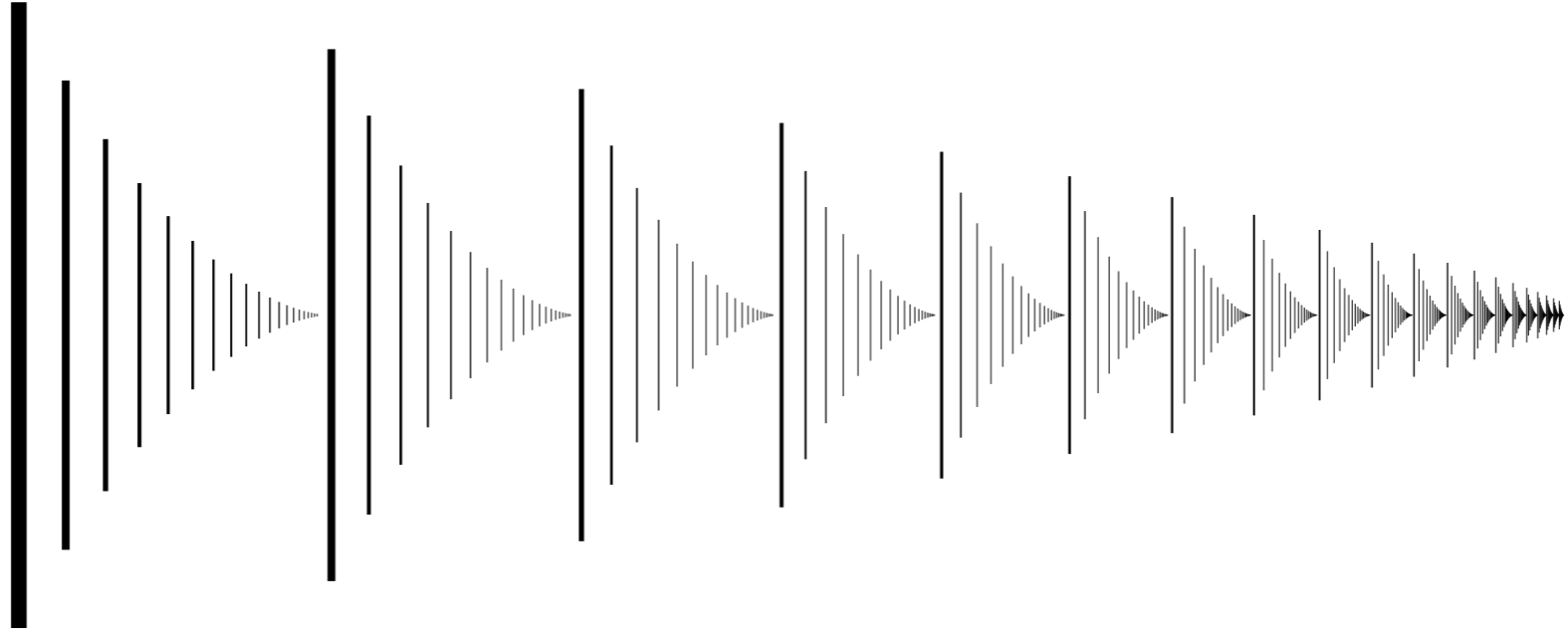


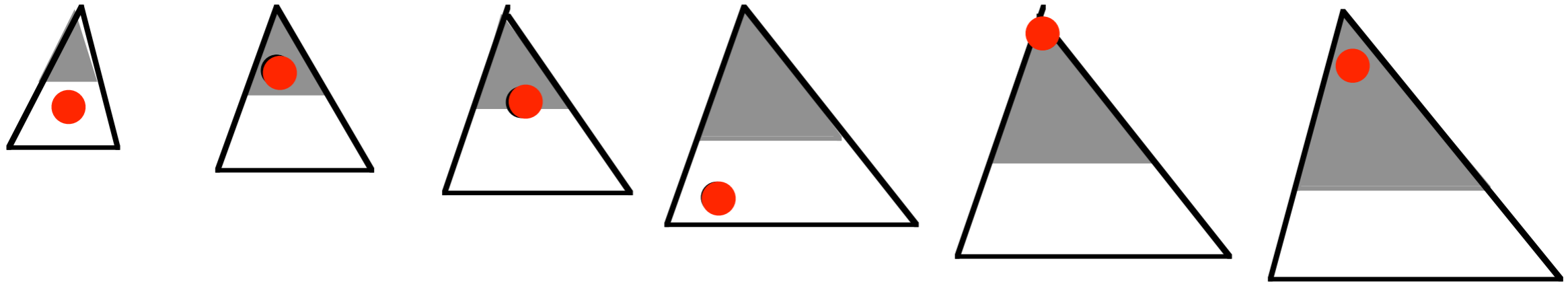
$$F(x) \rightarrow P(x, F(S(x)))$$



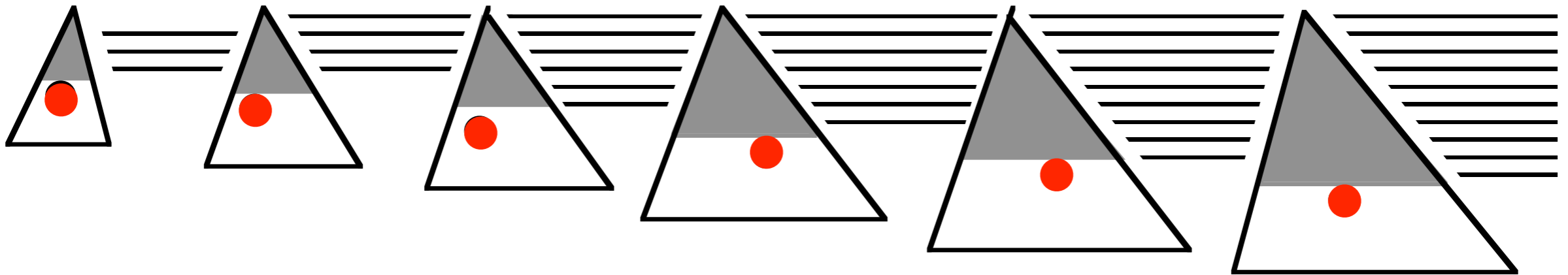
*Transfinite reduction sequence of length  $\omega + \omega$*

$$F(x) \rightarrow P(x, F(S(x)))$$





*Cauchy converging reduction sequence: activity may occur everywhere*



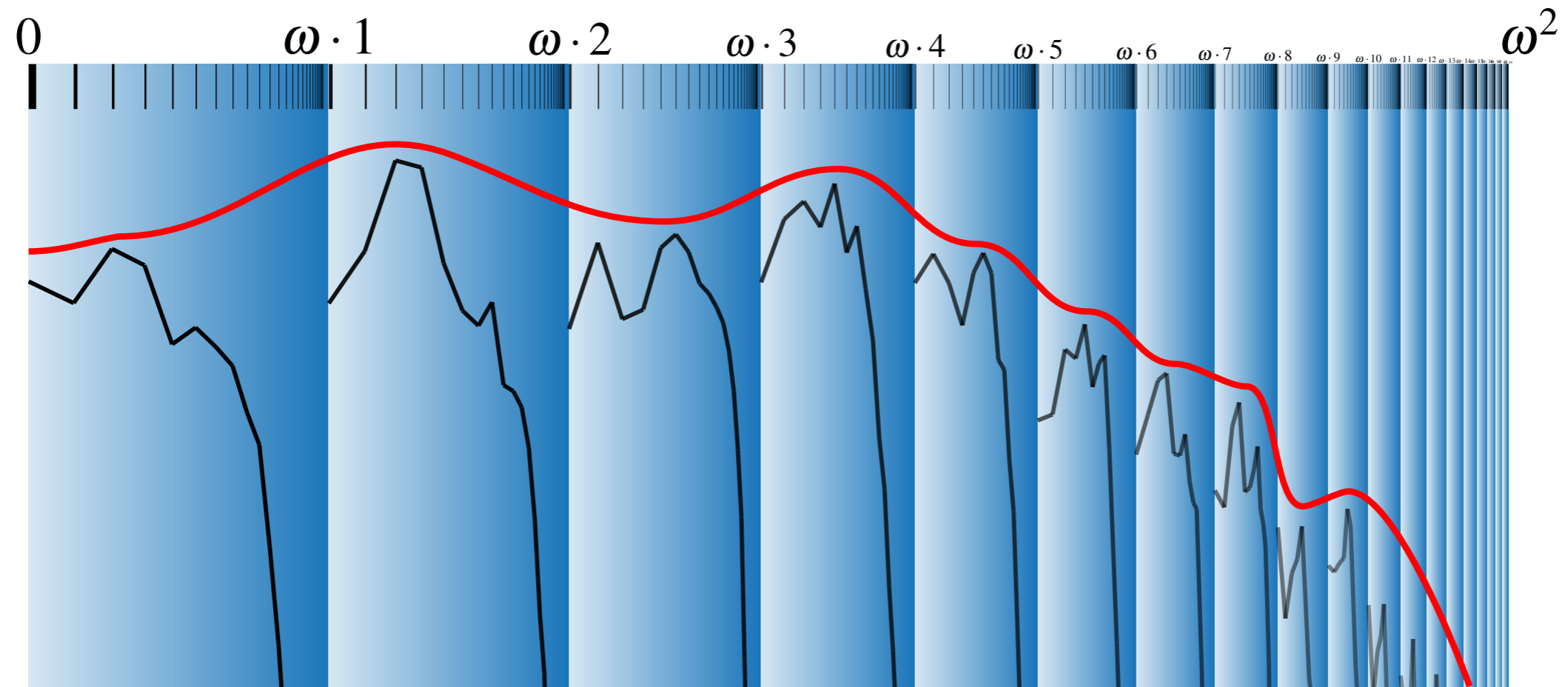
*Strongly converging reduction sequence, with descendant relations*

*difference between CC and SC: looping terms*

Kennaway-de Vries 1992; De Vrijer, Grabmayer, Endrullis, Hendriks, Simonsen 2012



*strong convergence: redex depth to infinity*

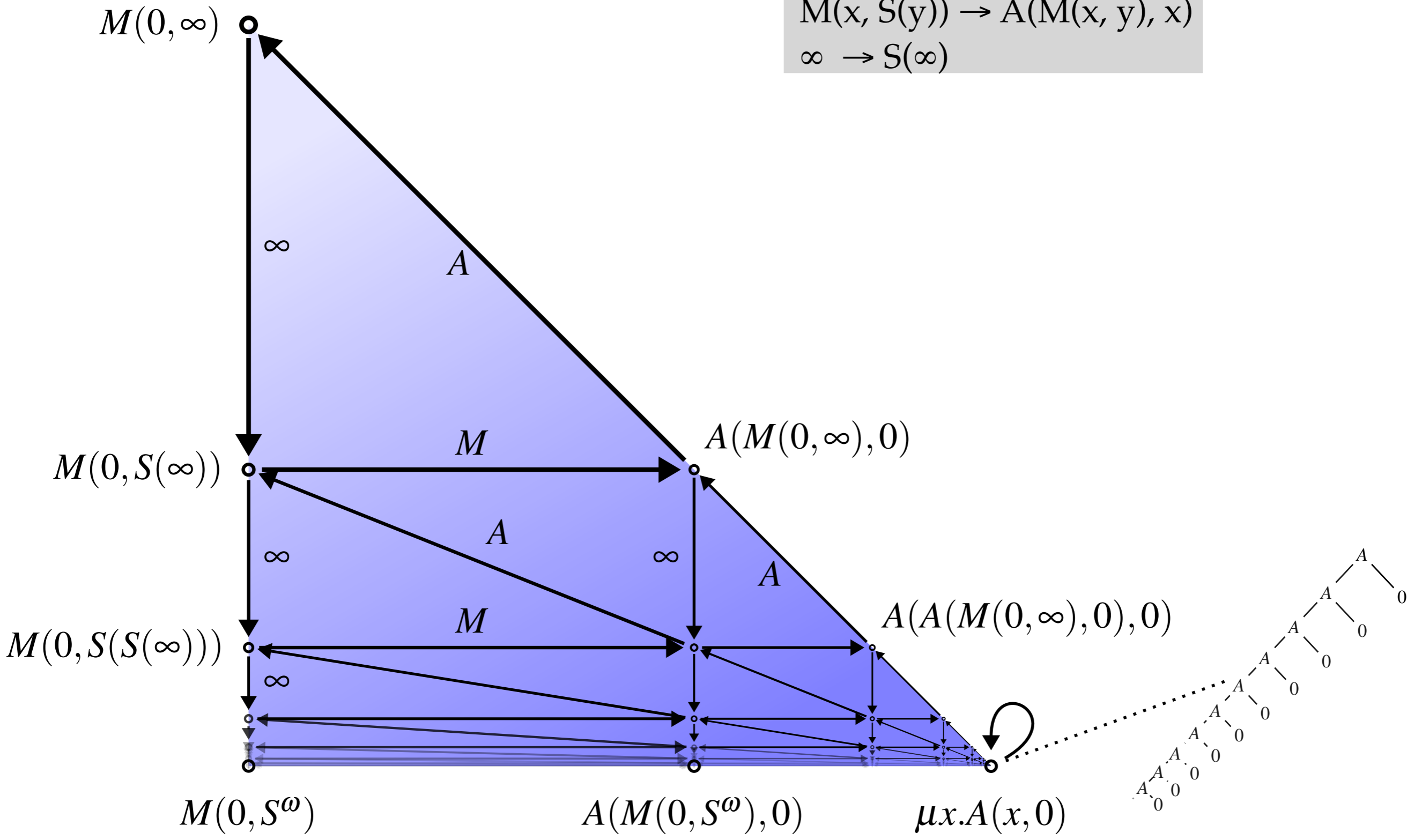


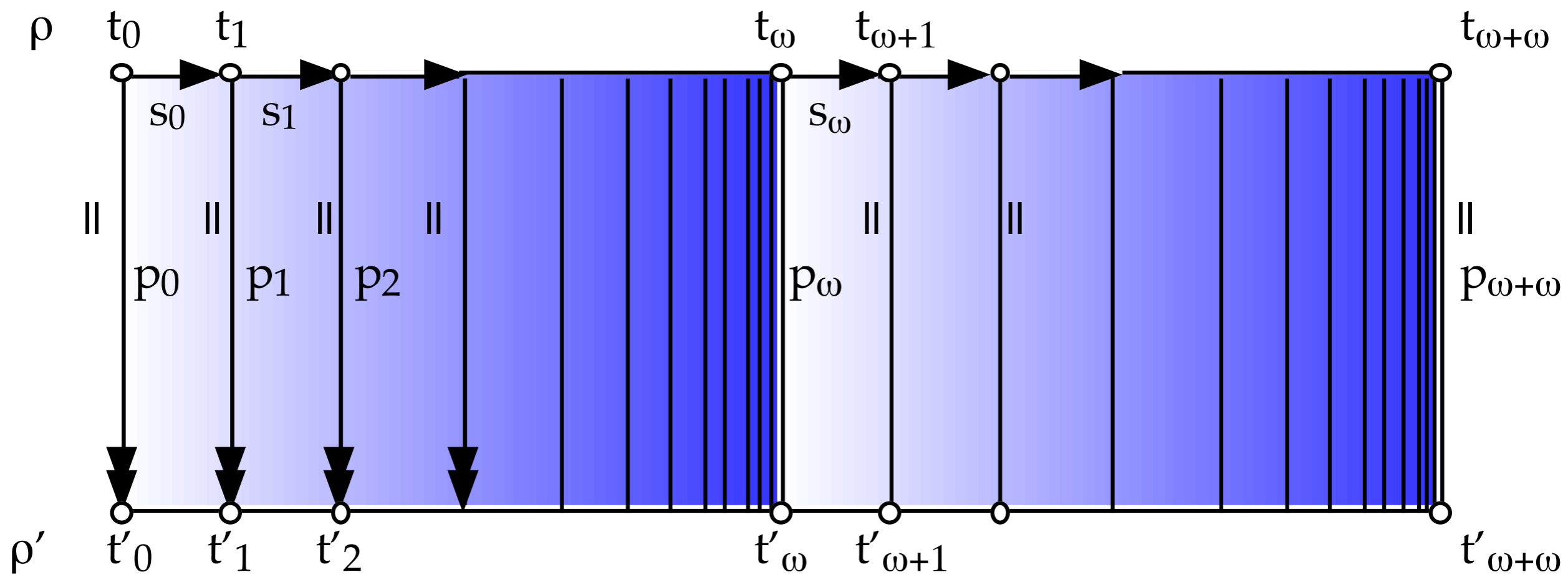
— convergence of depths towards  $\omega^2$

<b>Finitary rewriting</b>	<b>Infinitary or transfinite rewriting</b>
finite reduction	strongly convergent reduction
infinite reduction	divergent reduction (“stagnating”)
normal form	possibly infinite normal form
CR: two cointial finite reductions can be prolonged to a common term	$CR^\infty$ : two cointial strongly convergent reductions can be prolonged by strongly convergent reductions to a common term
UN: two cointial reductions ending in normal forms, end in the same normal form	$UN^\infty$ : two cointial strongly convergent reductions ending in (possibly infinite) normal forms, end in the same normal form
SN: all reductions lead eventually to a normal form	$SN^\infty$ : all reductions lead eventually to a possibly infinite normal form, equivalently: there is no divergent reduction
WN: there is a finite reduction to a normal form	$WN^\infty$ : there is a strongly convergent reduction to a possibly infinite normal form

# zero times infinity

$A(x, 0) \rightarrow x$   
 $A(x, S(y)) \rightarrow S(A(x, y))$   
 $M(x, 0) \rightarrow 0$   
 $M(x, S(y)) \rightarrow A(M(x, y), x)$   
 $\infty \rightarrow S(\infty)$

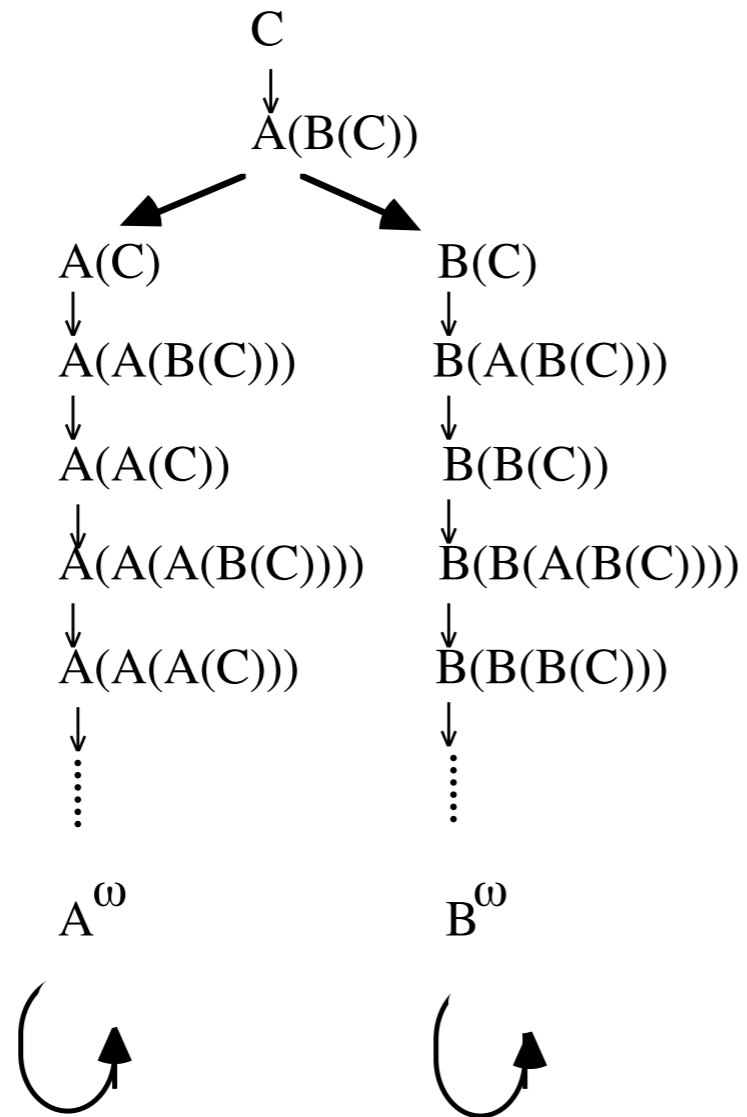




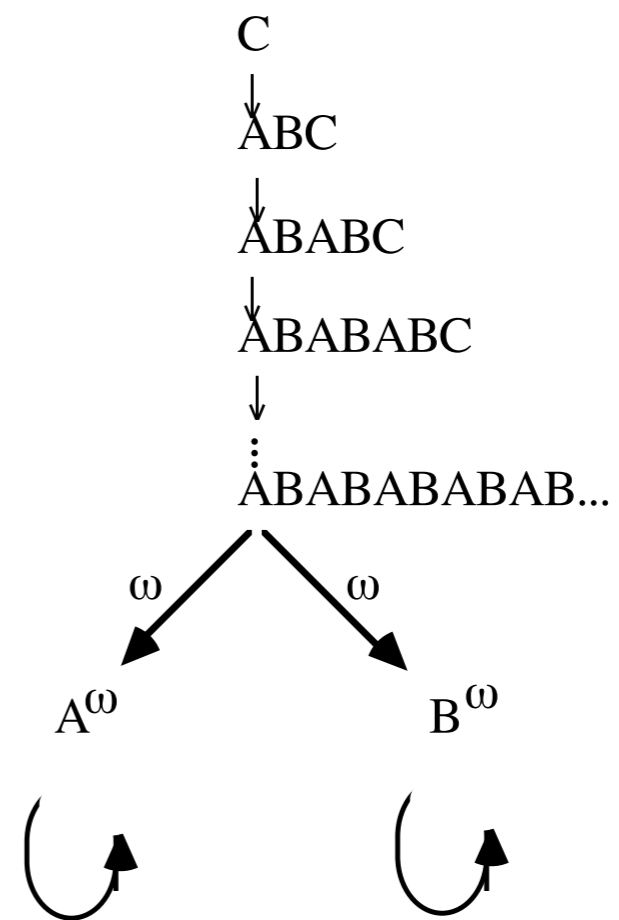
not  $CR^\infty$

$$\begin{aligned} A(x) &\rightarrow x \\ B(x) &\rightarrow x \\ C &\rightarrow A(B(C)) \end{aligned}$$

(a)



(b)



*Failure of infinitary confluence*

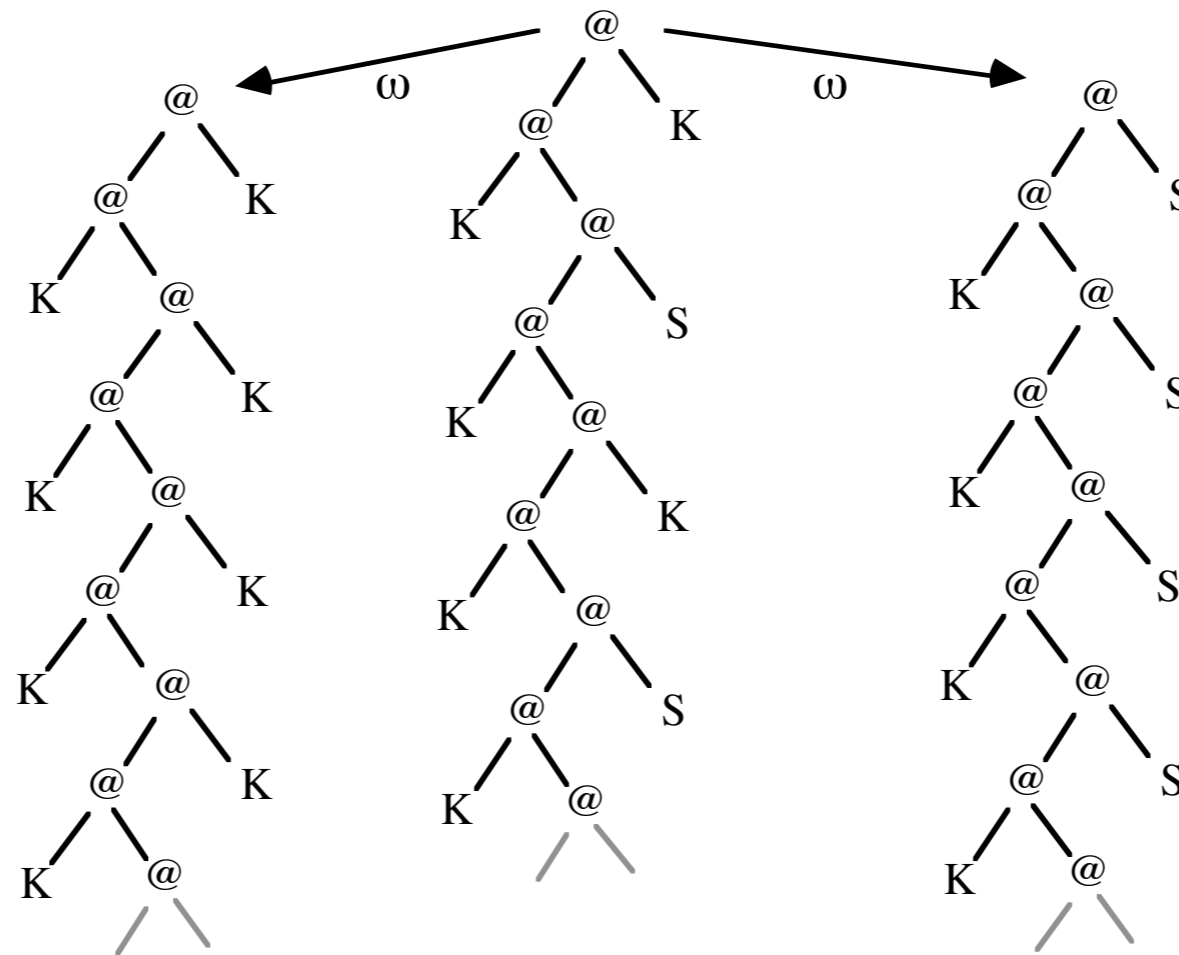


$Sxyz \rightarrow xz(yz)$   
 $Kxy \rightarrow x$

$@(@(@(S, x), y), z) \rightarrow @(@(x, z), @(y, z))$   
 $@(@(K, x), y) \rightarrow x$



*collapsing contexts*



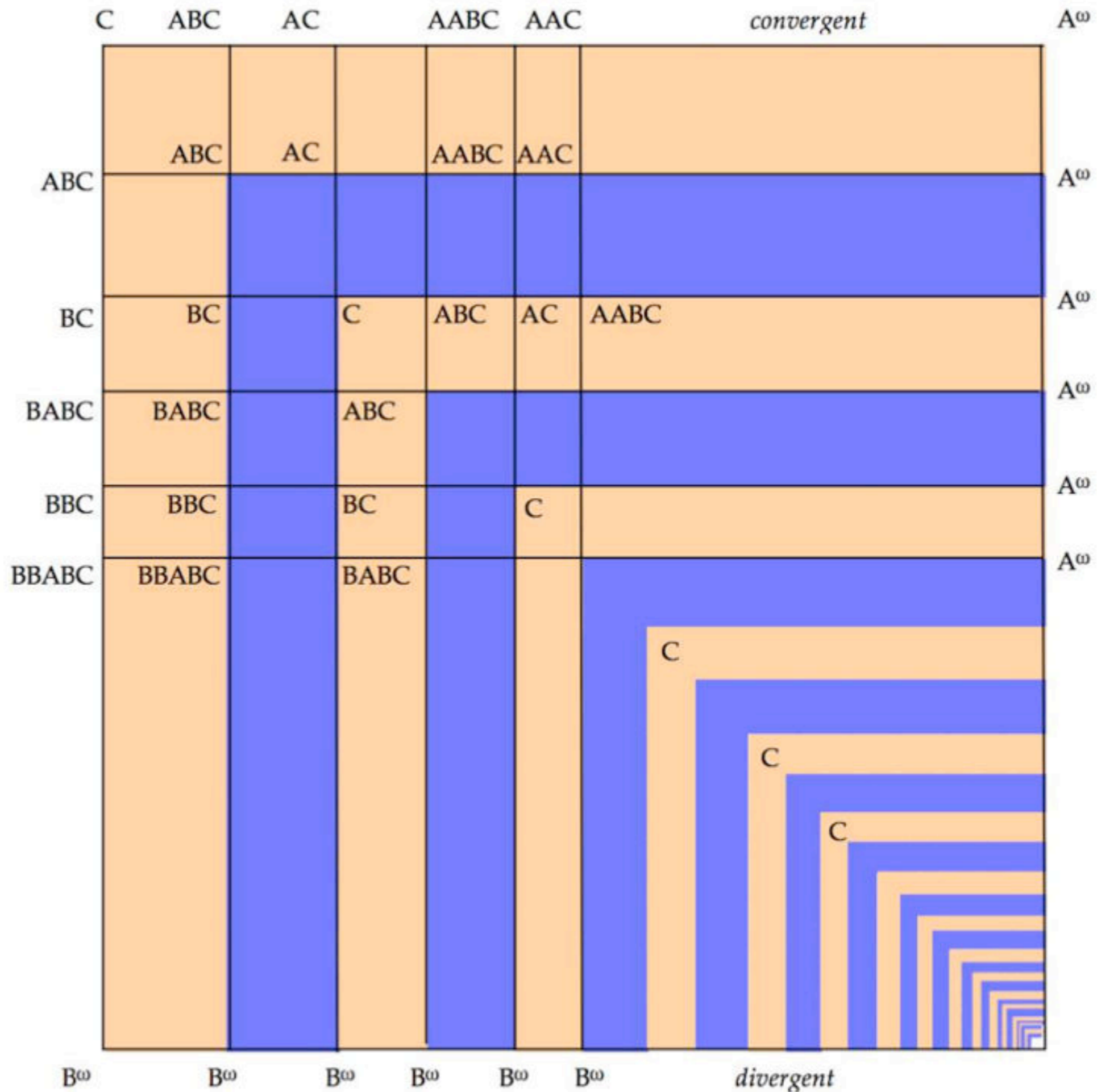
*Failure of infinitary confluence for Combinatory Logic*

# Failure of $CR^\infty$

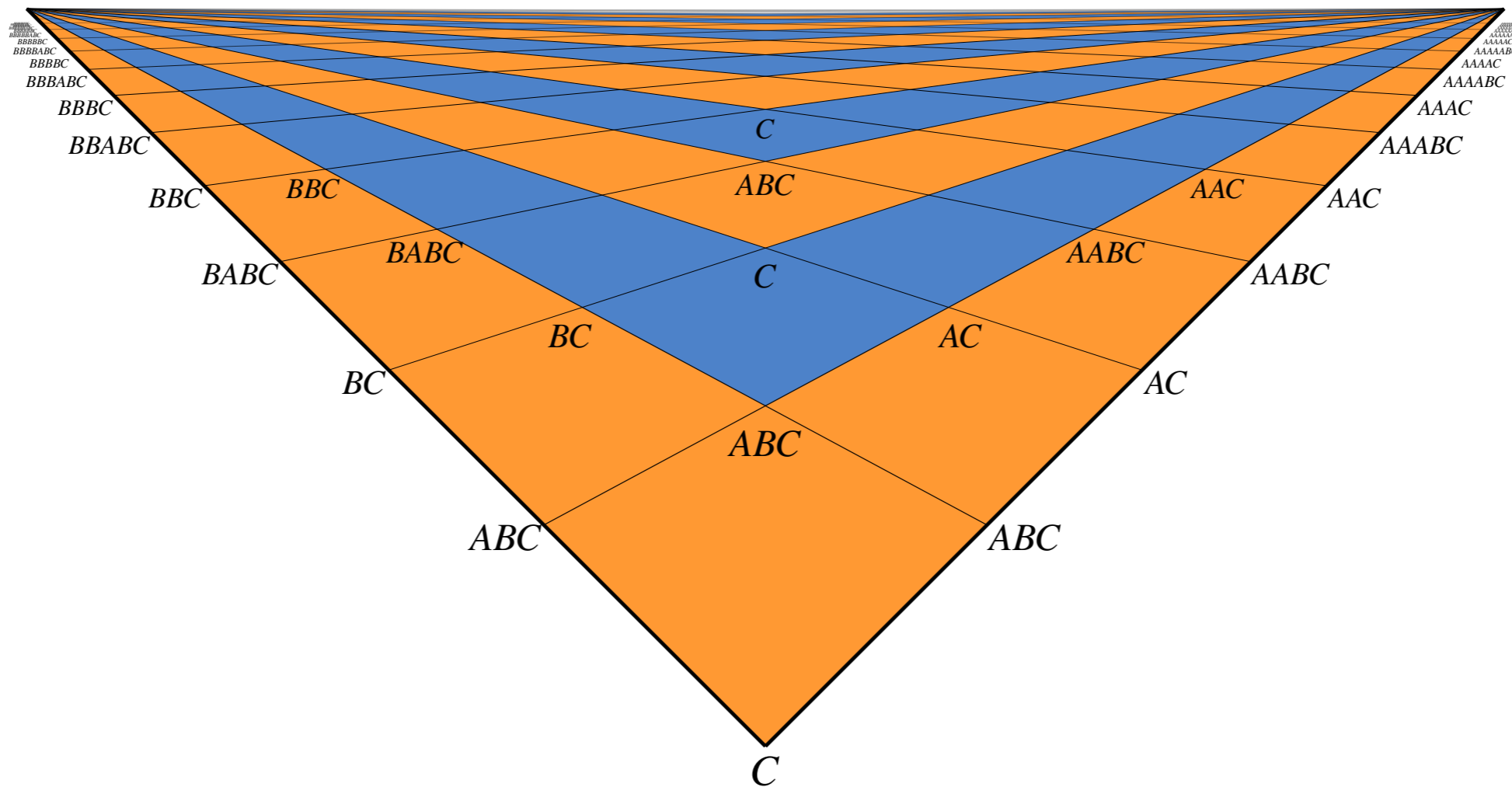
$$A(x) \rightarrow x$$

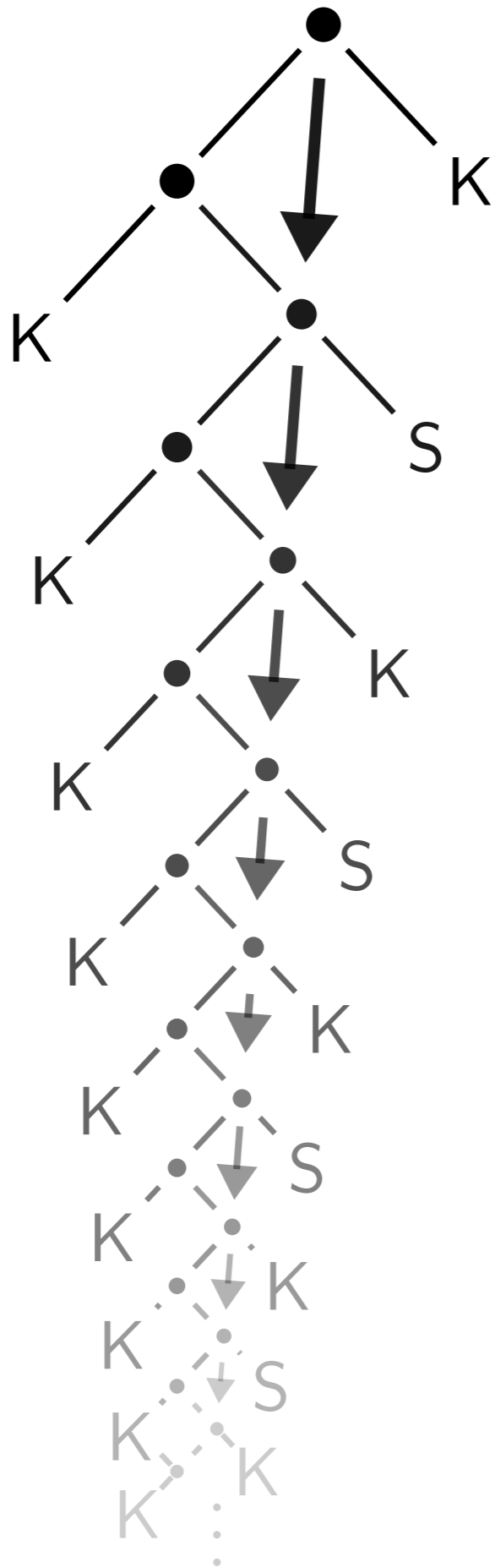
$$B(x) \rightarrow x$$

$$C \rightarrow A(B(C))$$



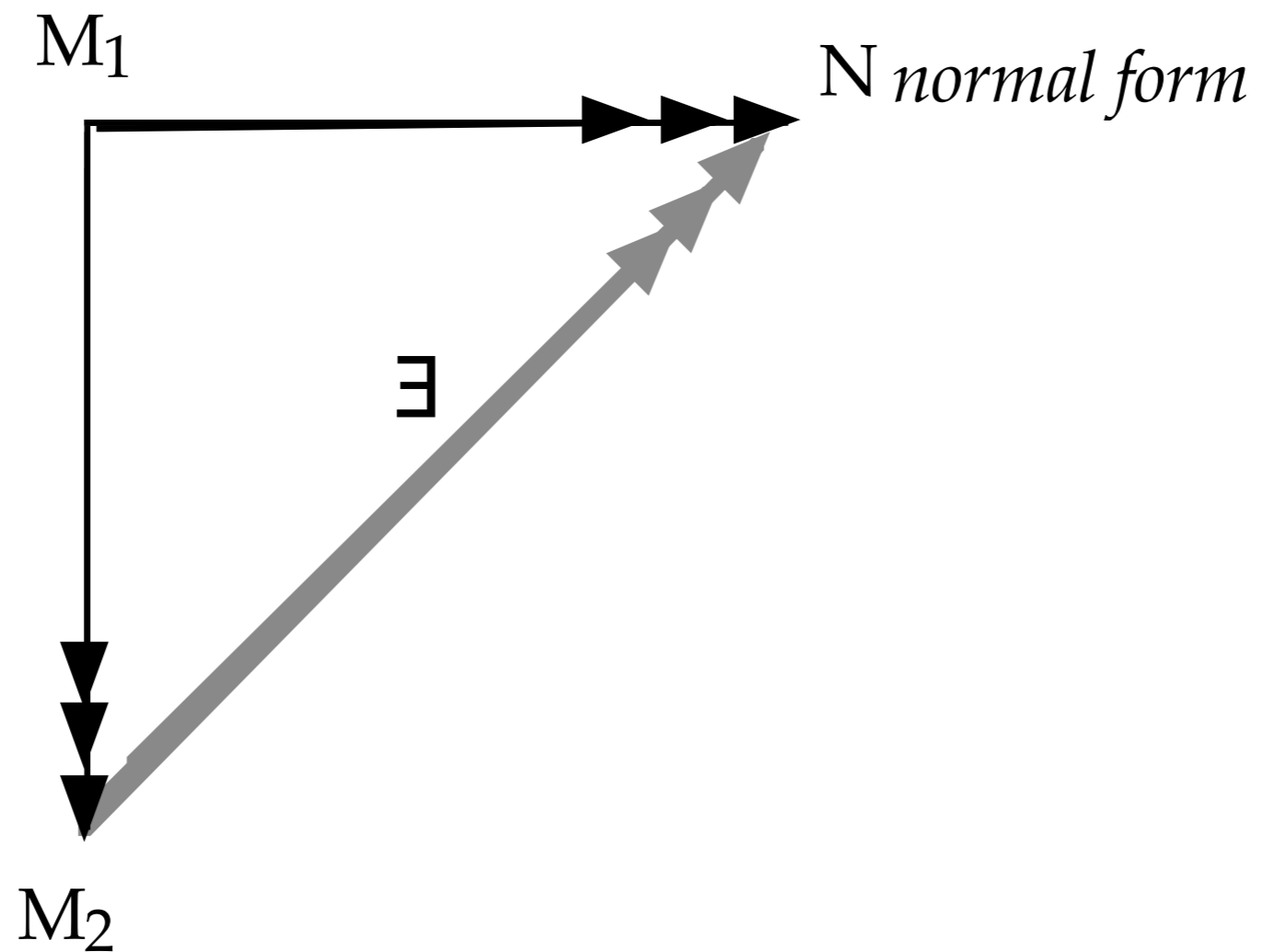
*ABC-counterexample in perspective:  
euclidean distance = tree distance*



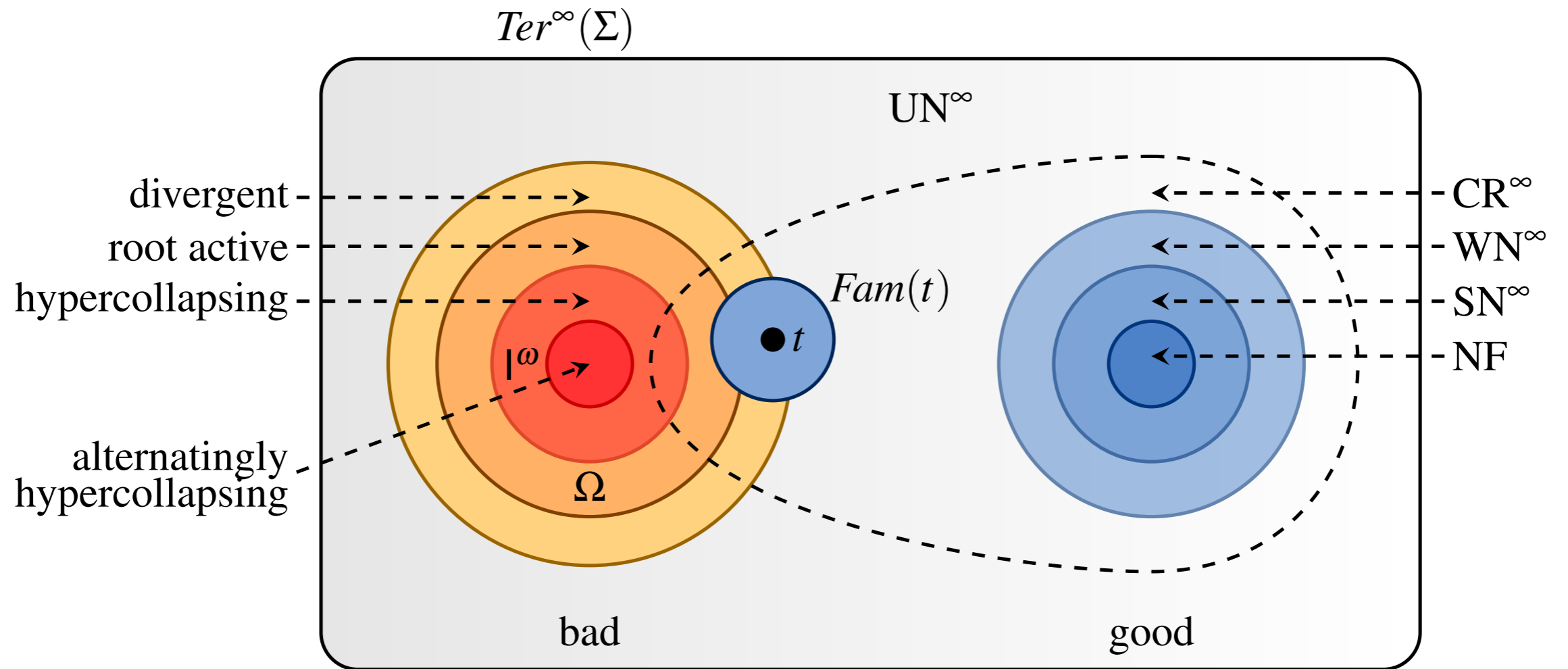


*Example 2.4.* The ‘*ABC*-example’ that we saw in the preceding example also works in the much more important rewrite system Combinatory Logic CL, with the usual three basic combinators  $I, K, S$  and their corresponding reductions rules (see, e.g., Barendregt [2]), and also in infinitary  $\lambda$ -calculus that we will consider in more detail in the next section. The figure on the right, with the infinite collapsing tower of two different *collapsing contexts*  $K \square K$  and  $K \square S$  shows how the *ABC*-counterexample can be simulated using a fixed-point construction in those calculi. To see that this is indeed a  $CR^\infty$ -counterexample, note that  $\mu x.K(KxS)K \twoheadrightarrow \mu x.KxS$  and also  $\mu x.K(KxS)K \twoheadrightarrow \mu x.KxK$ , while  $\mu x.KxS$  and  $\mu x.KxK$  only reduce to themselves (in any countable ordinal number of steps, by the way).

*Ketema-Simonsen, with  $UN^\infty$  as corollary*



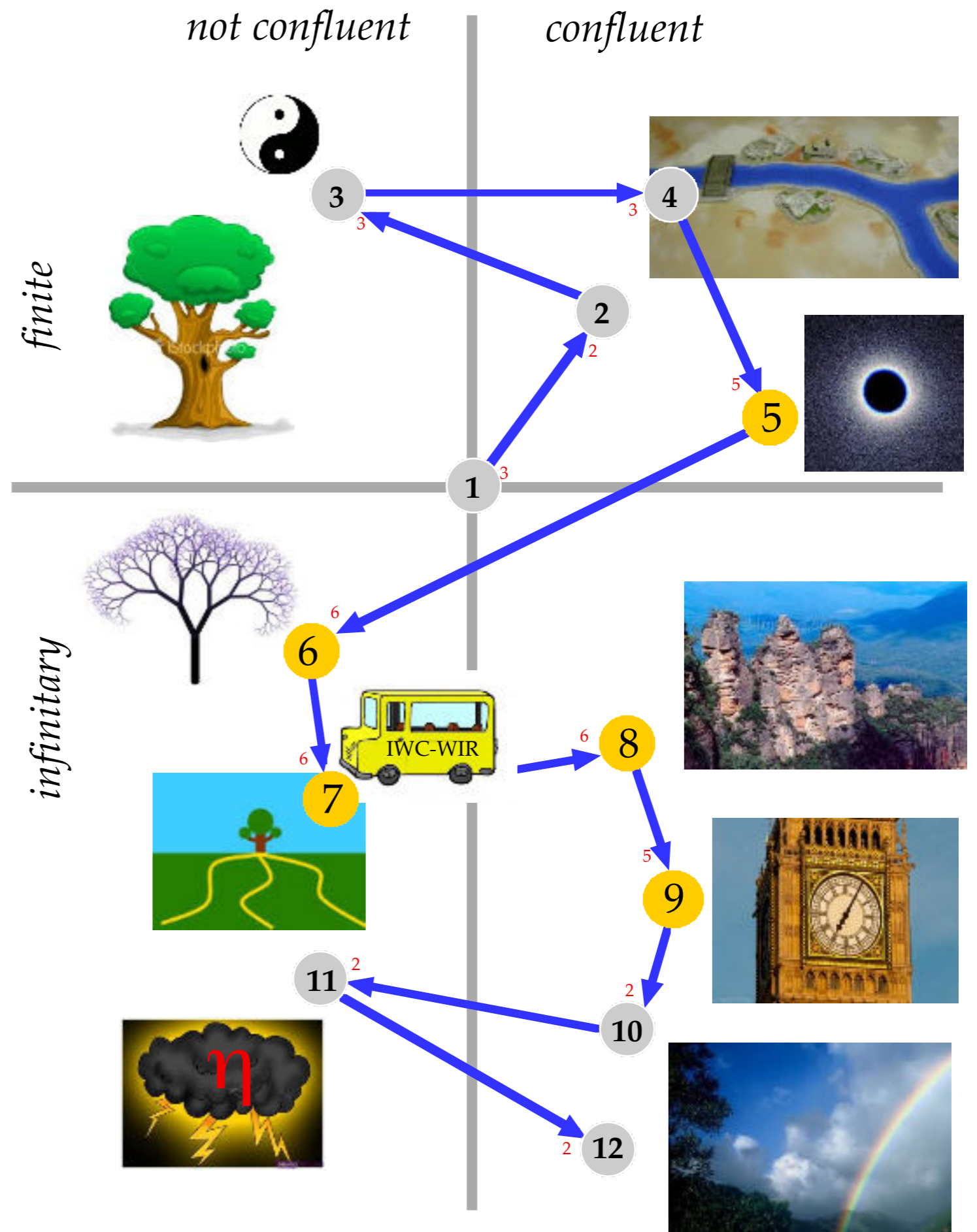




*For all terms  $t$  in an orthogonal TRS, we have*

$$Fam(t) \cap \mathbf{HC} = \emptyset \quad \Rightarrow \quad \mathbf{CR}^\infty(t)$$

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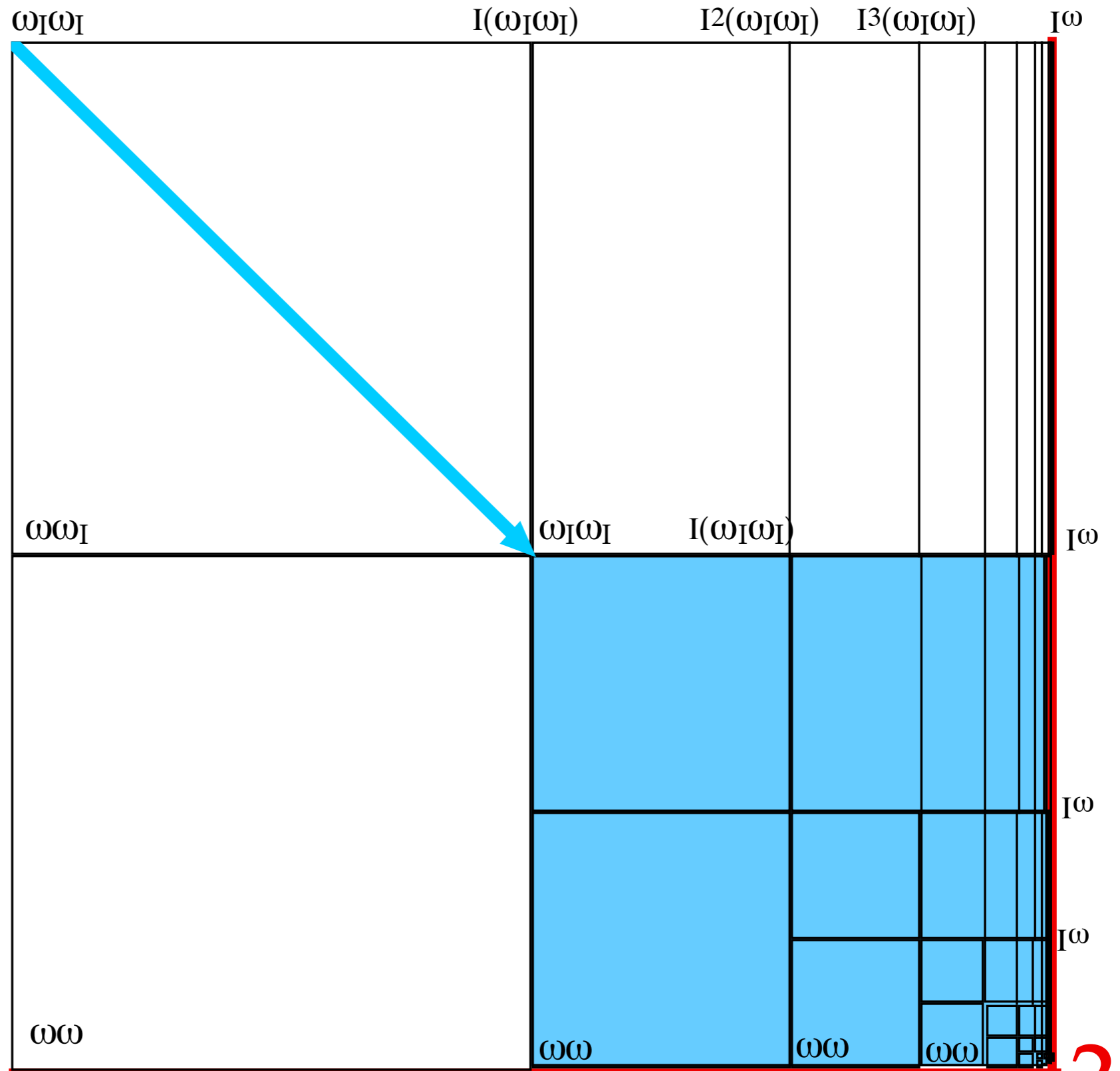
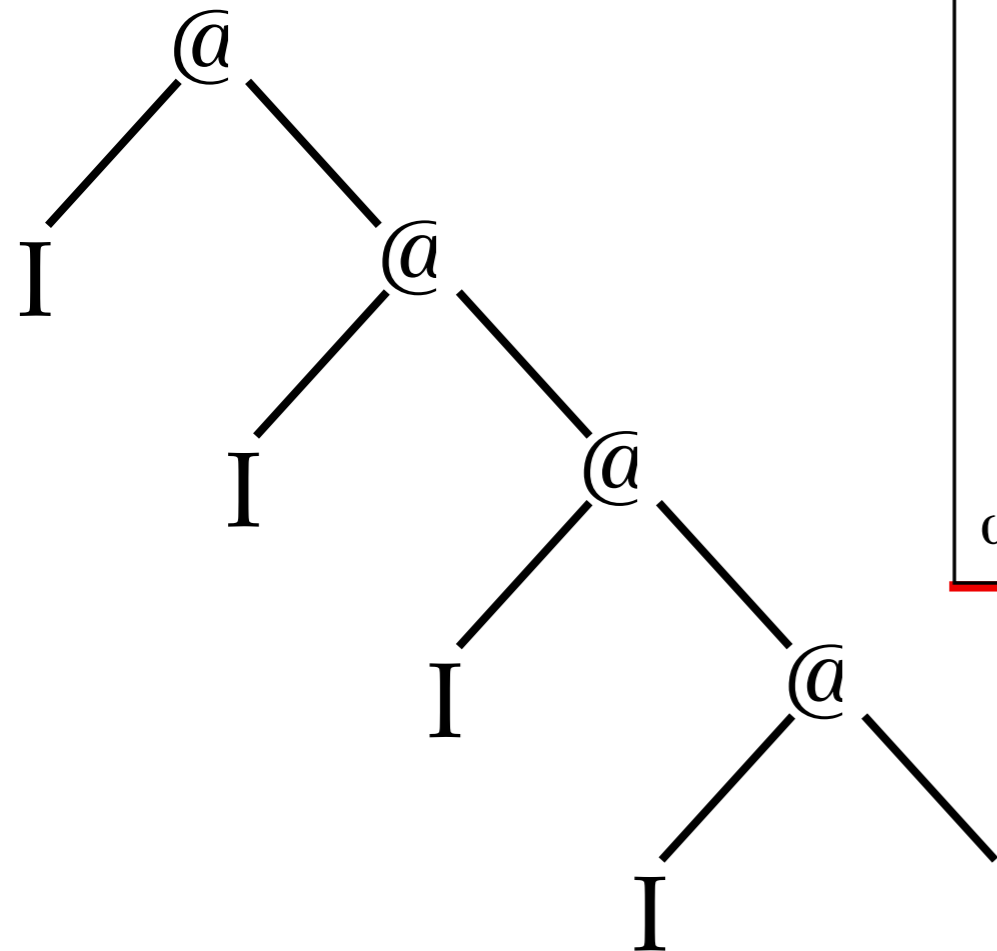
$\lambda^\infty$ : not  $PML^\infty$

$$\omega_I \equiv (\lambda x. I(xx))$$

$$\omega \equiv \lambda x. xx$$

$$YI \rightarrow \omega_I \omega_I$$

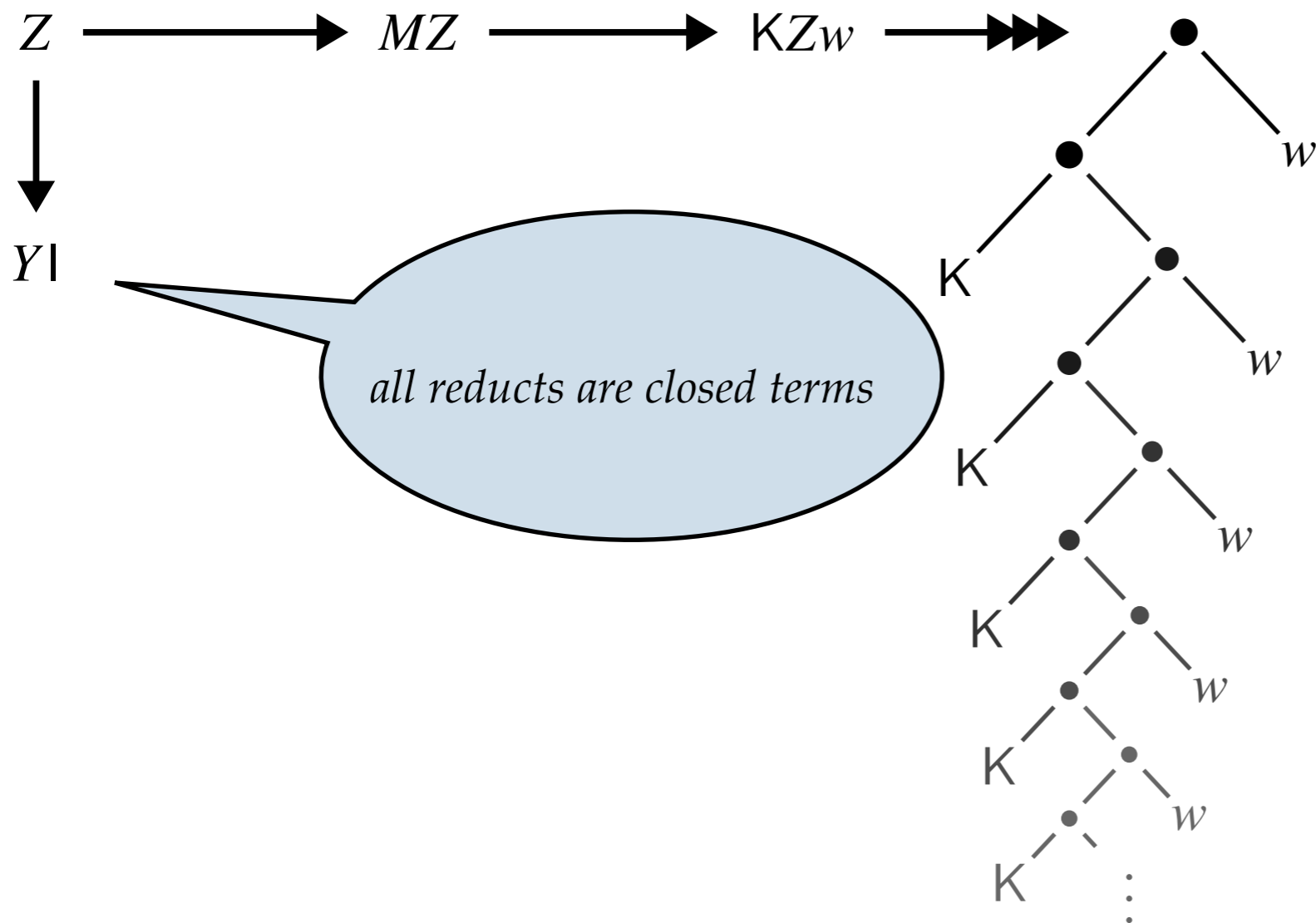
$$I^\omega \equiv$$



*For infinitary lambda calculus  
Parallel Moves Lemma  $PML^\infty$   
fails, hence also  $CR^\infty$*

# another counterexample

Let  $M \equiv \lambda x.Kxw$  and  $Z = YM$ . Then



*all reducts are closed terms*

*curious: in the limit,  $w$  is fixed as a free variable, all reducts are open terms*

# A SIMPLE PROOF

BY

$\neq_{\beta}$  ?

BYS

BYI

BYSI

$BYI \equiv (\lambda abc. a(bc)) YI$

$BYSI \equiv (\lambda abc. a(bc)) YSI$



$\lambda c. Y(Ic)$

$\lambda c. Yc$



Y

$\neq_{\beta}$  !

Y(SI)

*Curry's fpc*

*Turing's fpc*

$Y_0: \lambda f. (x.f(xx)(\lambda x.f(xx)))$

$Y_1: (\lambda ab. b(aab)) (\lambda ab. b(aab))$

$Y_0(SI) \twoheadrightarrow Y_1$

Exercise. Prove that  $Y_0 \neq_{\beta} Y_1$



# INFINITARY LAMBDA CALCULUS SUBSUMES SCOTT'S INDUCTION RULE

$$\frac{\Gamma, ax \sqsubseteq bx \vdash a(ux) \sqsubseteq b(ux)}{\Gamma, a\perp \sqsubseteq b\perp \vdash a(Yu) \sqsubseteq b(Yu)}$$

$$Yx \rightarrow \rightarrow x(Yx) \rightarrow \rightarrow x^2(Yx) \rightarrow^\omega x^\omega \equiv x(x(x(x\dots$$

$$BY \equiv (\lambda abc. a(bc)) Y \quad =_\infty$$

$=_\infty$

$\neq_\beta$

$$BYS \equiv (\lambda abc. a(bc)) YS$$

$$\lambda bc. Y(bc)$$

$\omega$

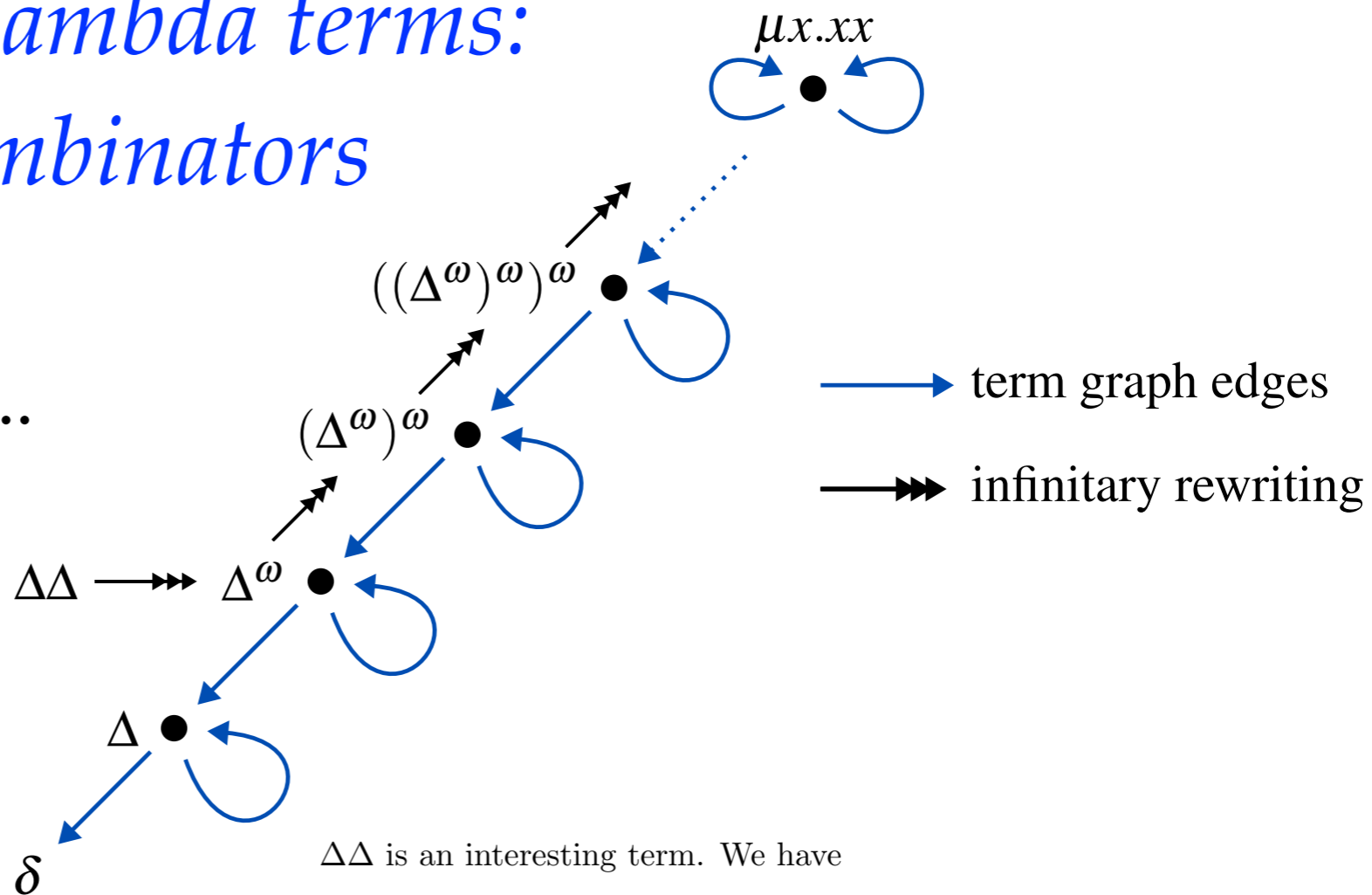
$$\lambda bc. (bc)^\omega \equiv \lambda cz. (cz)^\omega$$

$$\begin{aligned} &\downarrow \\ &\lambda c. Y(Sc) \\ &\downarrow \\ &\lambda c. Sc(Y(Sc)) \\ &\downarrow \\ &\lambda cz. cz(Y(Sc)z) \\ &\downarrow \\ &\lambda cz. cz(cz(Y(Sc)z)) \end{aligned}$$

$\omega$

# playing with infinite lambda terms: infinite fixed point combinators

twinkle =  $\Delta = \delta^\omega = \delta(\delta(\delta(\delta \dots$



$\Delta\Delta$  is an interesting term. We have

$$\Delta\Delta \twoheadrightarrow \Delta^\omega \twoheadrightarrow (\Delta^\omega)^\omega \twoheadrightarrow ((\Delta^\omega)^\omega)^\omega \twoheadrightarrow \dots$$

See Figure 8. Somewhat surprisingly,  $\Delta\Delta$  does have a normal form, viz.  $\mu x.xx$ ; and moreover  $\Delta\Delta$  has the property  $SN^\infty$ . To see that  $\mu x.xx$  is indeed the normal form, one may consider the reduction

$$\Delta\Delta \twoheadrightarrow (\Delta^\omega)^\omega \equiv \Delta^\omega((\Delta^\omega)^\omega) \twoheadrightarrow (\Delta^\omega)^\omega((\Delta^\omega)^\omega) \twoheadrightarrow \dots$$

and check that the reductions involved do not employ root redexes. (Only in the reduction  $\Delta\Delta \twoheadrightarrow \Delta^\omega$  a root step is present; in the ‘later’ reductions there are no root steps.) In fact we have a strongly convergent reduction

$$\Delta\Delta \twoheadrightarrow \Delta^\omega \twoheadrightarrow (\Delta^\omega)^\omega \twoheadrightarrow ((\Delta^\omega)^\omega)^\omega \twoheadrightarrow \dots \twoheadrightarrow \mu x.xx$$

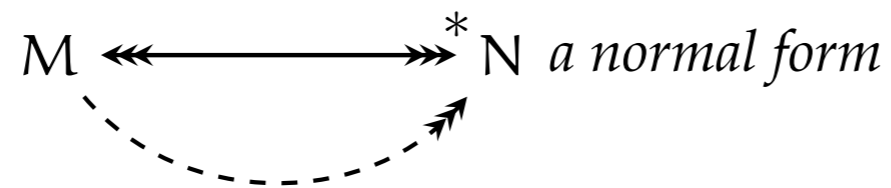
The term  $\Delta\Delta$  has uncountably many reducts. It has reductions of any countable ordinal length. It is  $SN^\infty$  with  $\mu x.xx$  as its unique normal form. This normal form is in fact a Berarducci tree. The example of  $\Delta\Delta$  was also mentioned in [4].  $SN^\infty$  can be proved as follows: We have  $CR^\infty$  as there are no collapsing rules in this TRS, which is a fragment (sub-TRS) of CL. Since there is a normal form, we have  $WN^\infty$ . Hence,  $SN^\infty$  follows by the equivalence  $SN^\infty \iff WN^\infty$  as global properties of TRSs.

$$\Delta x \equiv \delta \Delta x \rightarrow_\beta \rightarrow_\beta x(\Delta x)$$

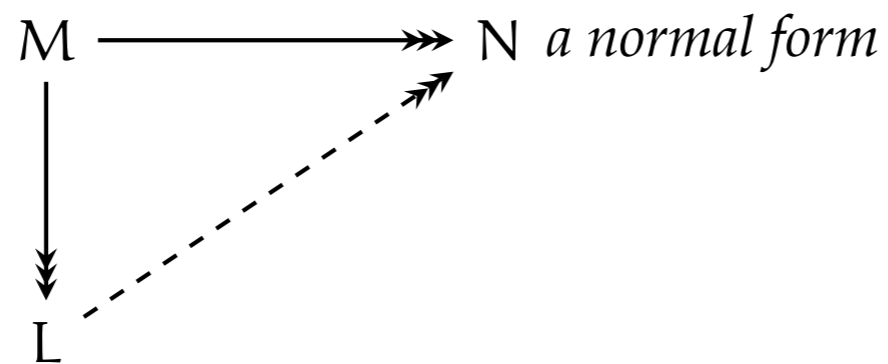
$(SS)^\omega SSSI$ , another infinite fpc



The infinitary  $\beta$ -reduction  $\twoheadrightarrow_{\beta}$  has the infinitary normal form property  $\text{NF}^{\infty}$ , that is, for all  $M, N \in \text{Ter}^{\infty}(\lambda)$  with  $N$  a normal form and  $M (\twoheadleftarrow_{\beta} \cup \twoheadrightarrow_{\beta})^* N$  we have  $M \twoheadrightarrow_{\beta} N$ . In a picture:

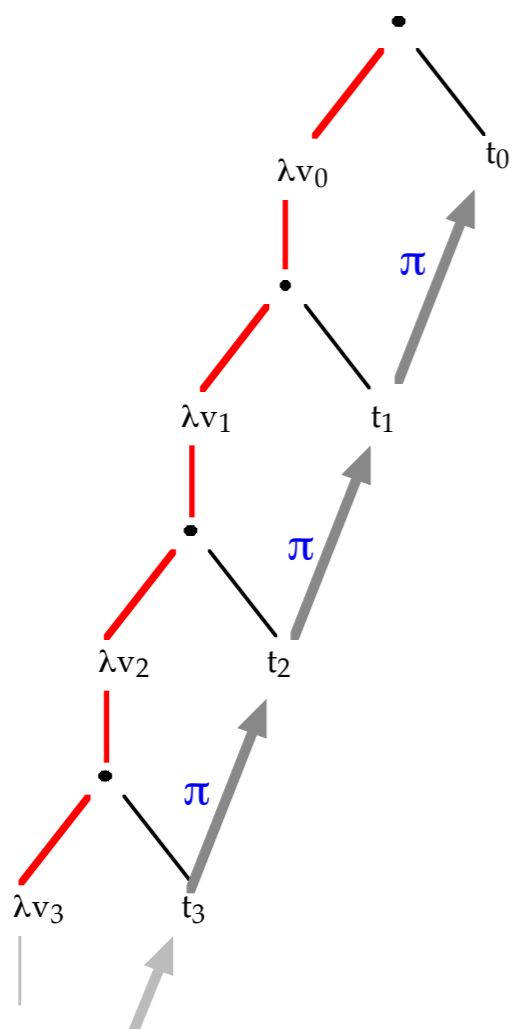


Actually the following property is sufficient:

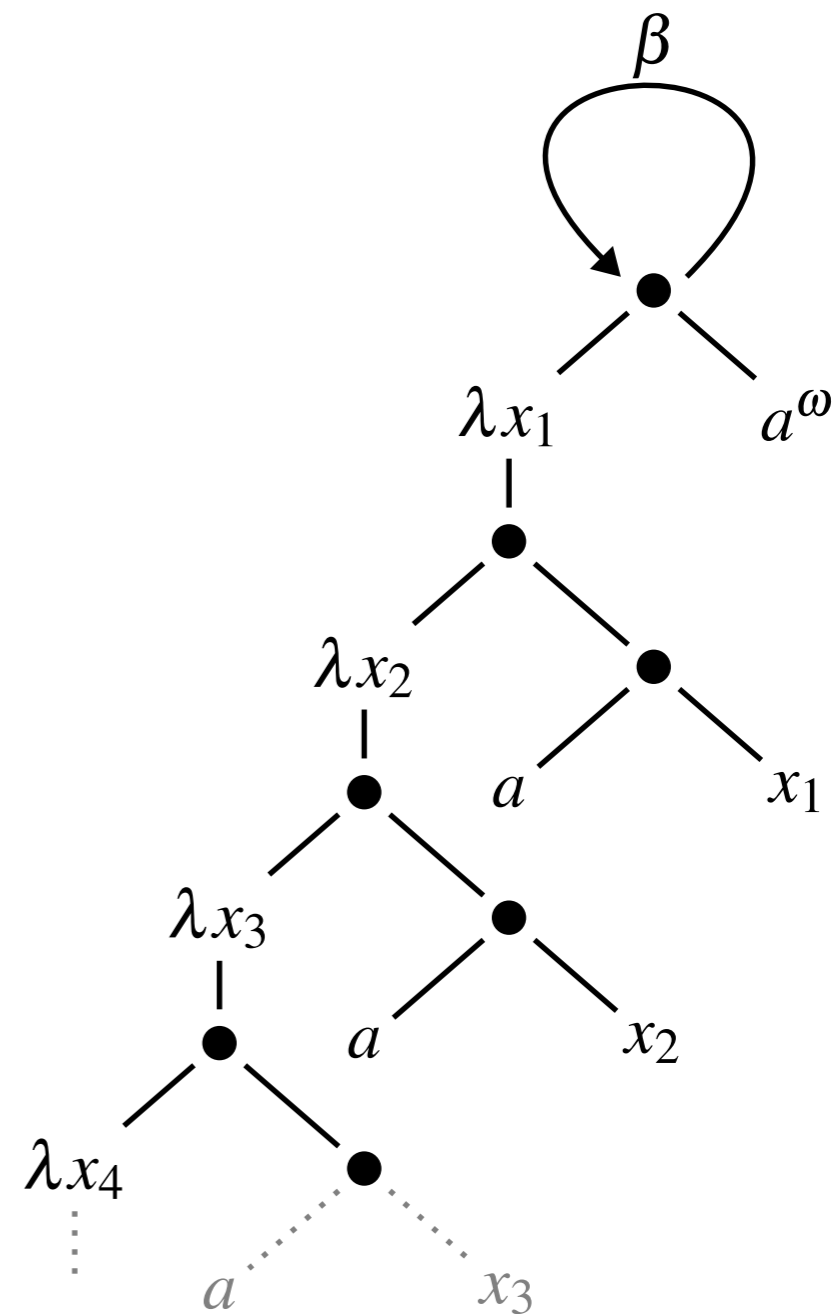


We obtain infinitary unique normal forms  $\text{UN}^{\infty}$  as a direct corollary.

# playing with infinite lambda terms: looping lambda terms



looping term with infinite *spine* (red) and cascade projective sequence, with projection  $\pi$ .



**Theorem 13.2.6.** *In infinitary  $\lambda$ -calculus, a term is root looping if and only if it is of one of the following forms:*

- (i)  $\Omega$
- (ii)  $I^\omega$
- (iii)  $BB$  where  $B$  is the infinite solution of  $B = \lambda x.xB$ ,
- (iv)  $(\lambda v_0.(\lambda v_1.(\lambda v_2.\dots)t_2)t_1)t_0$  such that  $t_i$  is obtained from  $t_{i+1}$  by replacing  $v_0$  by  $t_0$  and all variables  $v_{j+1}$  by  $v_j$ . We call such a term a cascade.

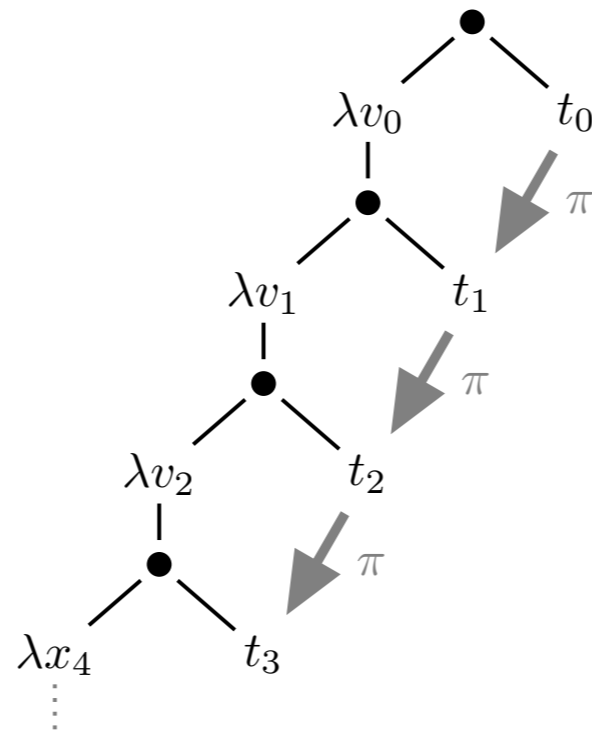
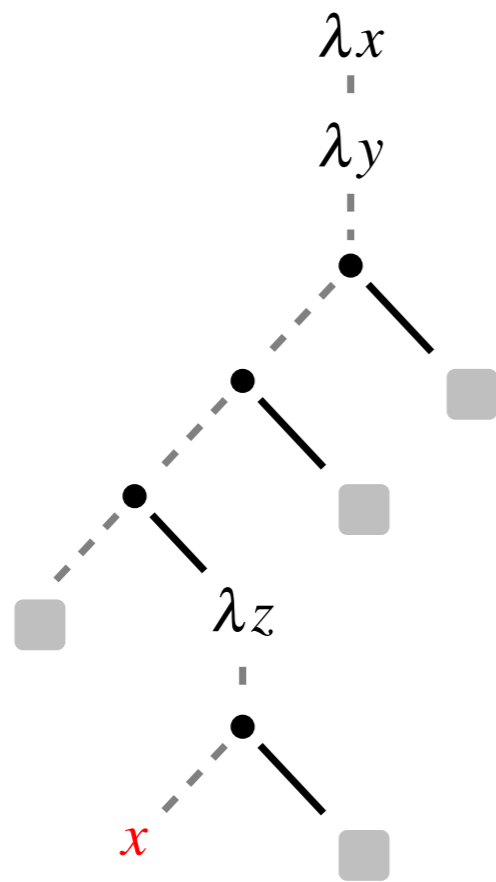
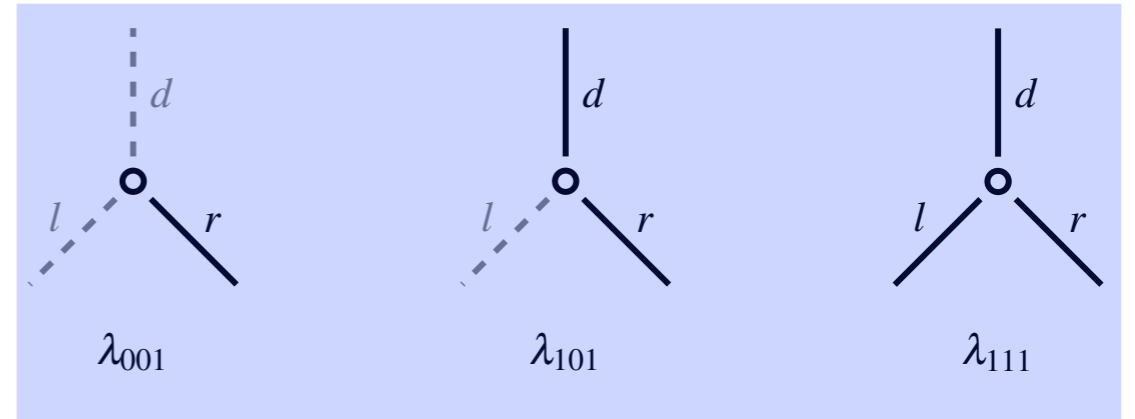
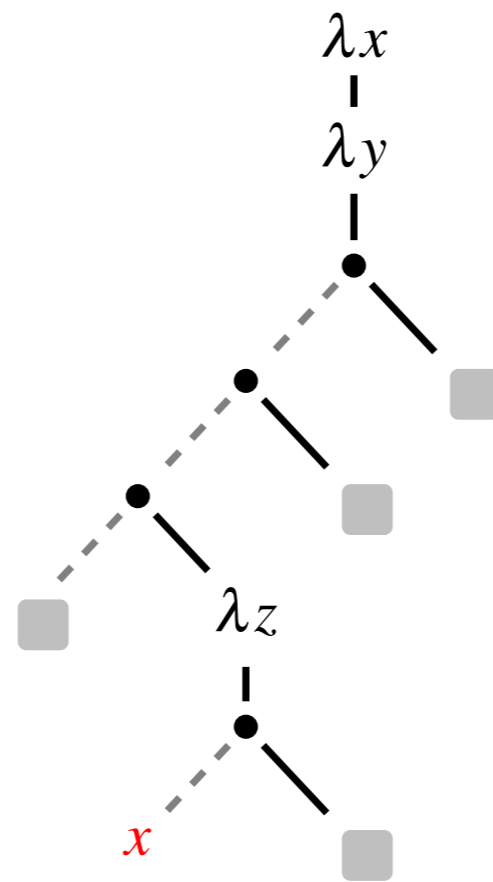


Figure 13.2: *The shape of cascades; here  $\pi$  stands for replacing all variables  $v_j$  by  $v_{j+1}$  followed by replacing an arbitrary (possibly infinite) number of occurrences of  $t_0$  by  $v_0$ .*

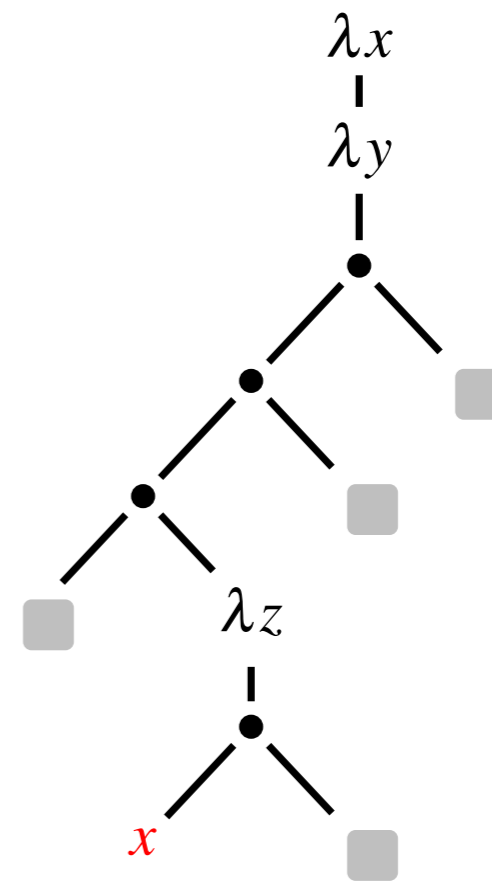
# *different ways to count depth*



001-depth 1  
*{l, d}-steps don't count*



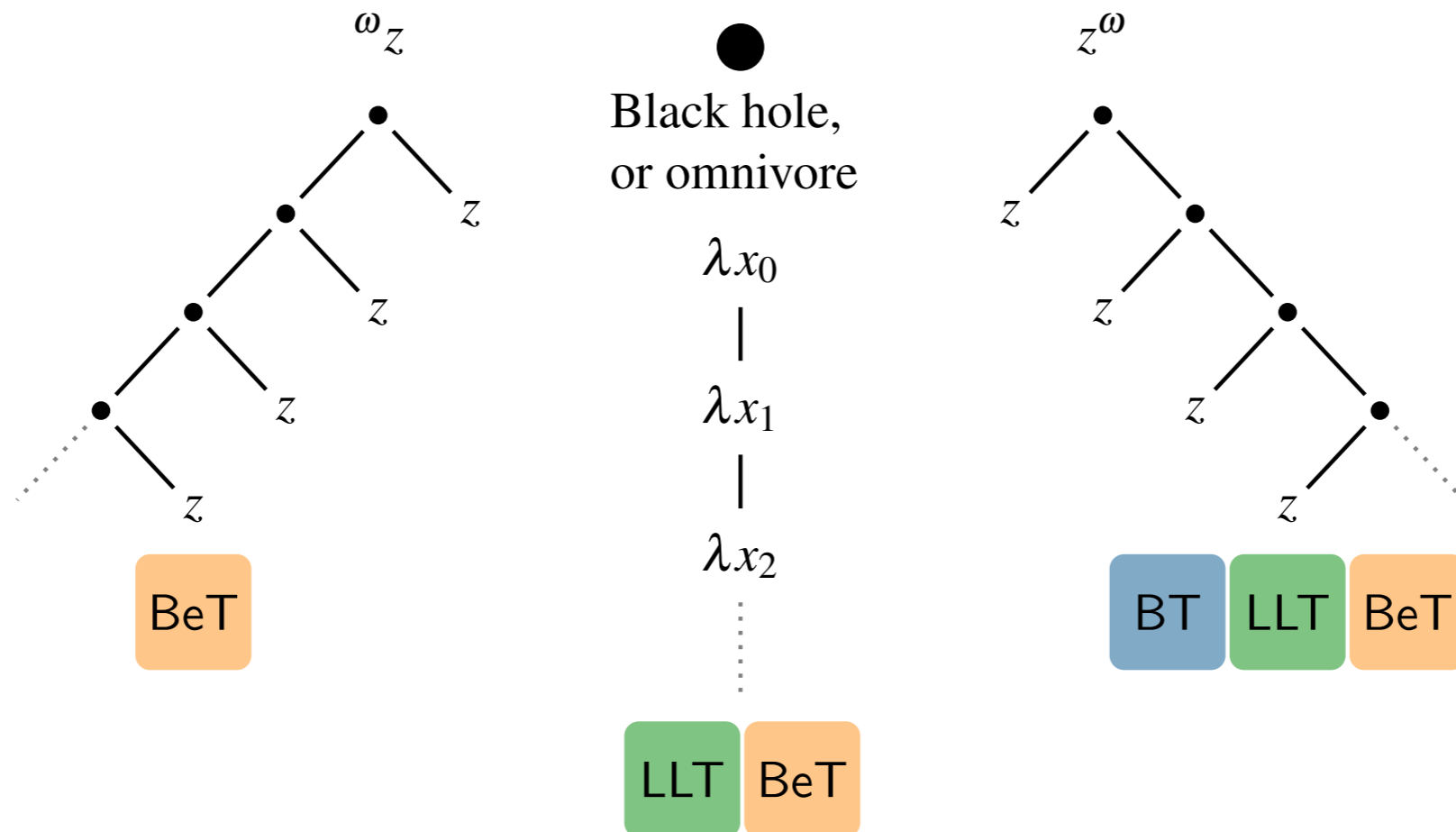
101-depth 4  
*{l}-steps don't count*



111-depth 7  
*all steps count*

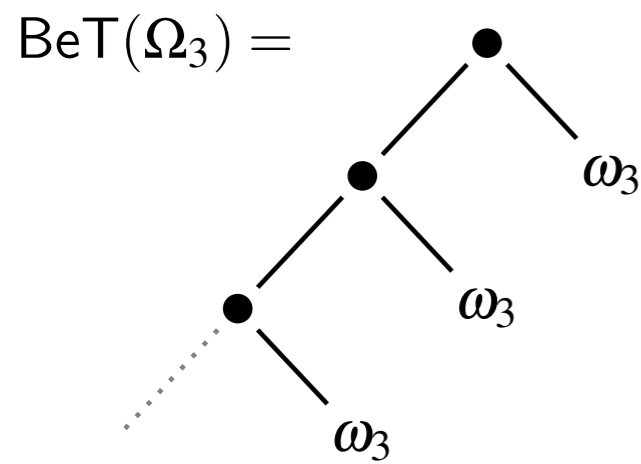


*typical terms in the three domains*

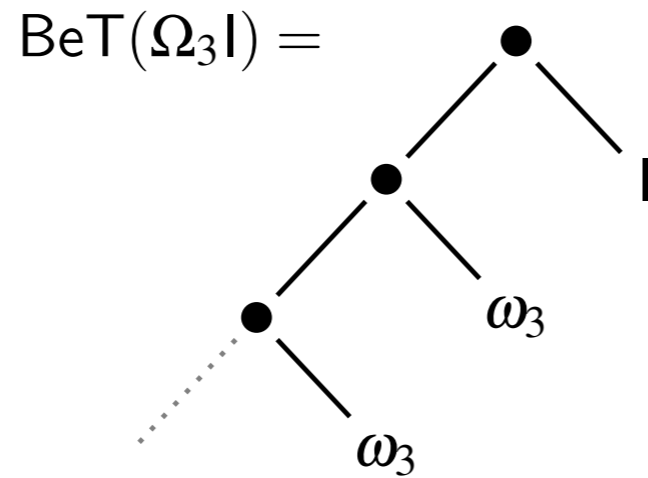


*a tool for consistency analysis*

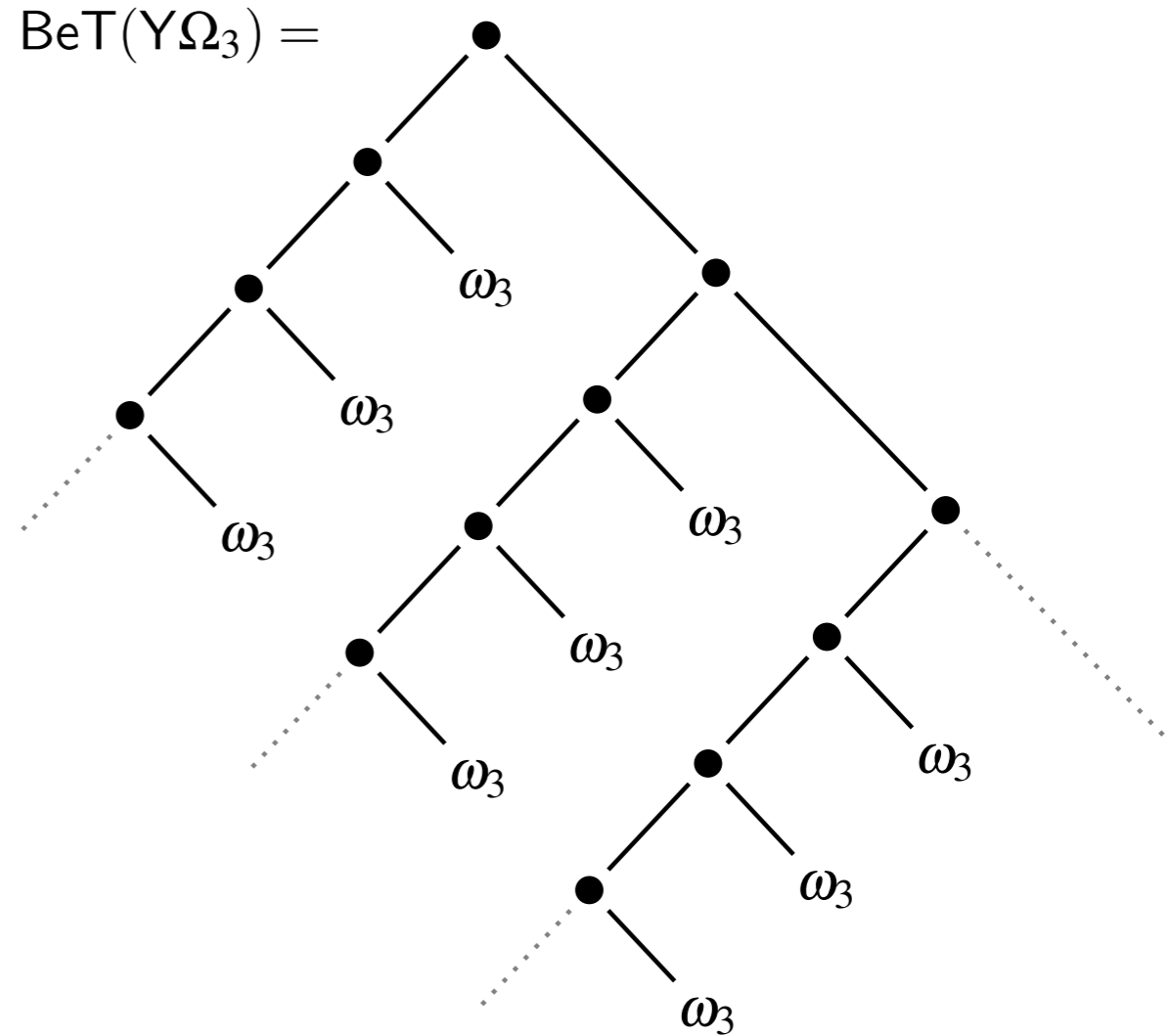
# Berarducci Trees



*not easy*

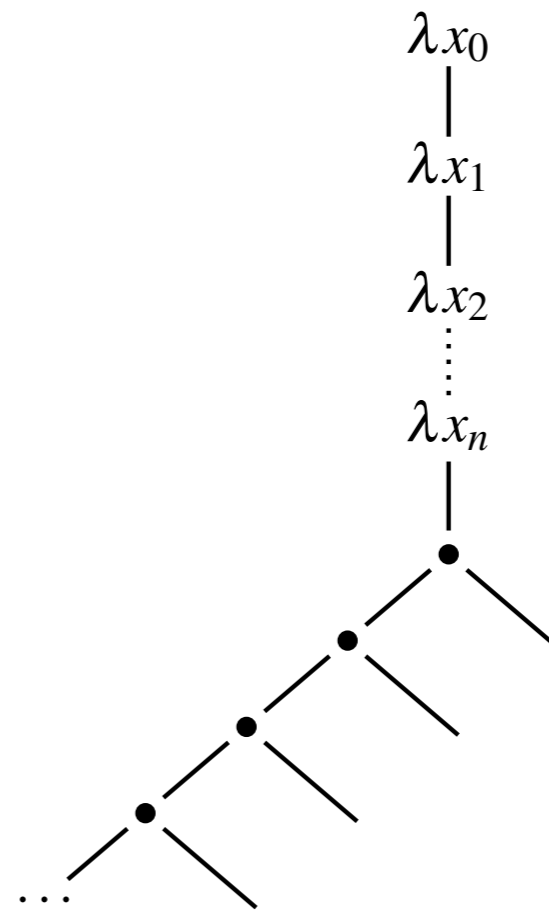
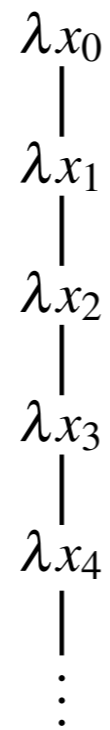
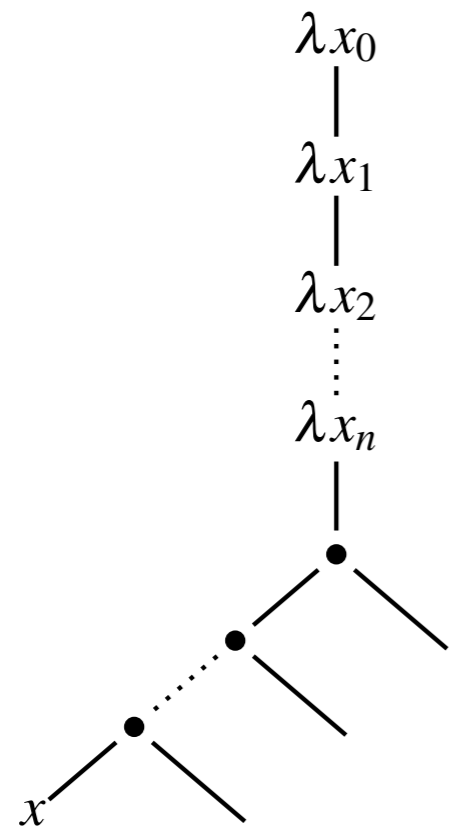


*easy*

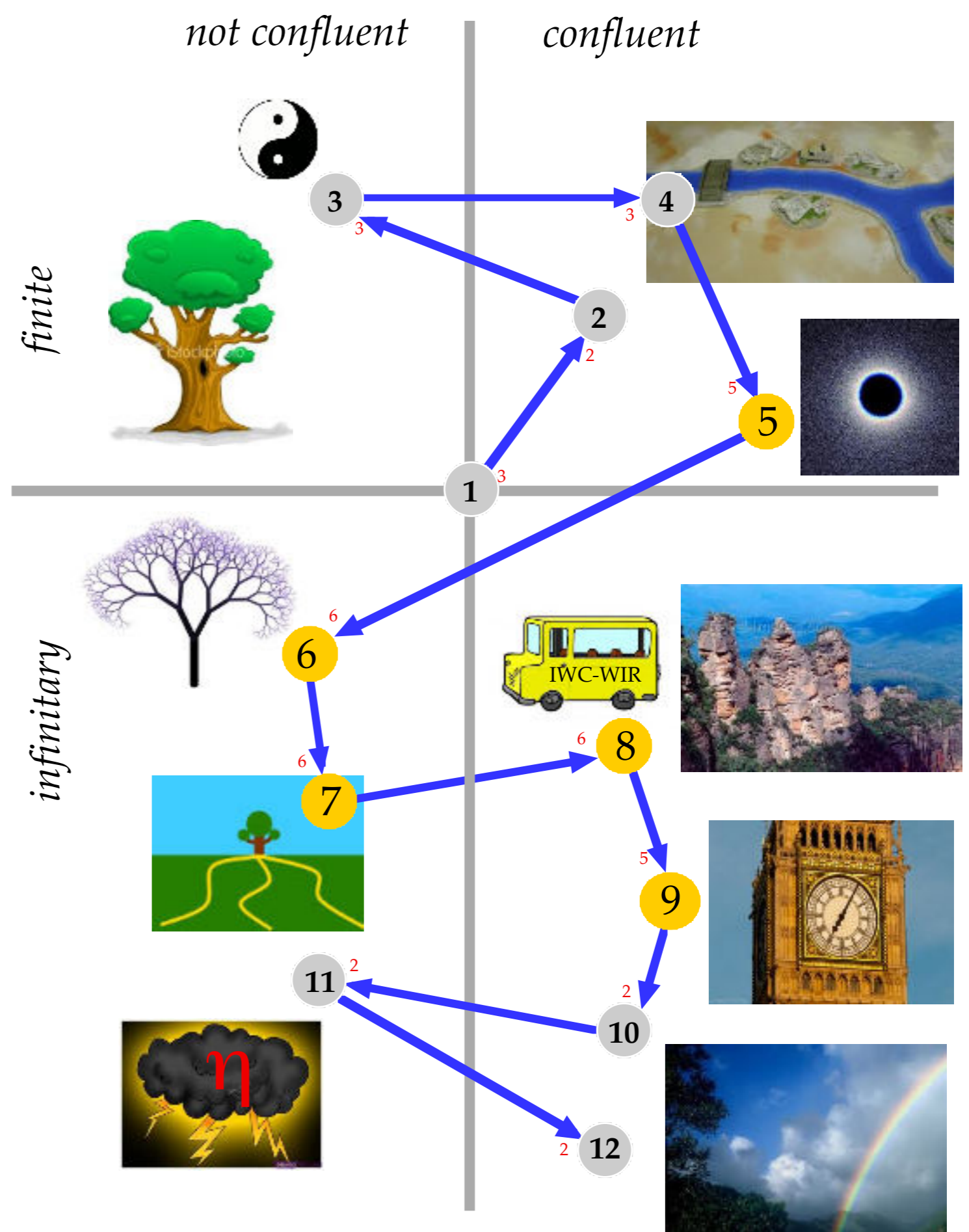


*easy for closed normal forms;*  
*open problem for general terms*

# *building blocks for infinitary lambda normal forms*



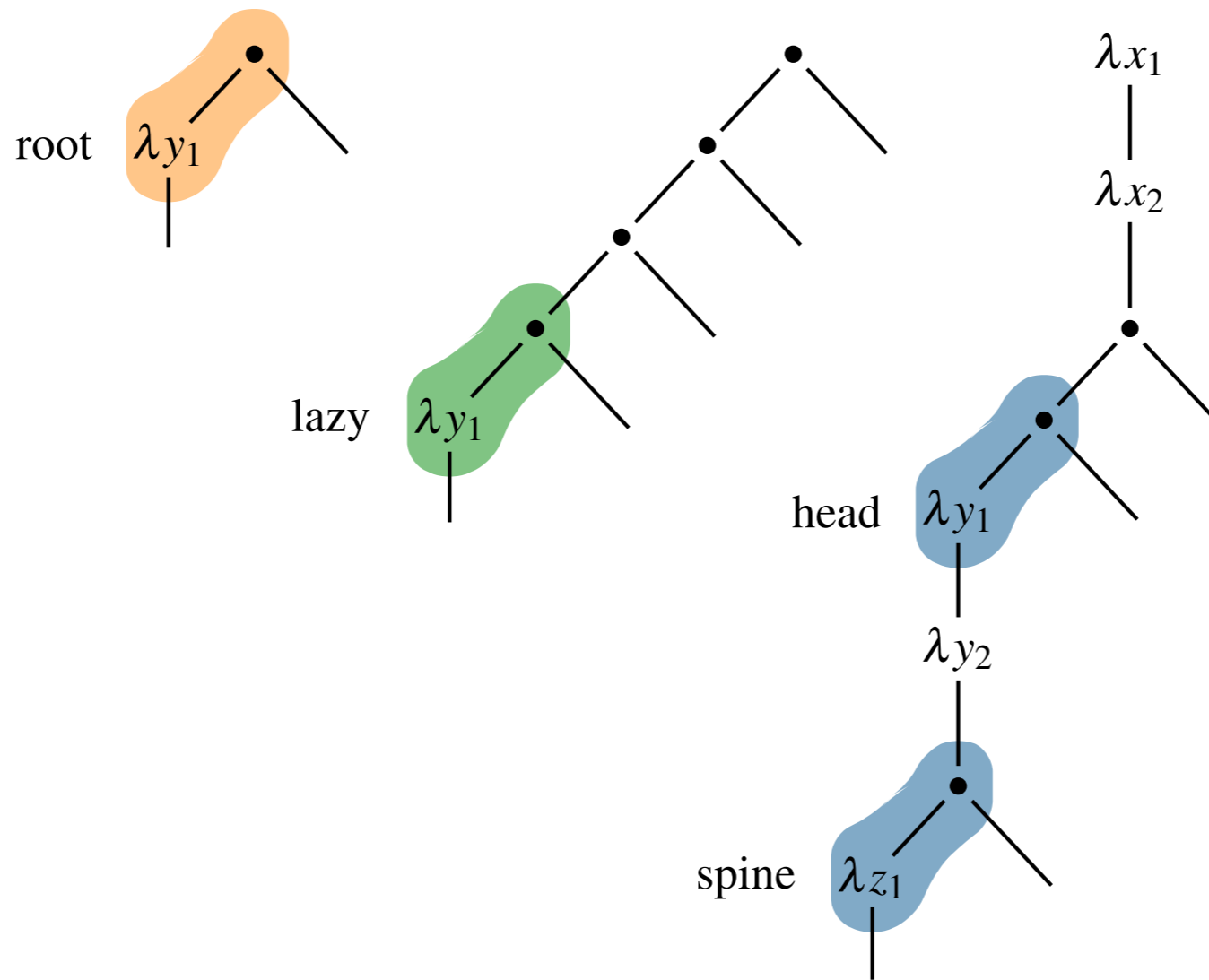
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# Coinductive definition of BT, LLT, BeT

$$\begin{aligned}
 \text{BT}(M) &= \begin{cases} \lambda \vec{x}. y \text{ BT}(M_1) \dots \text{BT}(M_m) & \text{if } M \text{ has hnf } \lambda \vec{x}. y M_1 \dots M_m, \\ \perp & \text{otherwise.} \end{cases} \\
 \text{LLT}(M) &= \begin{cases} x \text{ LLT}(M_1) \dots \text{LLT}(M_m) & \text{if } M \text{ has whnf } x M_1 \dots M_m, \\ \lambda x. \text{LLT}(M') & \text{if } M \text{ has whnf } \lambda x. M', \\ \perp & \text{otherwise.} \end{cases} \\
 \text{BeT}(M) &= \begin{cases} y & \text{if } M \twoheadrightarrow y, \\ \lambda x. \text{BeT}(N) & \text{if } M \twoheadrightarrow \lambda x. N, \\ \text{BeT}(M_1) \text{ BeT}(M_2) & \text{if } M \twoheadrightarrow M_1 M_2 \text{ such that } M_1 \text{ is of order 0,} \\ \perp & \text{in all other cases (i.e., when } M \text{ is mute).} \end{cases}
 \end{aligned}$$

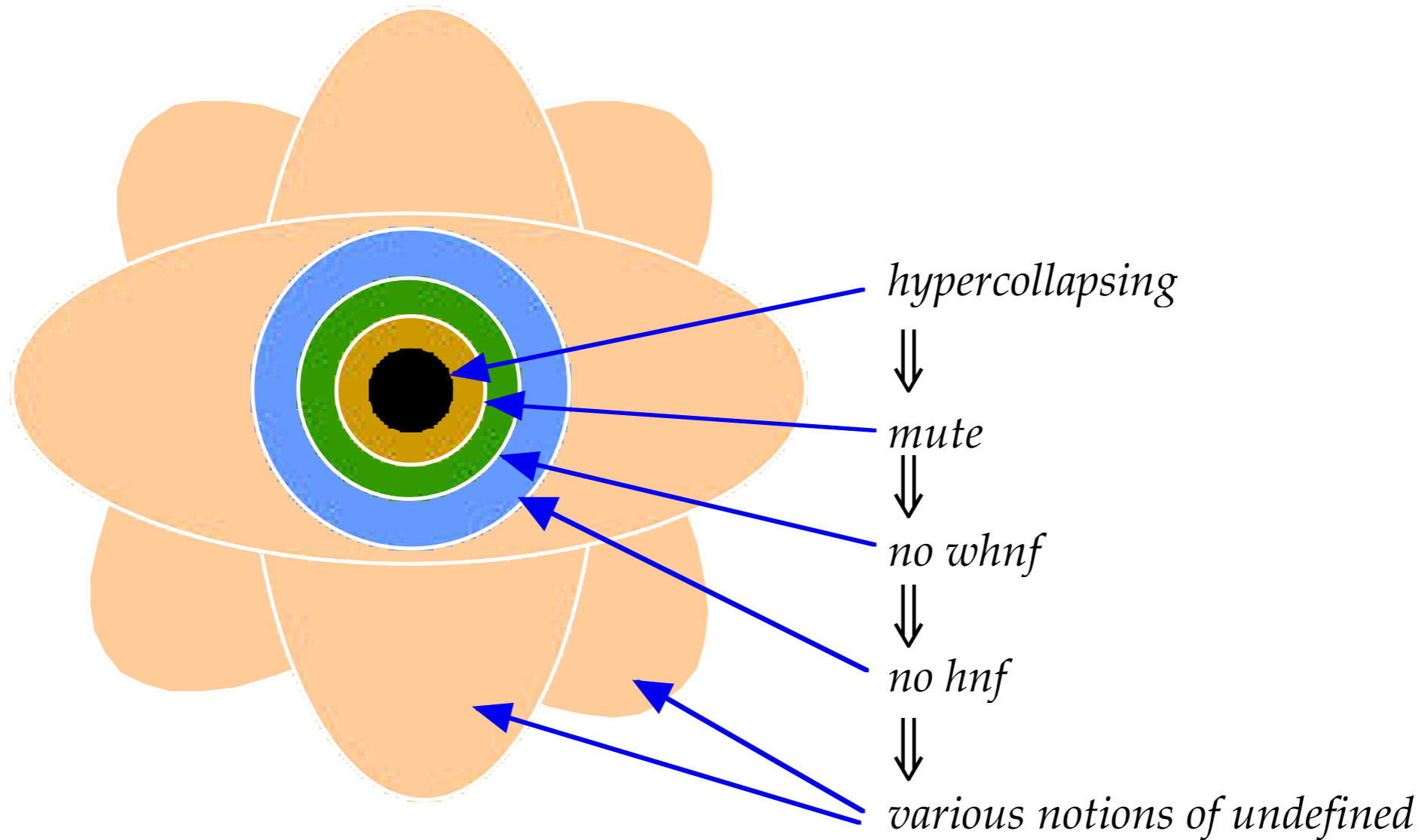
# *the typical redexes*



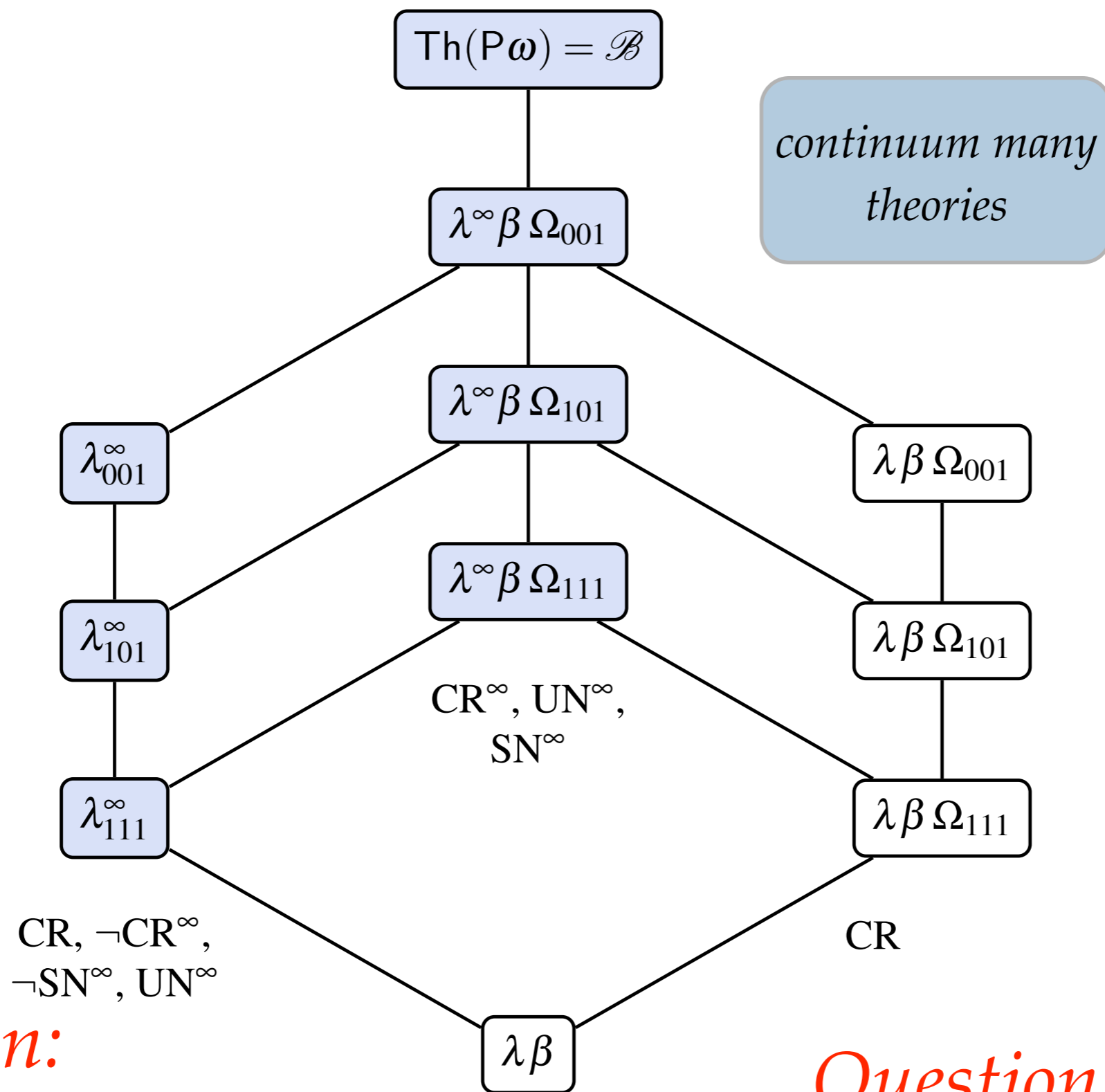
redex is root  $\Rightarrow$  lazy  $\Rightarrow$  head  $\Rightarrow$  spine



*notions of undefinedness, with a **caveat***



# *lambda theories compared*



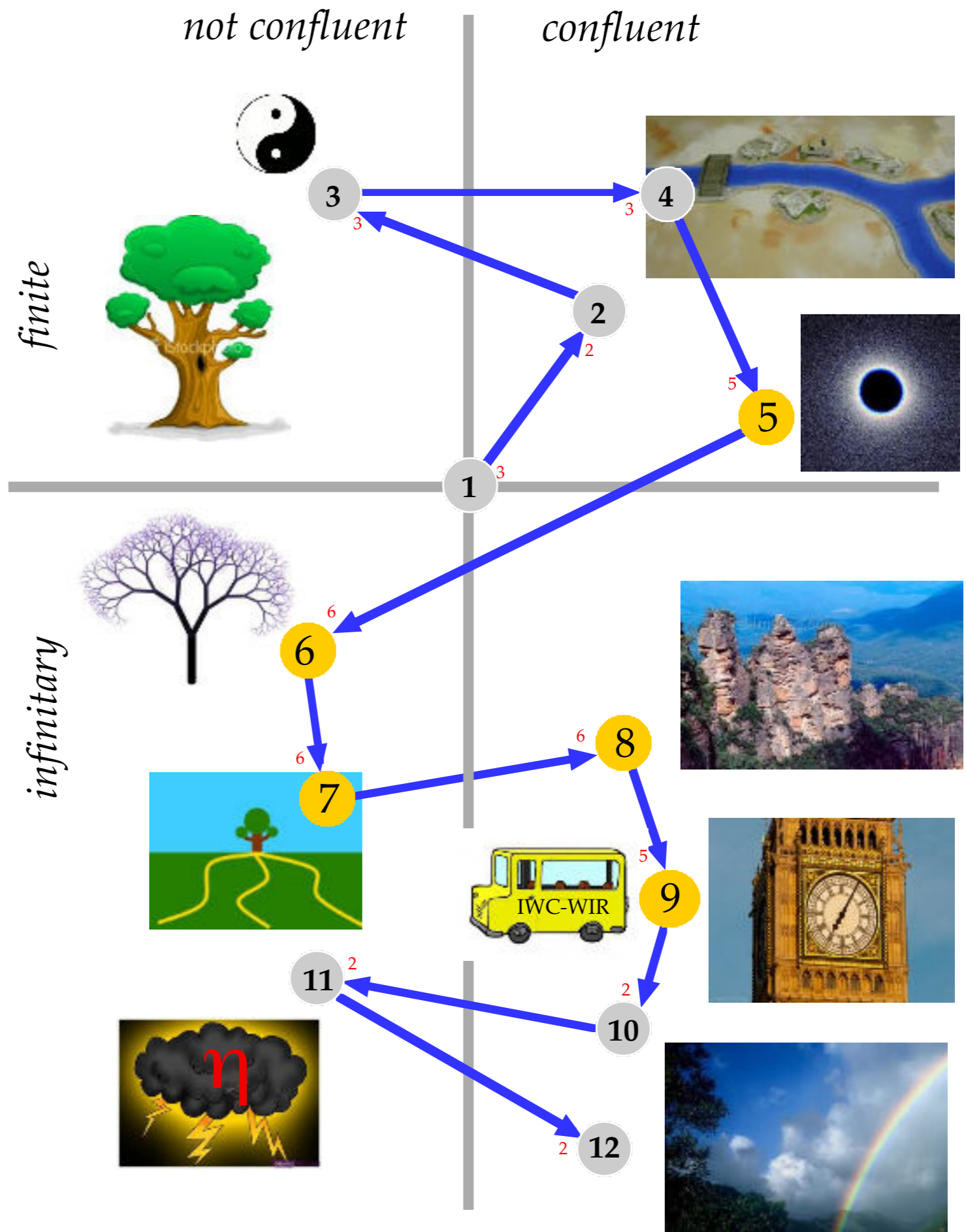
*Question:*

*Do we have LLT and BeT versions of  $\mathbf{P}\omega$  ?*

*Question:*

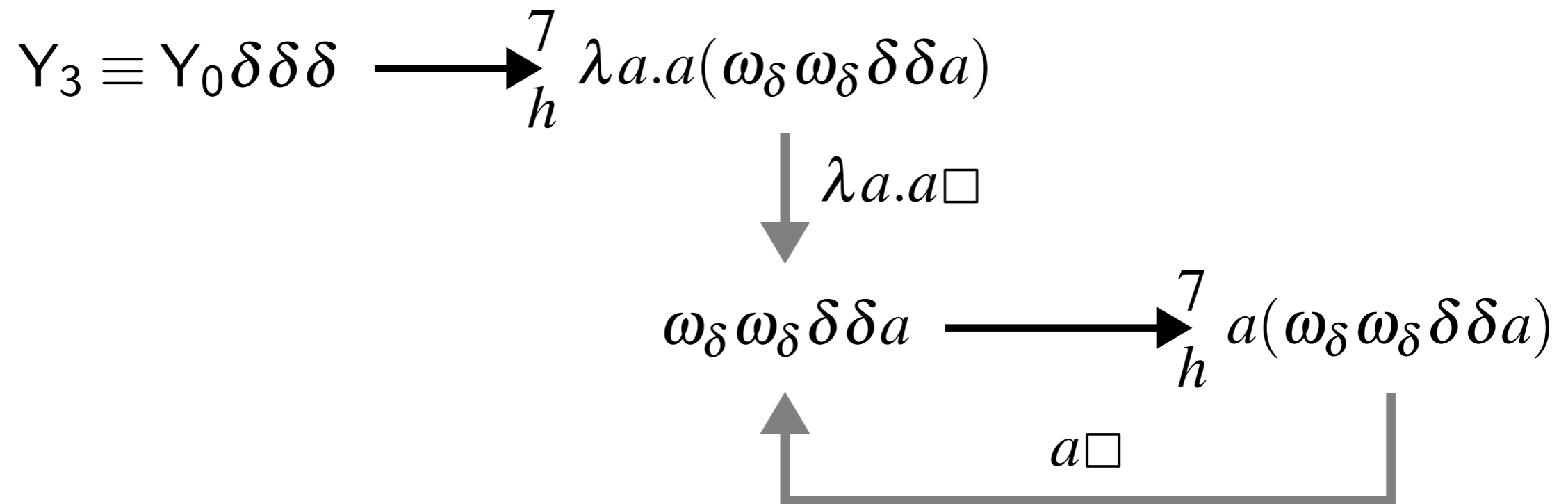
*can we interpret  $\lambda^\infty \beta \Omega$  in  $\mathbf{P}\omega$  ?*

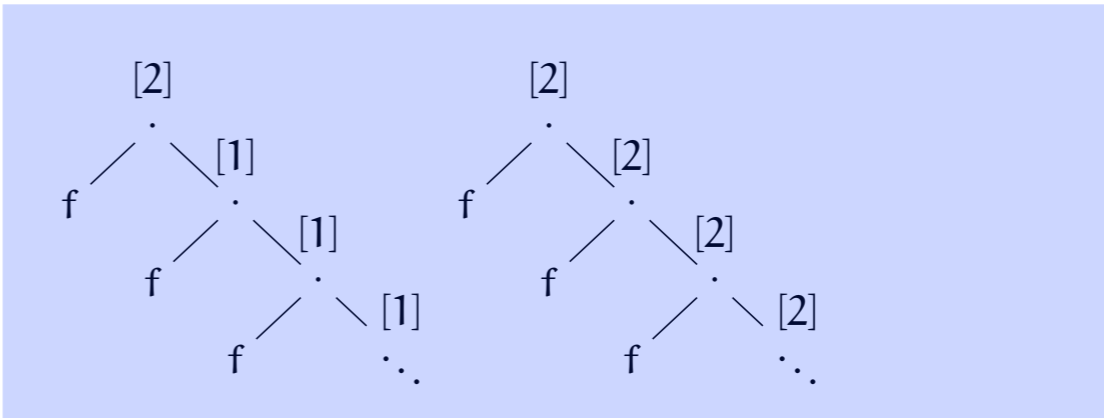
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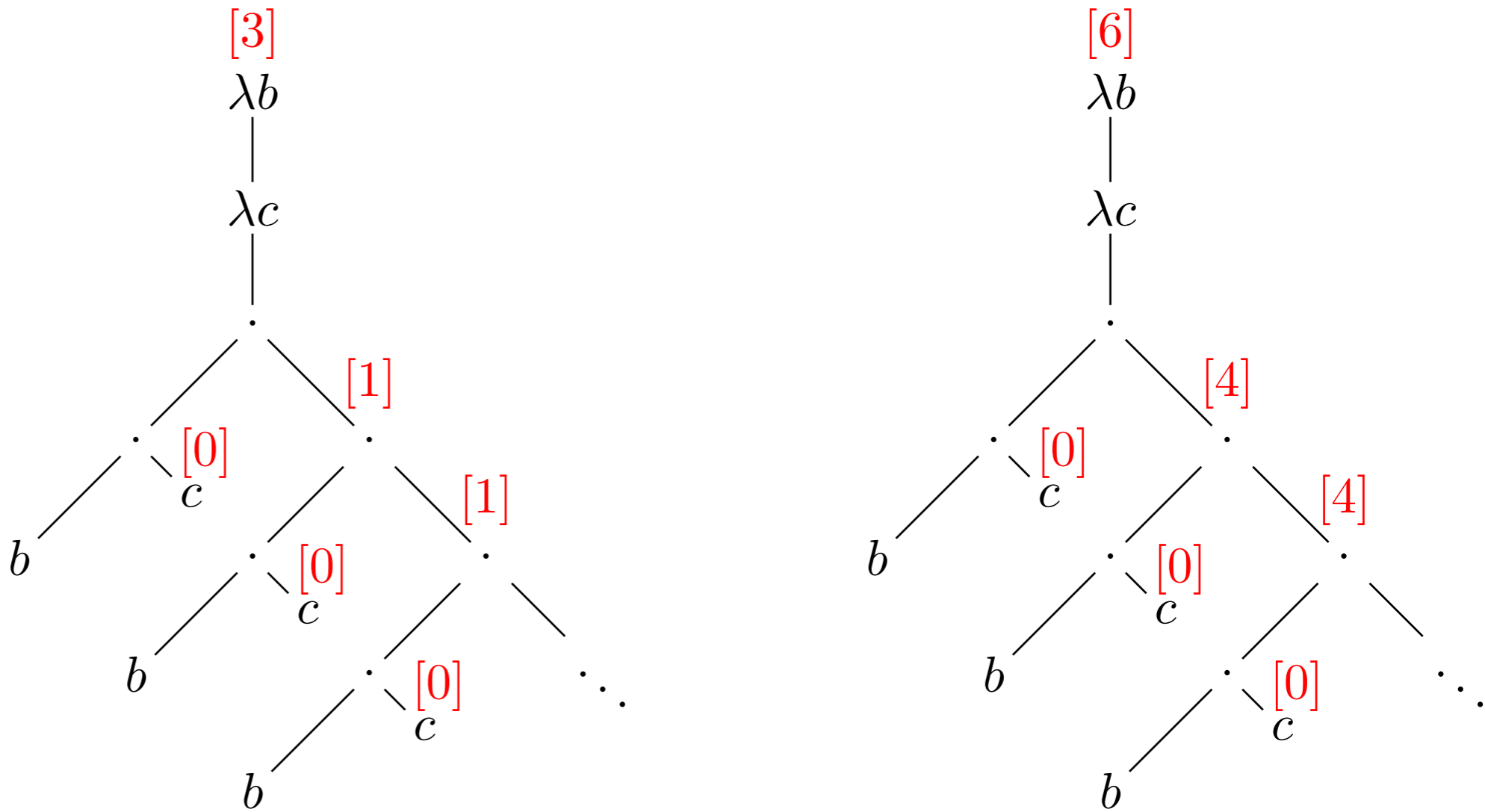
*clock behaviour of fpc in Böhm sequence of fpc's*

$Y_0, Y_0 \delta, Y_0 \delta\delta, Y_0 \delta\delta\delta, Y_0 \delta\delta\delta\delta, \dots$





*Clocked BT's of  $Y_0f$  and  $Y_1f$*



*Clocked Böhm trees of  $BY_0$  and  $BY_0S$ .*

## Clocked Lambda Calculus

$$(\lambda x.M)N \rightarrow \tau(M[x:=N])$$

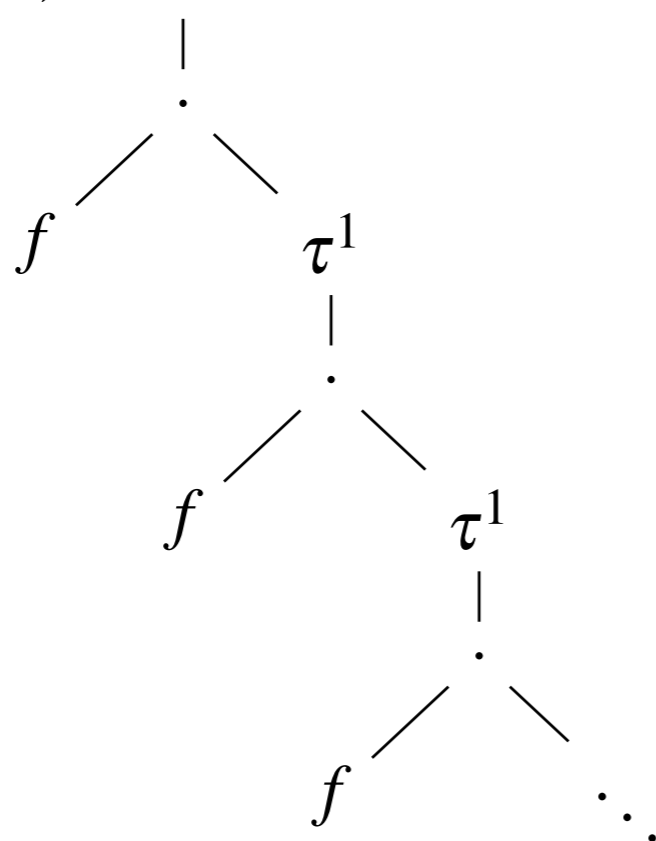
$$\tau(M)N \rightarrow \tau(MN)$$

The  $\tau$ 's are ticks of the clock (measure of efficiency).

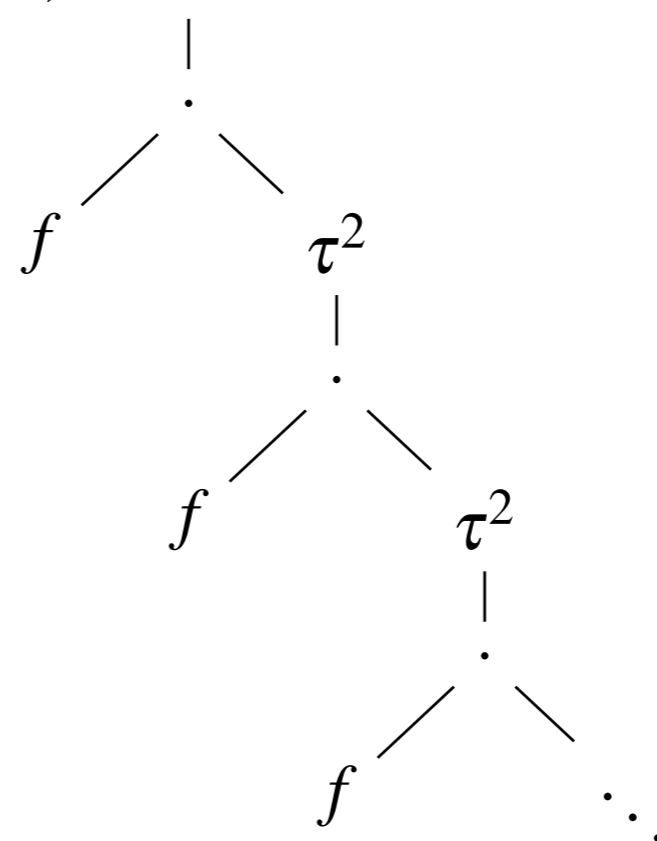
Properties: orthogonal,  $SN^\infty$ ,  $CR^\infty$ ,  $UN^\infty$

Normal forms are clocked Lévy–Longo trees:

$$nf(Y_0 f) \equiv \tau^2$$



$$nf(Y_1 f) \equiv \tau^2$$



different clock  
 $\Rightarrow Y_0 \neq Y_1$

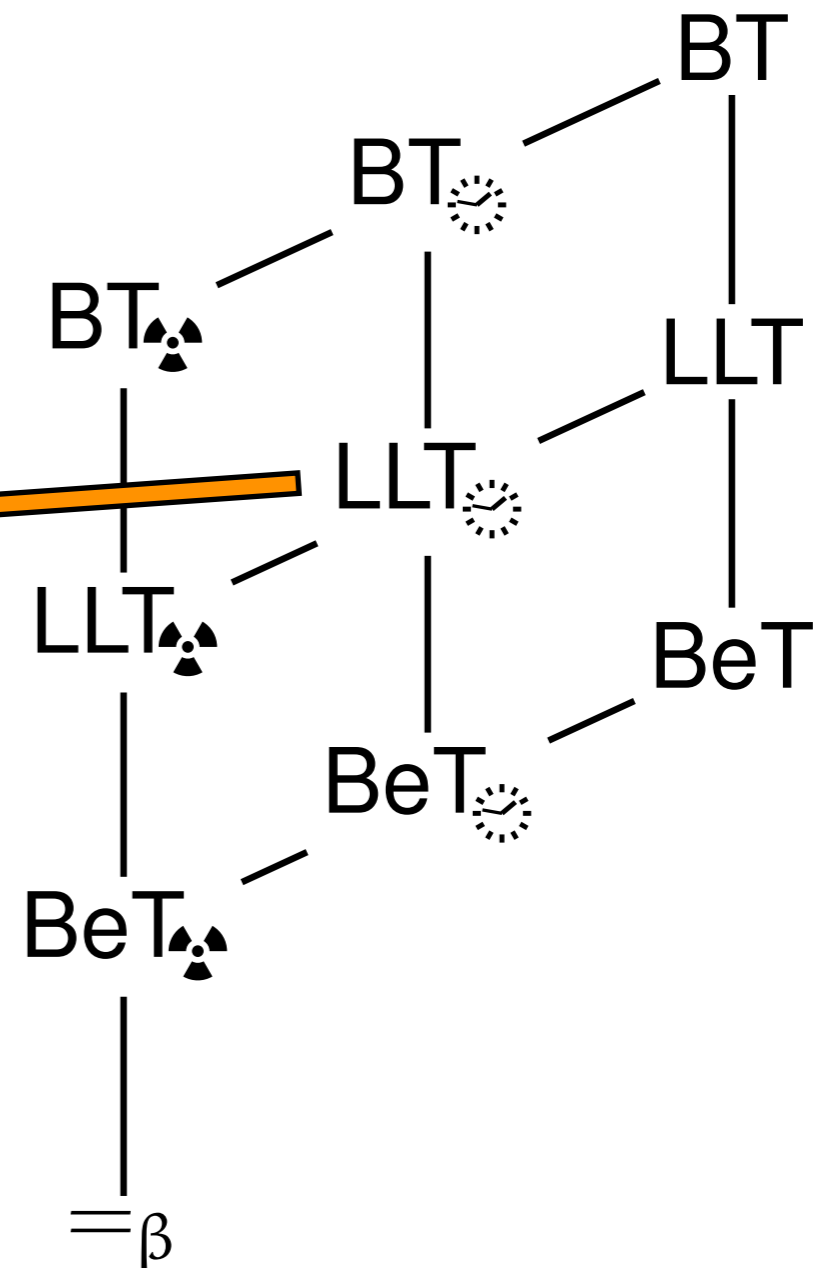


## *clocked lambda theories*

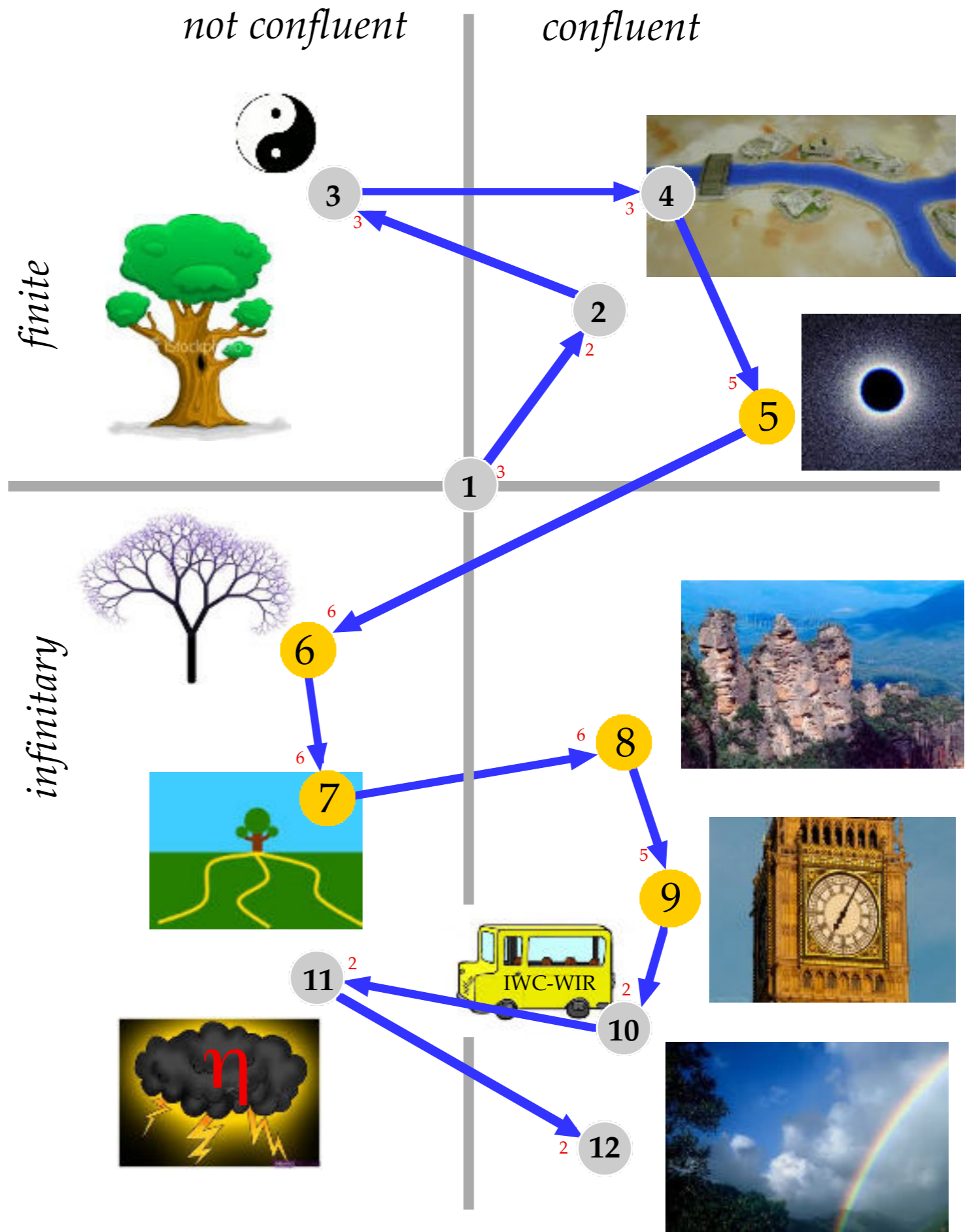
### *Exercise.*

(i) in  $\lambda^\infty\beta$  there is only one Ogre, Omnivore;

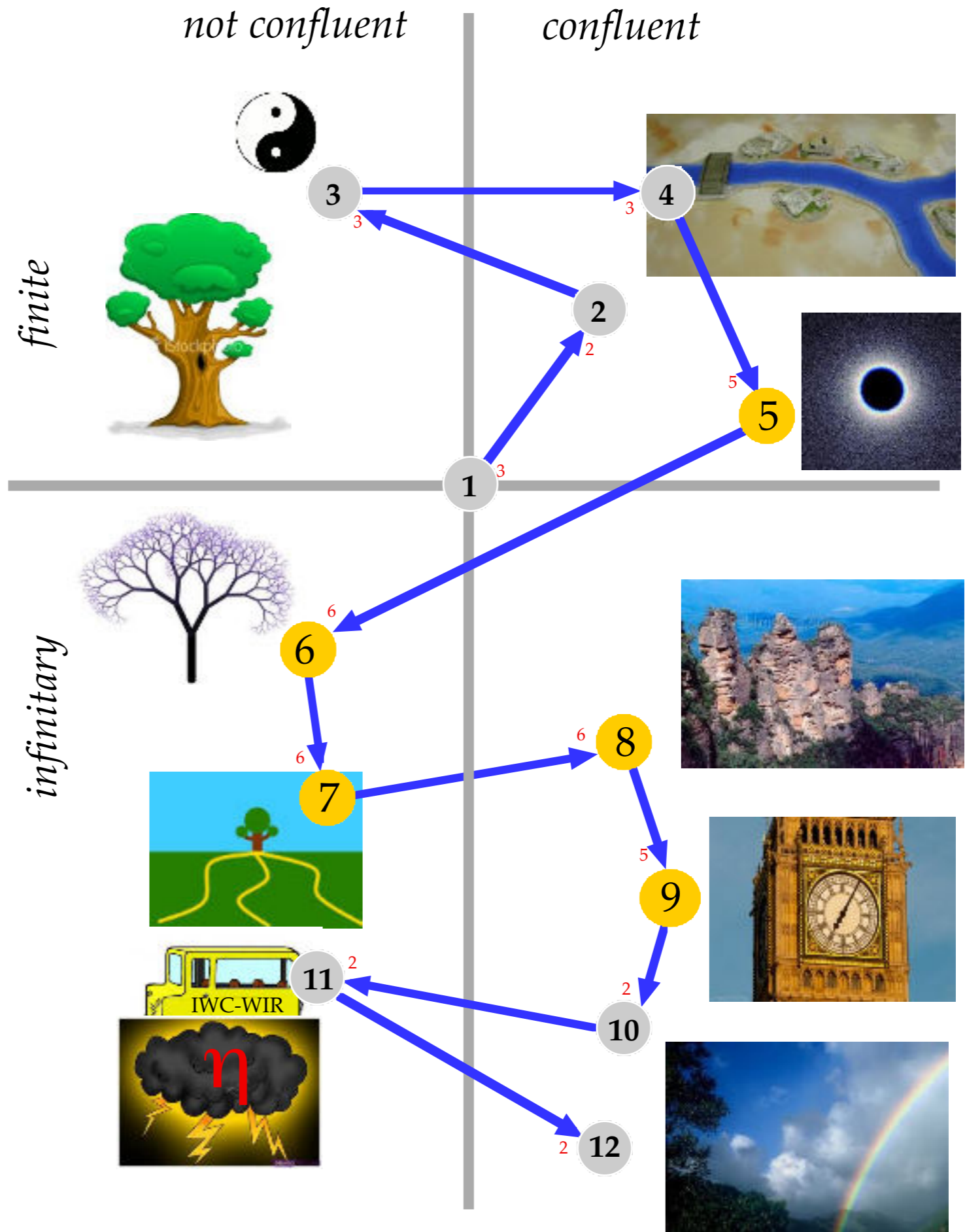
(ii) in  $\lambda\beta$  there are infinitely many,  
i.p. all  $Y_nK$  are different



1. Introduction
2. Lambda and CL: basic confluence
3. Surjective Pairing: confluence lost
4. Confluence of a higher order
5. Lambda with black holes: confluence
6. Confluence lost in infinity
7. The threefold path
8. Black holes to the rescue
9. The rhythm of lambda terms
10. Getting rid of ordinals
11. Infinity and eta: total breakdown
12. A lambda universe



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$$PS \rightarrow \varepsilon$$

$$SP \rightarrow \varepsilon$$

where  $\varepsilon$  is the empty word. This system has two trivial critical pairs:

$$P \leftarrow \overline{PSP} \rightarrow P$$

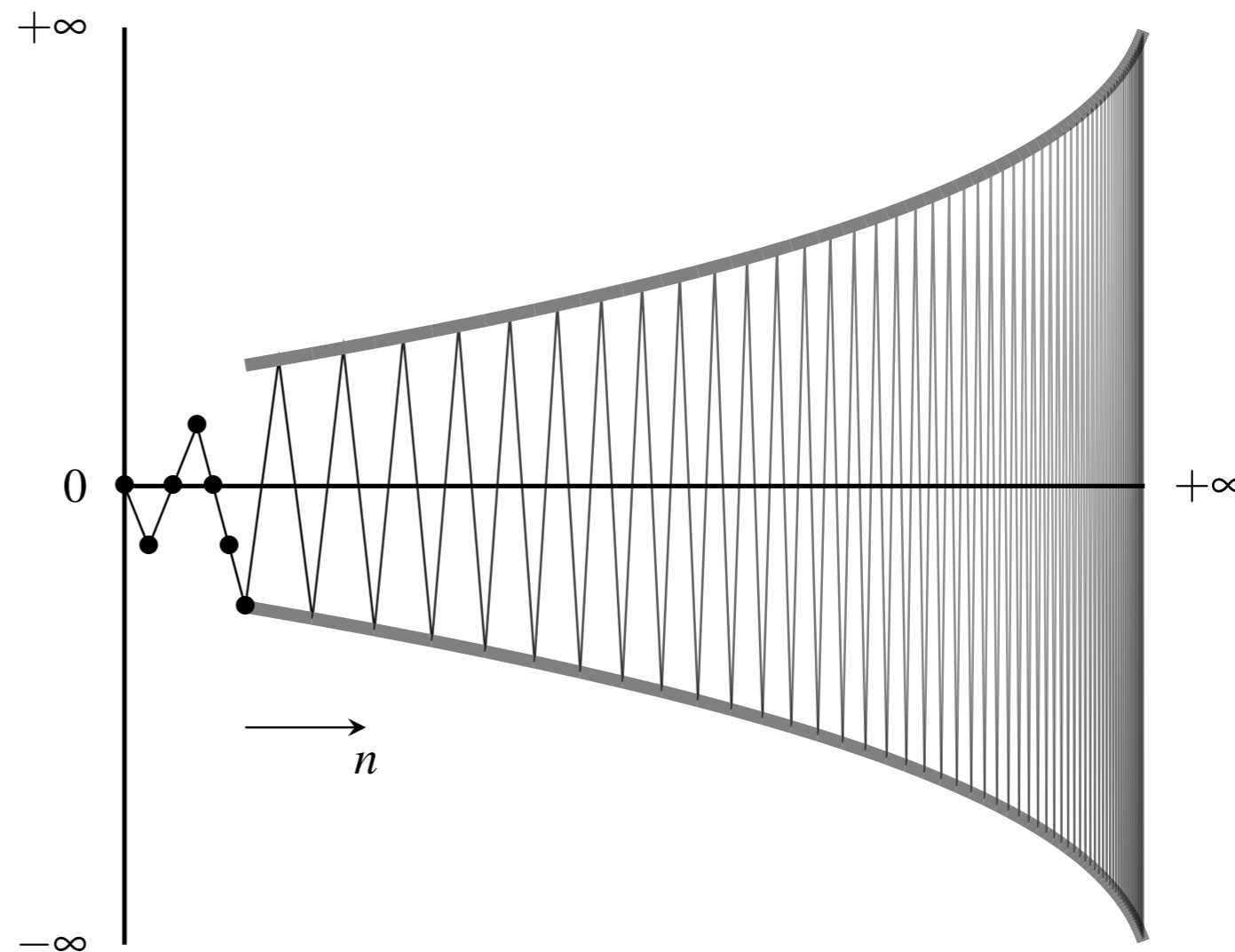
$$S \leftarrow \overline{SPS} \rightarrow S,$$

and hence is weakly orthogonal.

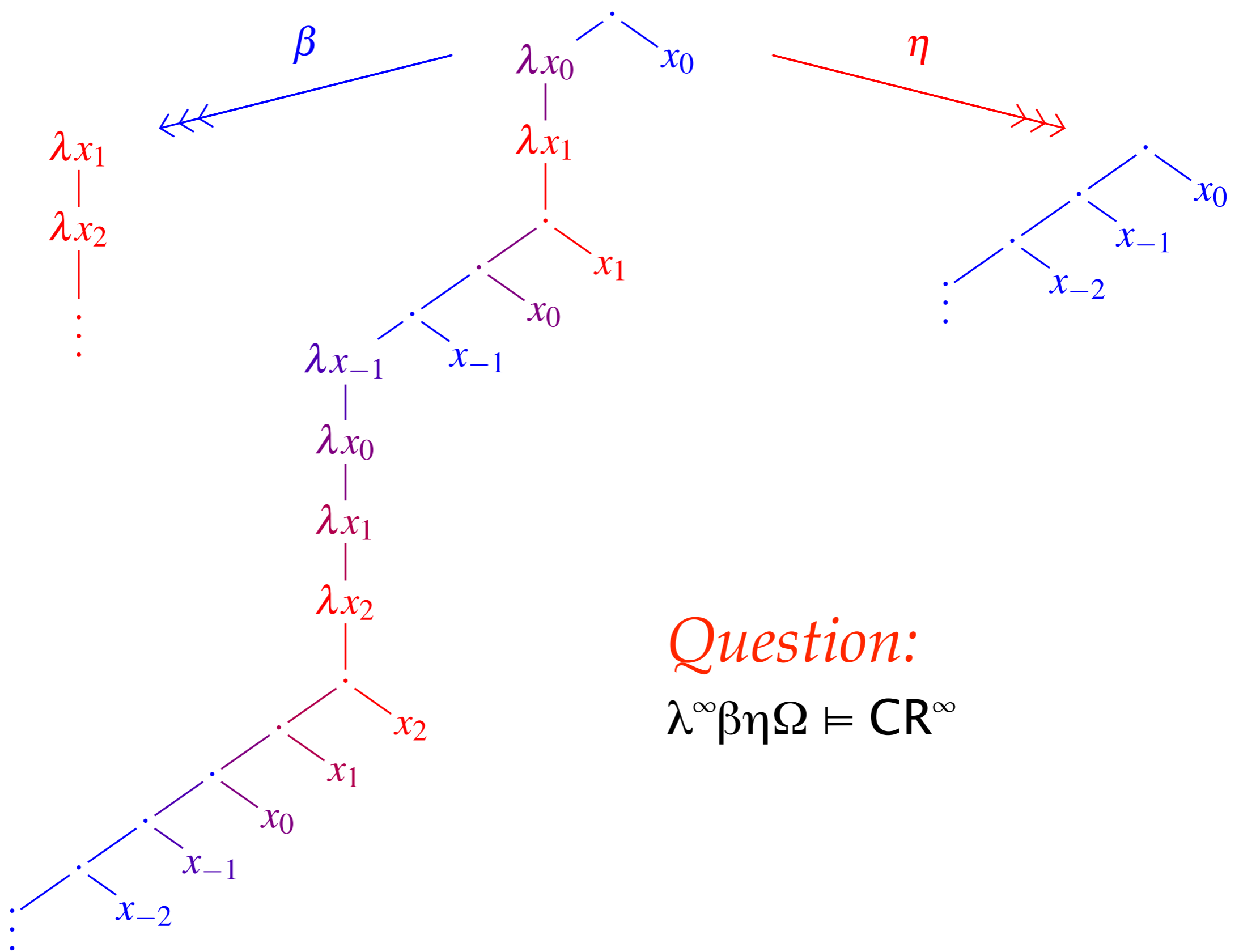
Now consider the term  $\psi$  defined as follows:

$$\psi = P \text{ SS PPP SSSS PPPPP SSSSSS } \dots$$

$$S^\omega \leftarrow \psi \rightarrow P^\omega$$



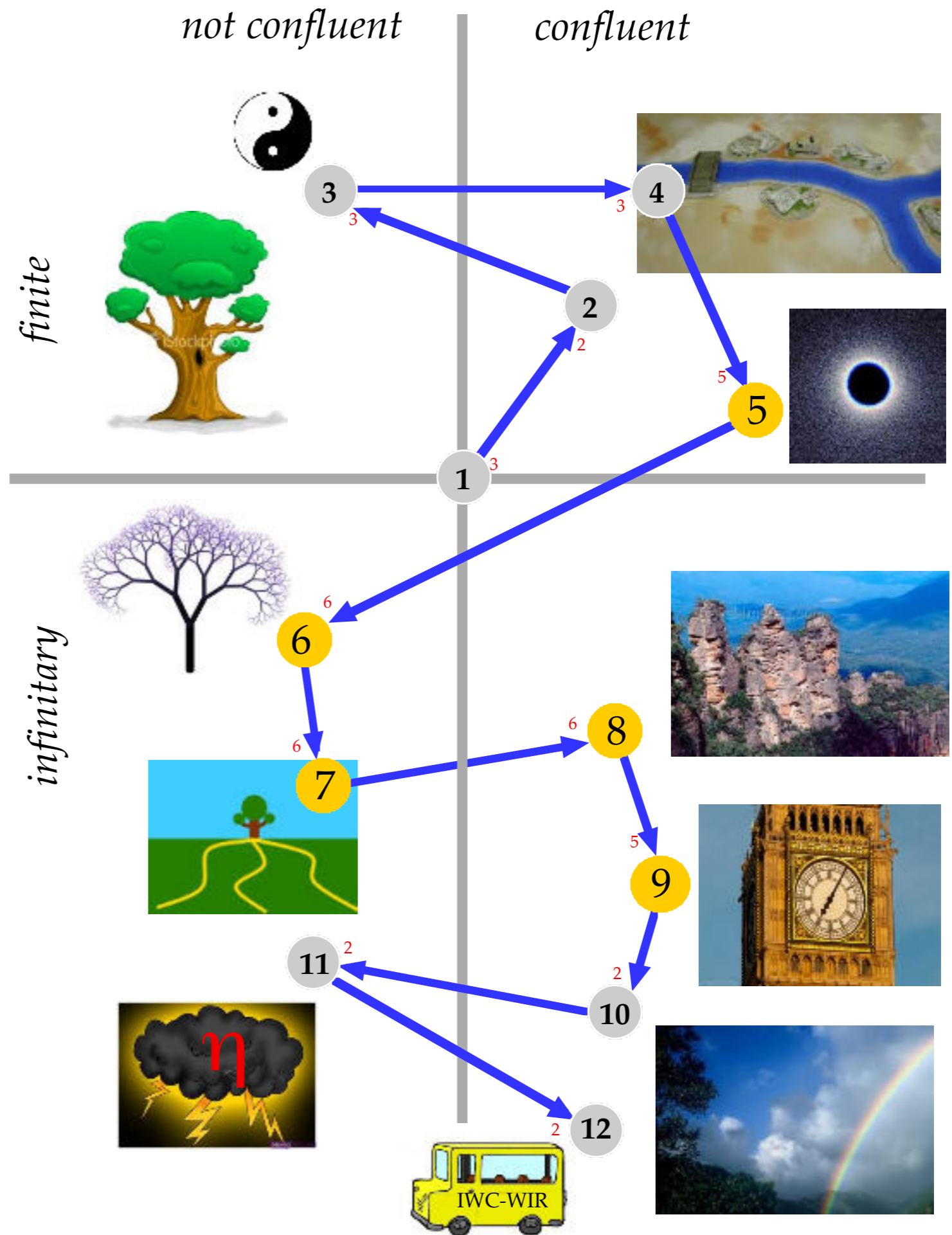
$$\lambda^\infty \beta \eta \not\equiv \mathbf{UN}^\infty$$



*Question:*  
 $\lambda^\infty \beta \eta \Omega \equiv \mathbf{CR}^\infty$



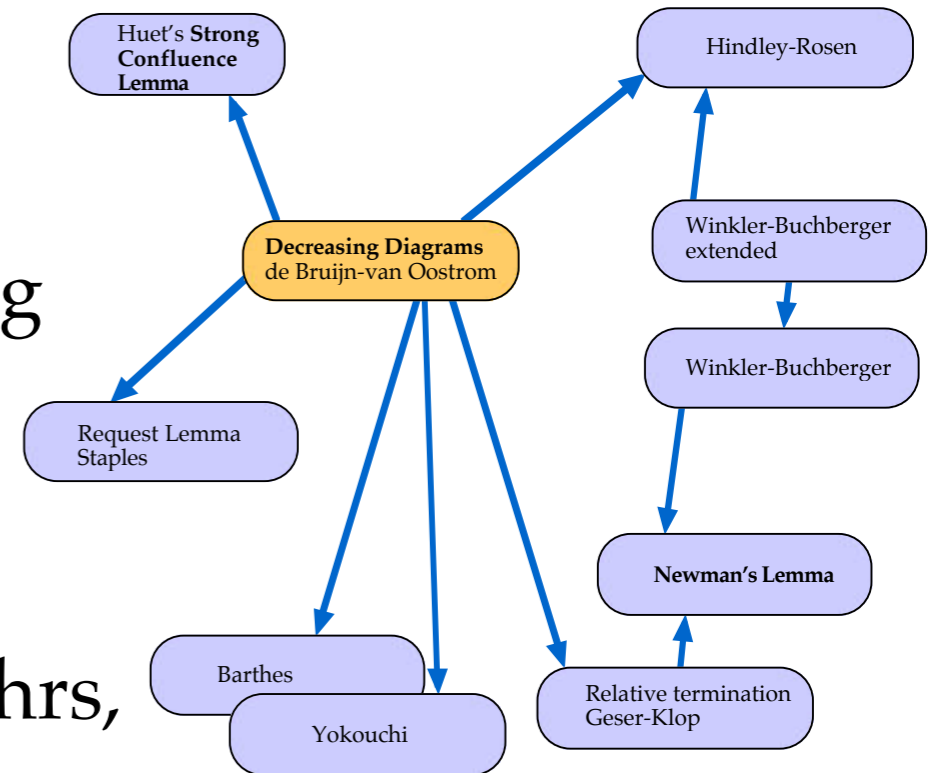
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# Other bus tours of the RTA-IWC-WIR company

Abstract rewriting and confluence, decreasing diagrams, De Bruijn - Van Oostrom

Alternative set-up of infinitary rewriting, Kahrs, ideal completion (Bahr), coinductive definition, Endrullis, Polonsky et al.



## Infinitary Rewriting Coinductively

$$\twoheadrightarrow = \mu x. \nu y. (\rightarrow_{\varepsilon} \cup \bar{x})^* \circ \bar{y}$$

$$\bar{R} = \{ \langle f(s_1, \dots, s_n), f(t_1, \dots, t_n) \rangle \mid s_1 R t_1, \dots, s_n R t_n \} \cup \text{id}$$



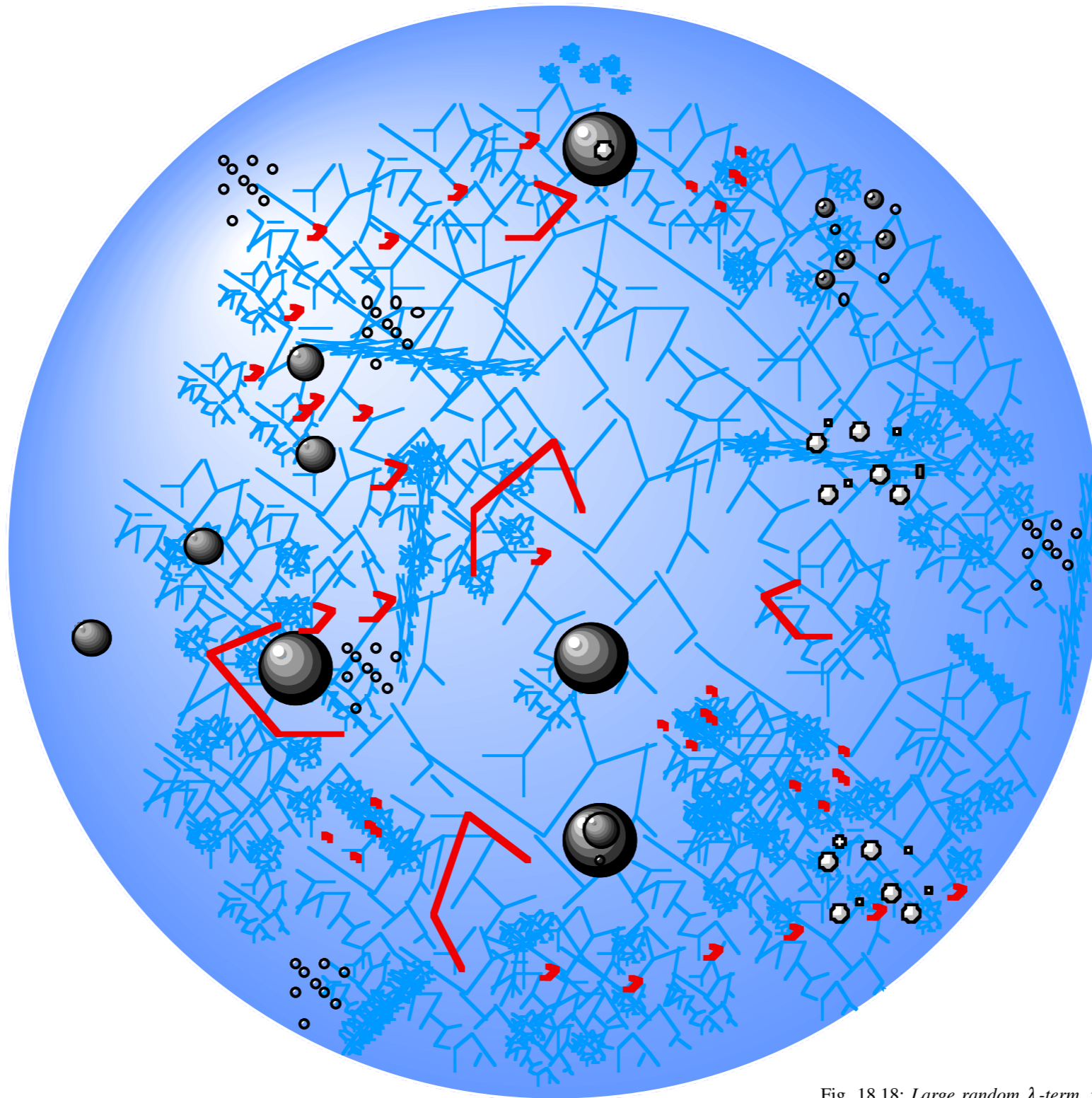
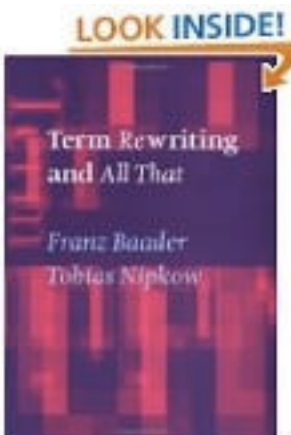


Fig. 18.18: Large random  $\lambda$ -term, viewed as a mini-cosmos, evolving non-deterministically by local changes due to  $\beta$ -steps; their patterns are the red configurations. In the final result the place and nature of the normalized parts of the structure, as well as the singularities formed by the unsolvable terms, the black holes, is 'predestined', independent of the actual evolution path to the normal form, an infinite  $\lambda\beta\Omega$ -term.



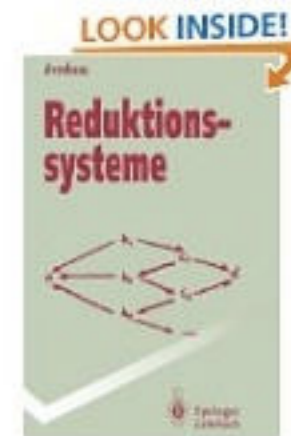
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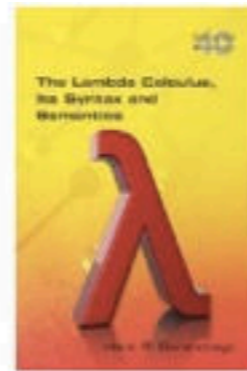
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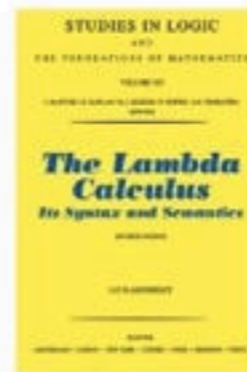
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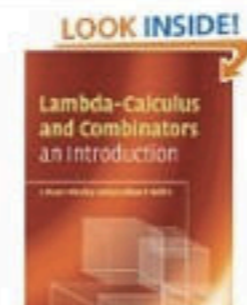
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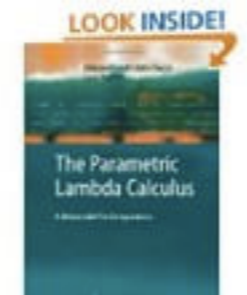


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