An Inkling of lambda calculus and rewriting:

black holes, causality, and graphs

Henk *Jan Willem* *Joerg & Roy*

Wolfram Research Seminar

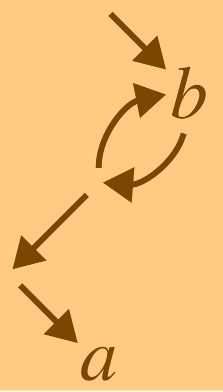
October 29, 2020

Proving Wolfram's *Causal Invariance Principle*

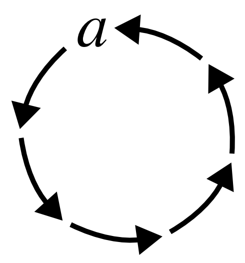
Jan Willem Klop

Wolfram Research Seminar
October 29, 2020

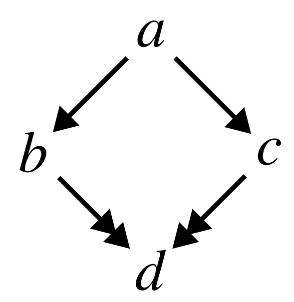
Abstract rewriting in 30 seconds or less



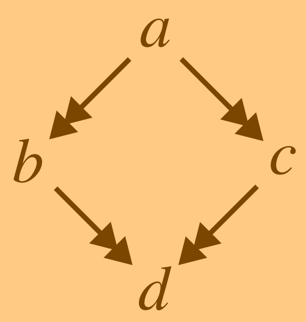
normal form



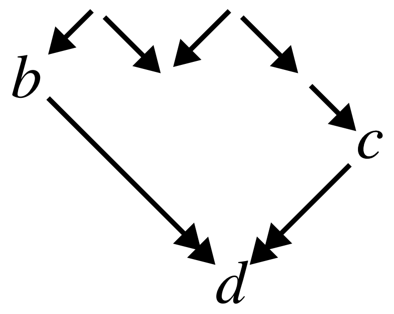
reduction cycle; loop if one step



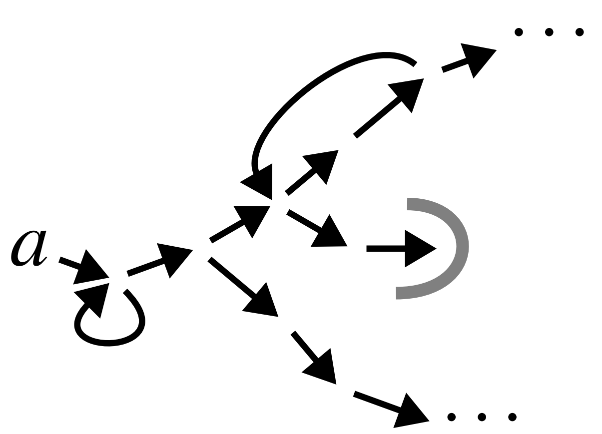
WCR, weakly Church-Rosser



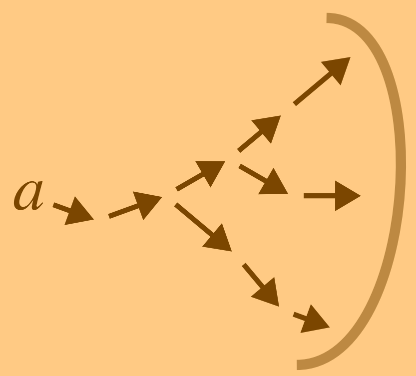
CR, Church-Rosser



equivalent: CR, Church-Rosser

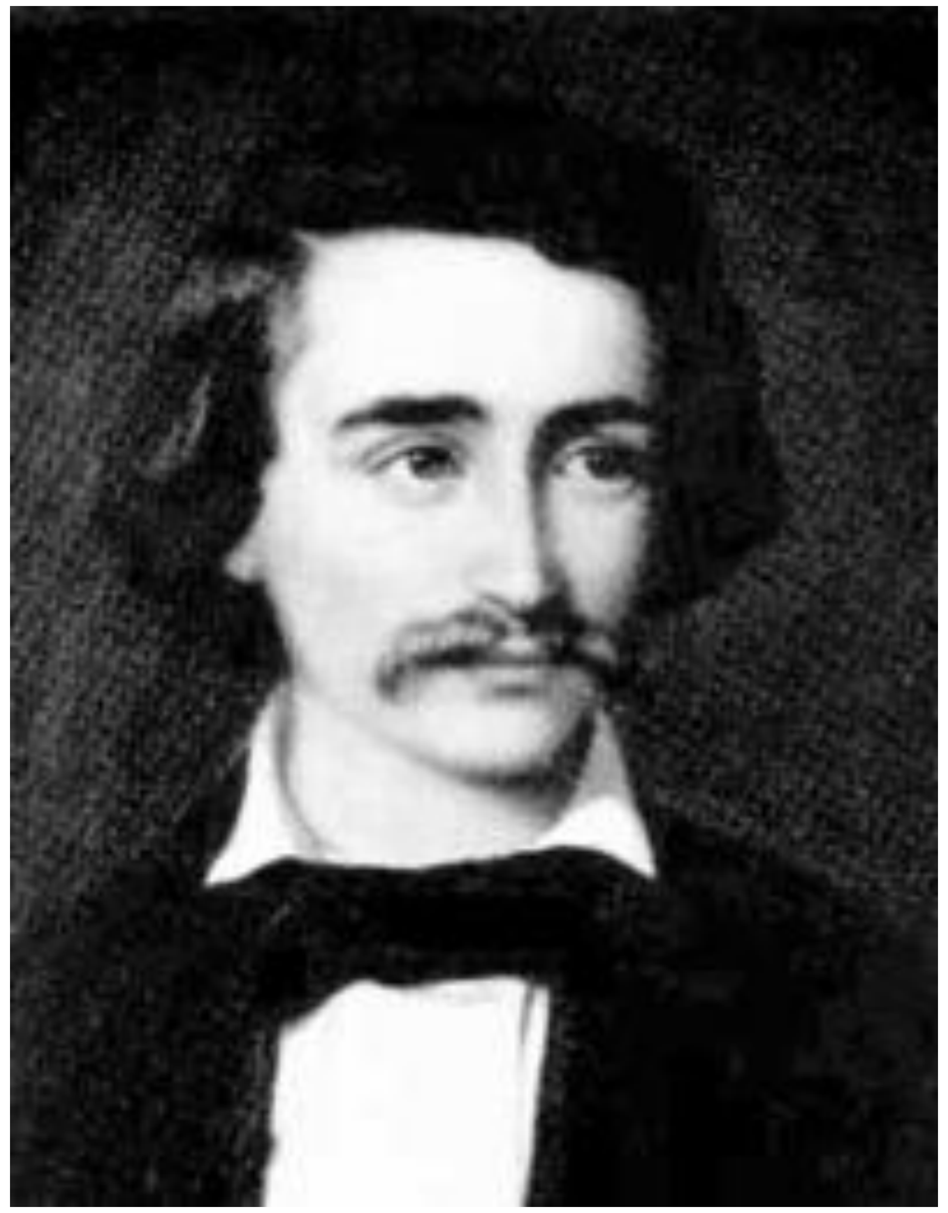


WN, weakly normalizing



SN, strongly normalizing;terminating; noetherian

Term rewriting in 30 seconds or less



Richard Dedekind, 1831-1916.

OK

orthogonal TRS

not OK

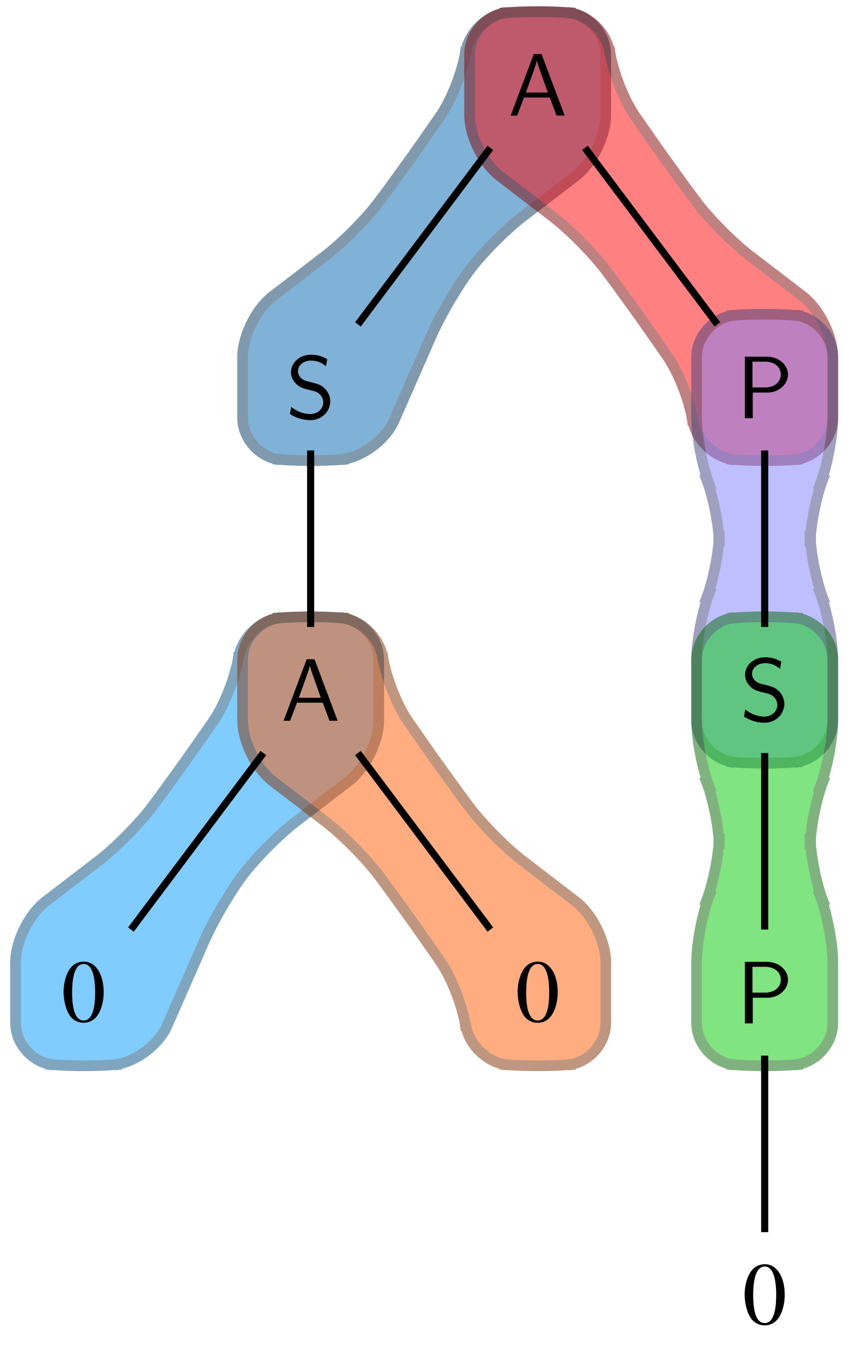
erasing rule

$$\mathsf{A}(x,0) \to x$$
$$\mathsf{A}(x,\mathsf{S}(y)) \to \mathsf{S}(\mathsf{A}(x,y))$$
$$\mathsf{M}(\underline{x,}0) \to 0$$
$$\mathsf{M}(x,\mathsf{S}(y)) \to \mathsf{A}(\mathsf{M}(x,y),x)$$

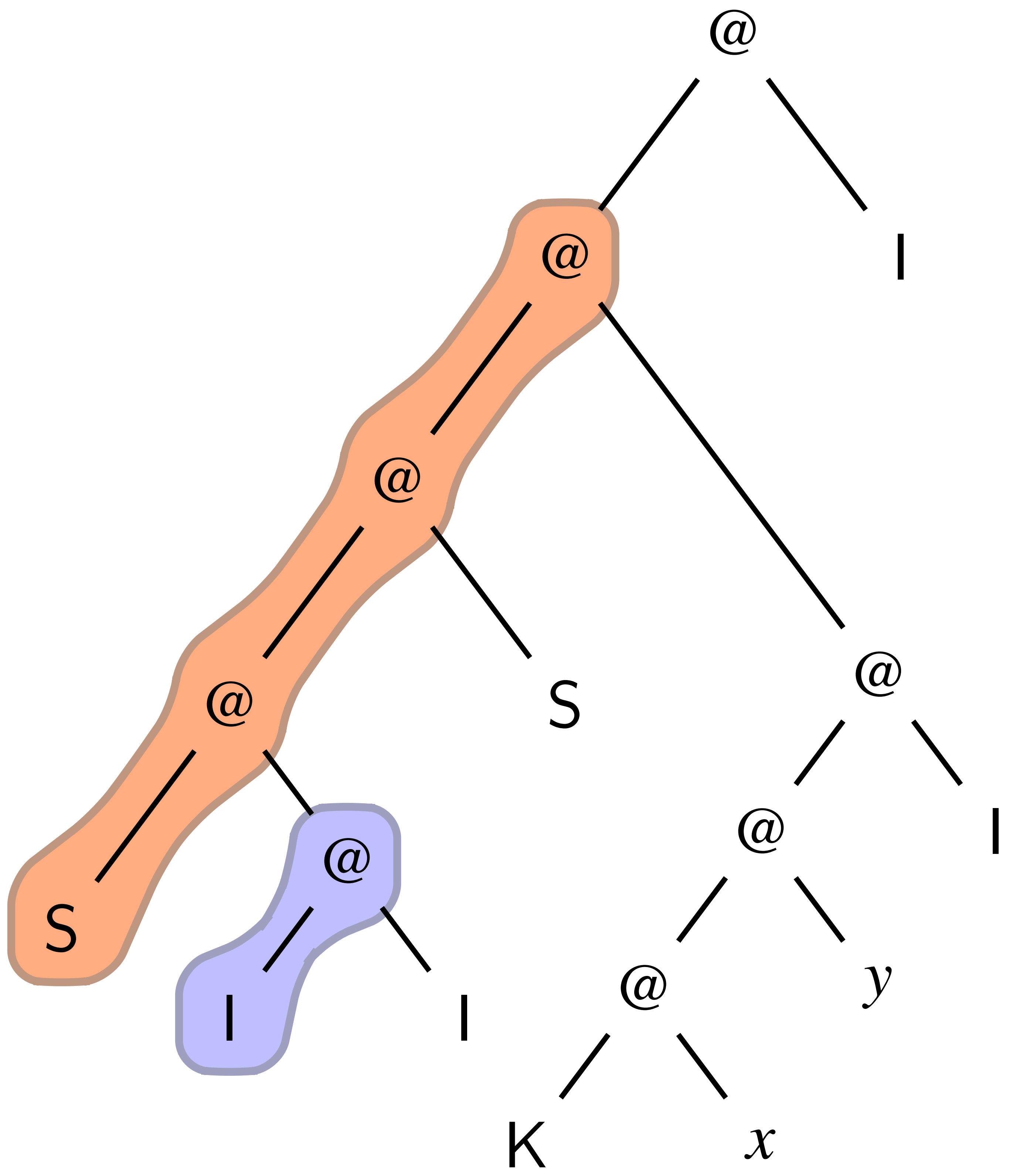
Dedekind's TRS for addition and multiplication of natural numbers.

Orthogonal rewriting

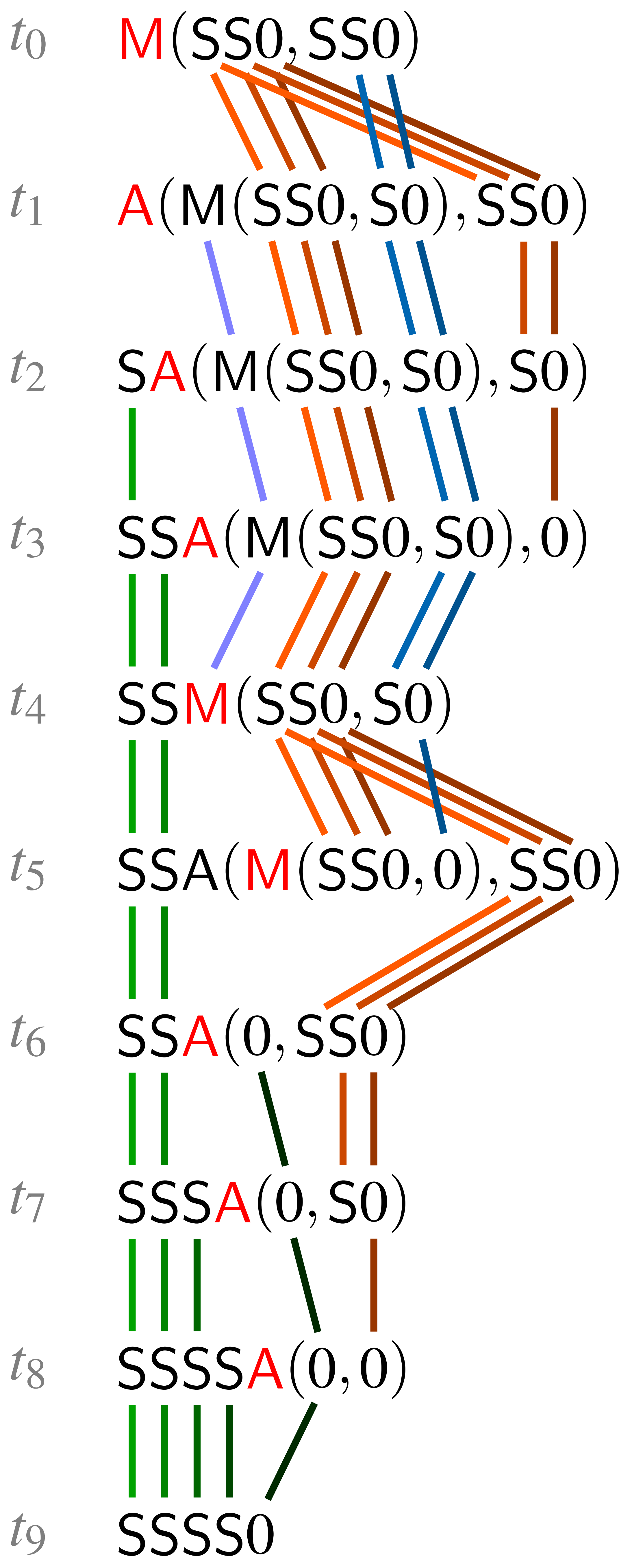
overlapping rules



non-overlapping rules in CL



descendant; creation; causation



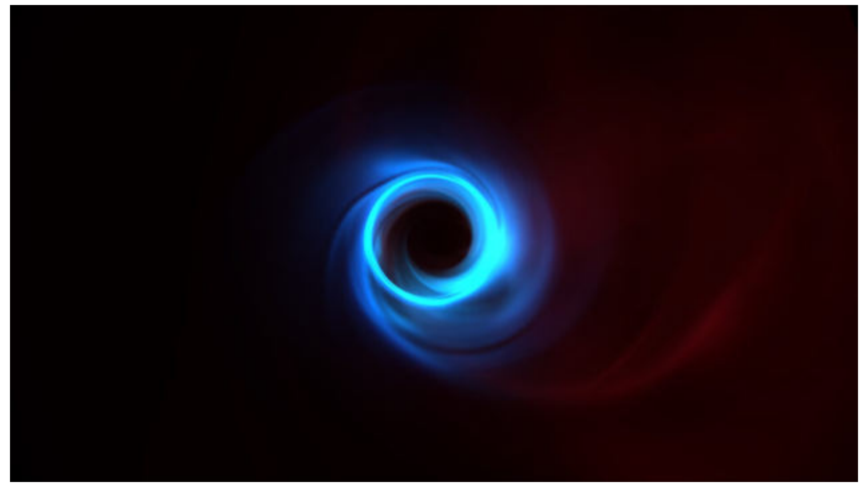
$$\ldots[(\lambda x.C[xB])\lambda y.A]\ldots \to_\beta \ldots[C^\sigma[(\lambda y.A)B^\sigma]]\ldots \tag{1}$$

$$\ldots[(\lambda x.x)(\lambda y.A)B]\ldots \to_\beta \ldots[(\lambda y.A)B]\ldots \tag{2}$$

$$\ldots[(\lambda x.\lambda y.A)DB]\ldots \to_\beta \ldots[(\lambda y.A^\sigma)B]\ldots \tag{3}$$

$$BAA \to ABA$$

$$BBA \to BAB$$



Black holes in term rewriting

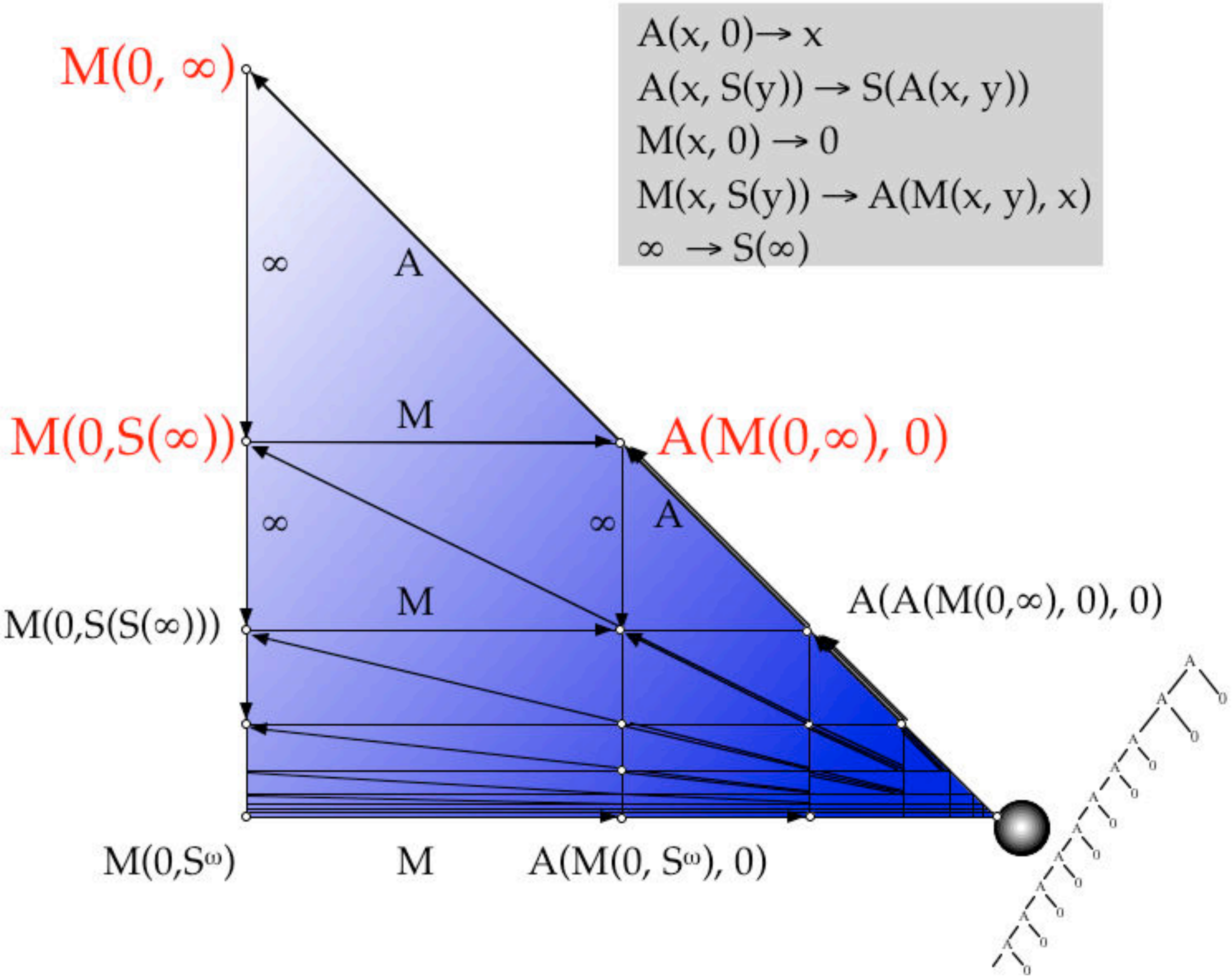
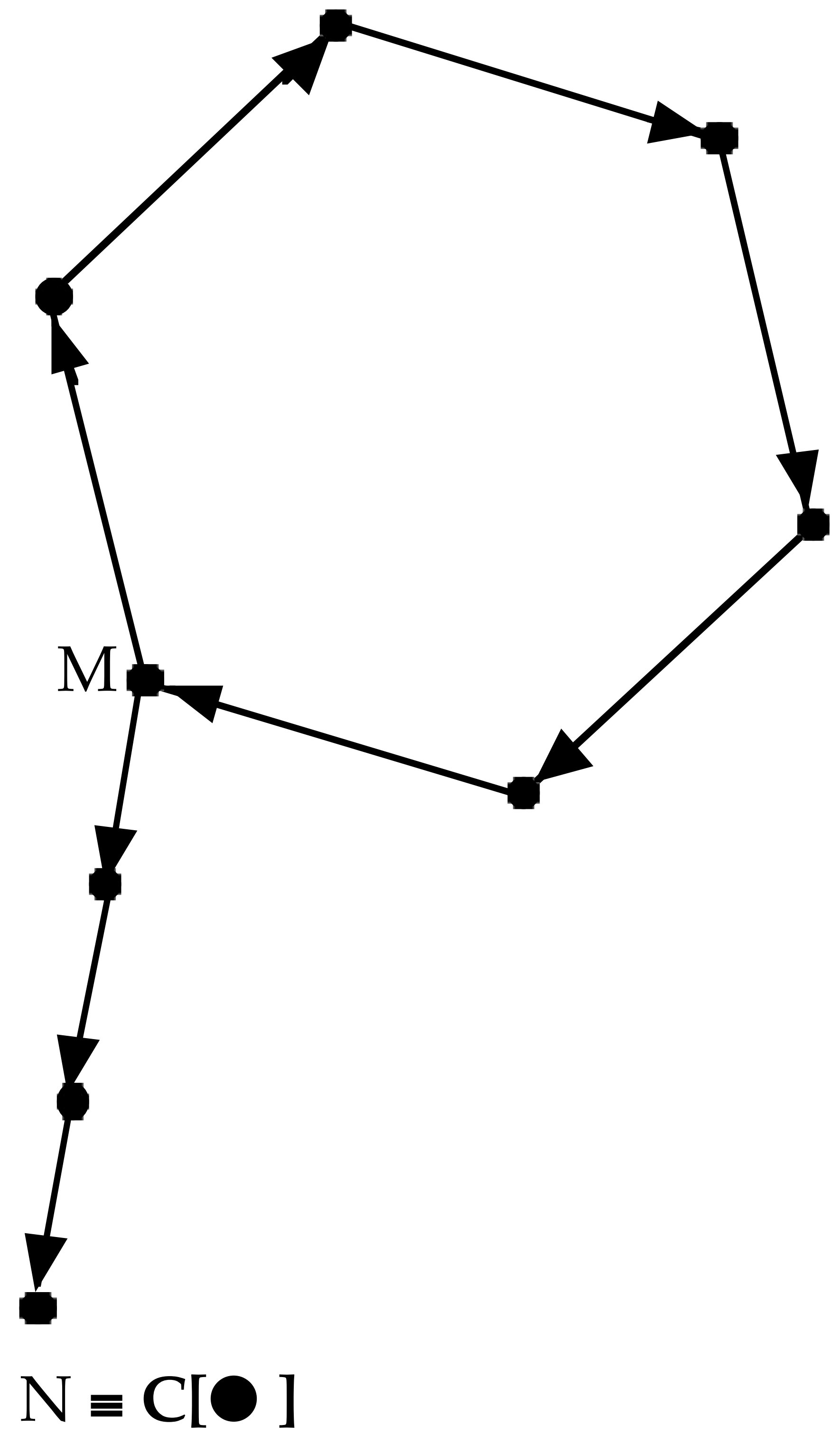
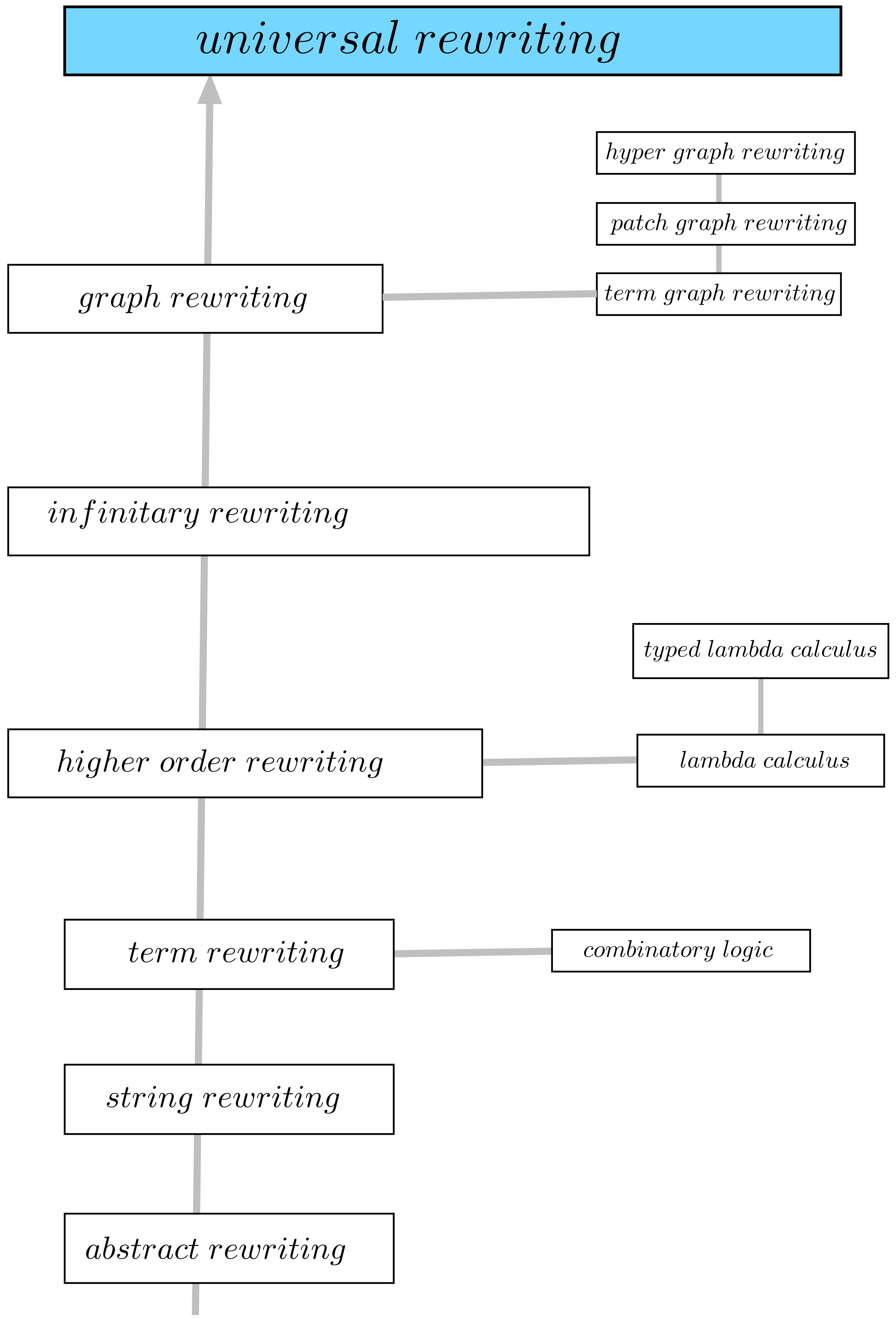


Figure 26: *Zero times infinity.*

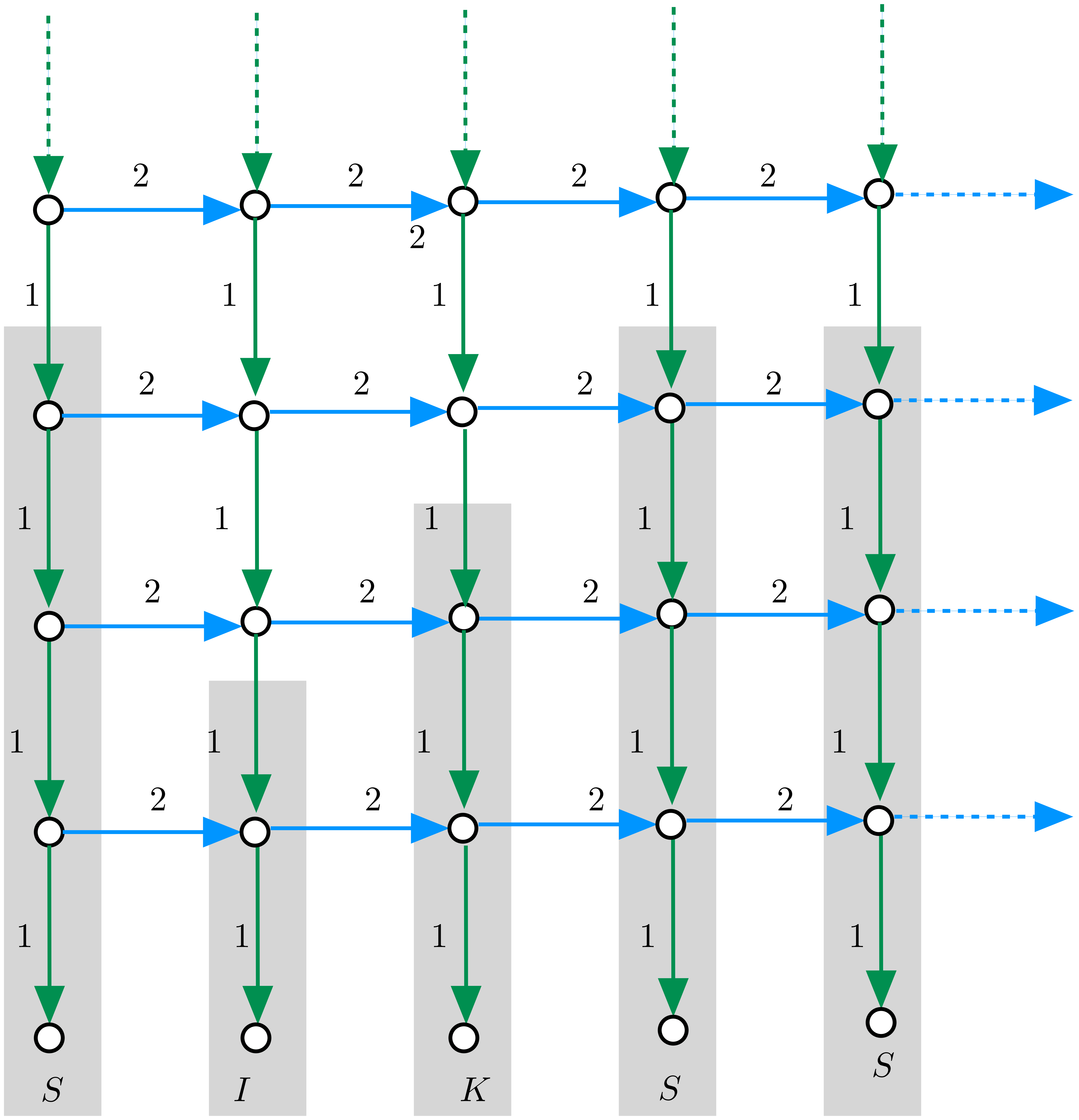
reduction cycles entail black holes



Theorem. Let the λ-term M admit a cyclic reduction. Then M is unsolvable, or else, a subterm of a reduct of M is unsolvable. In fact, even 'mute', the worst kind of unsolvable.



universal rewriting
hyper graph rewriting
patch graph rewriting
term graph rewriting
graph rewriting
infinitary rewriting
typed lambda calculus
higher order rewriting
lambda calculus
term rewriting
combinatory logic
string rewriting
abstract rewriting



96

plane filling CL term with redexes

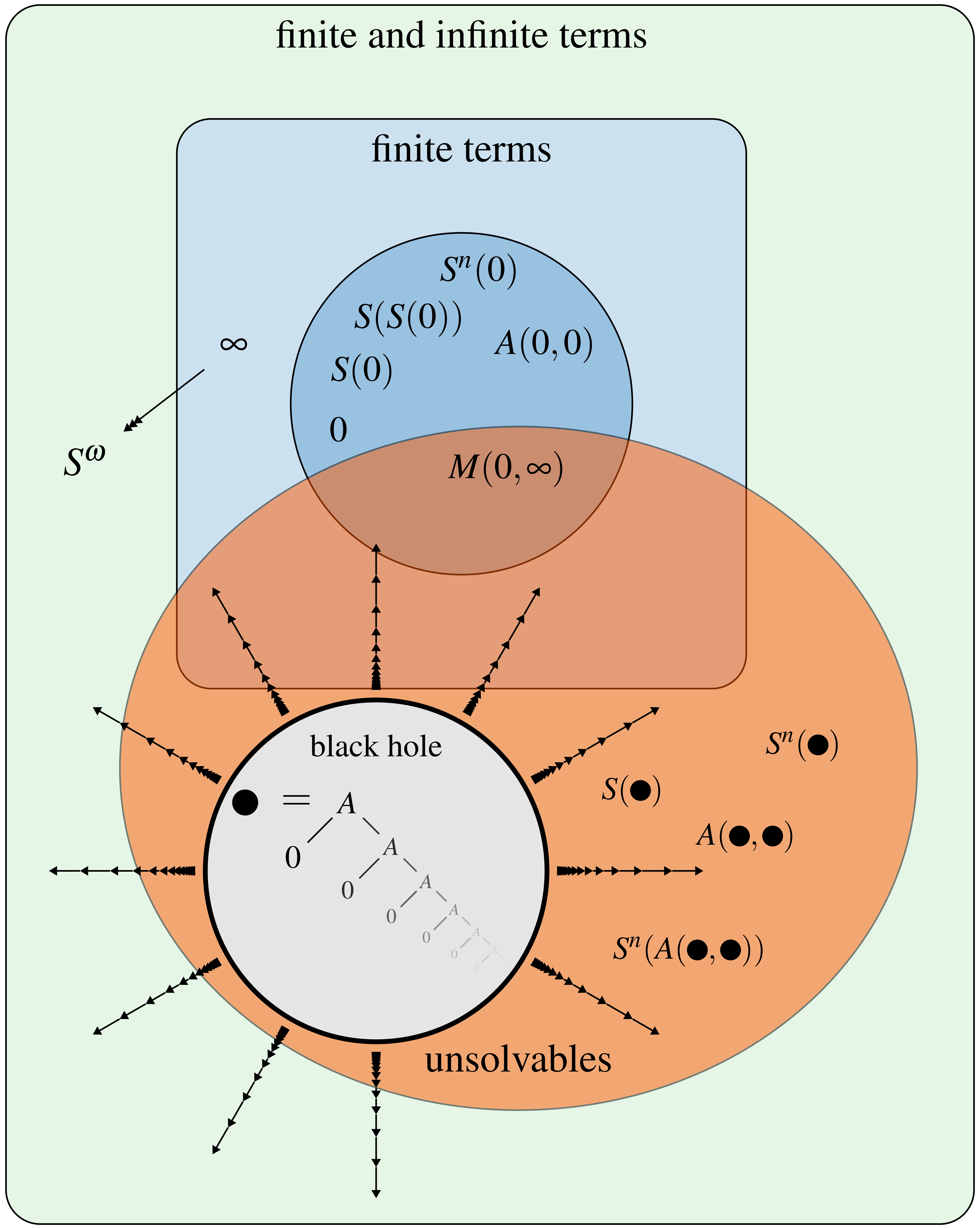


Figure 27: D$^\infty$, *the mini-universe of Dedekind's rules for natural numbers with infinity* ∞*. The outer space of infinite terms contains many black holes. The black holes are* Cauchy condensation points, *the limits of* expansions, *inverse reductions. The property* CR$^\infty$ *holds, because there is only one collapsing operator. The black hole term has a trivial causal graph, consisting of infinitely many unrelated points, a discrete partial order, as we will see later.*

Theorem 5.1. *(Wolfram, Causal Invariance)*

For orthogonal rewrite systems, given a normalizing reduction $M \twoheadrightarrow N$*, the partial order of 'causation' between redex firings is invariant among all reductions from* M *to its normal form* N*. Causation means that the latter redex firing uses symbols descending from symbols created by the former redex firing.*

Joerg: *non-erasing*

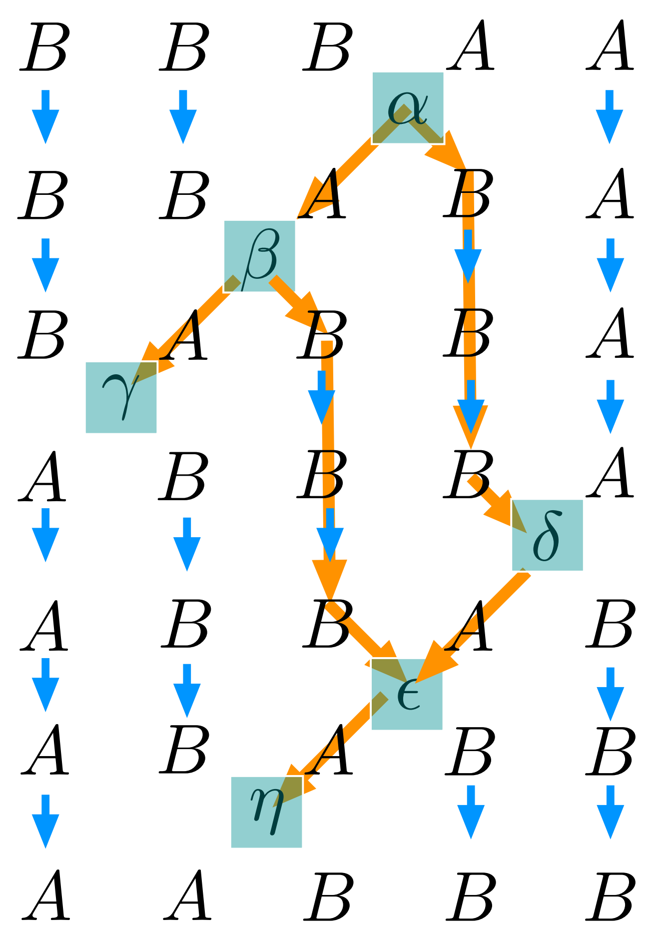


Figure 11: *Wolfram's Causal Invariance displayed in rewrite sequence* BBBAA ↠ AABBB *using the orthogonal string rewrite rule* BA → AB*, with redex firings* α, β, γ, δ, ϵ, η*. Yellow arrows are the 'caused by' relation. Blue arrows are the descendant relation. A 'caused' redex firing uses material* descending from *material* created by *an earlier 'causing' firing. The ensuing partial order is an invariant among all rewritings from the same begin term* BBBAA *to its normal form* AABBB*. This example is taken from Wolfram's writings.*

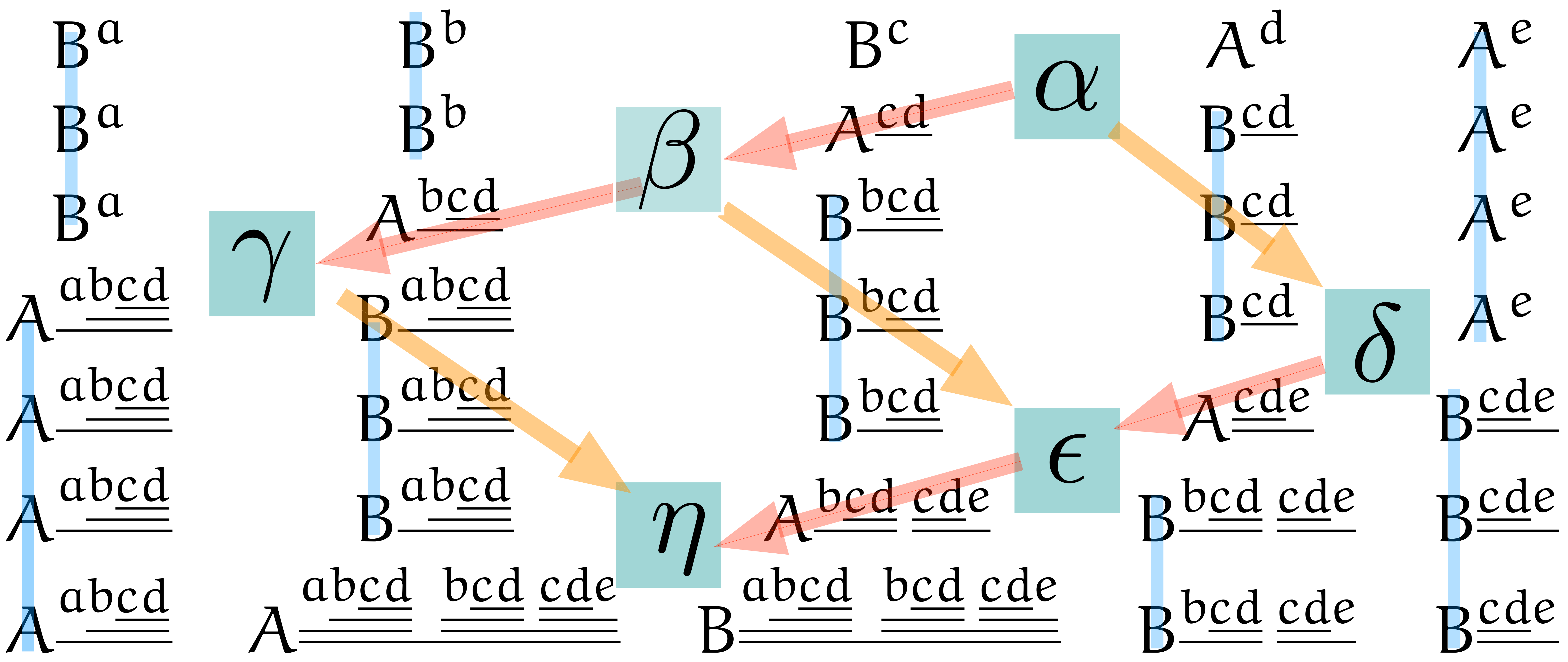


Figure 12: *Lattice of contracted redexes in orthogonal reduction from* BBBAA *to its normal form* AABBB*. Along the reduction Lévy labels record the development. The final labels encode the lattice structure, invariant under all the alternative normalizing reductions according to Wolfram's causal invariance principle. 'Causation steps' are given by the subterm relation among the* redex degrees*. Transparent blue lines are the* descendant *relation. Transparent red steps are the redex* creation *steps, also causal steps.*

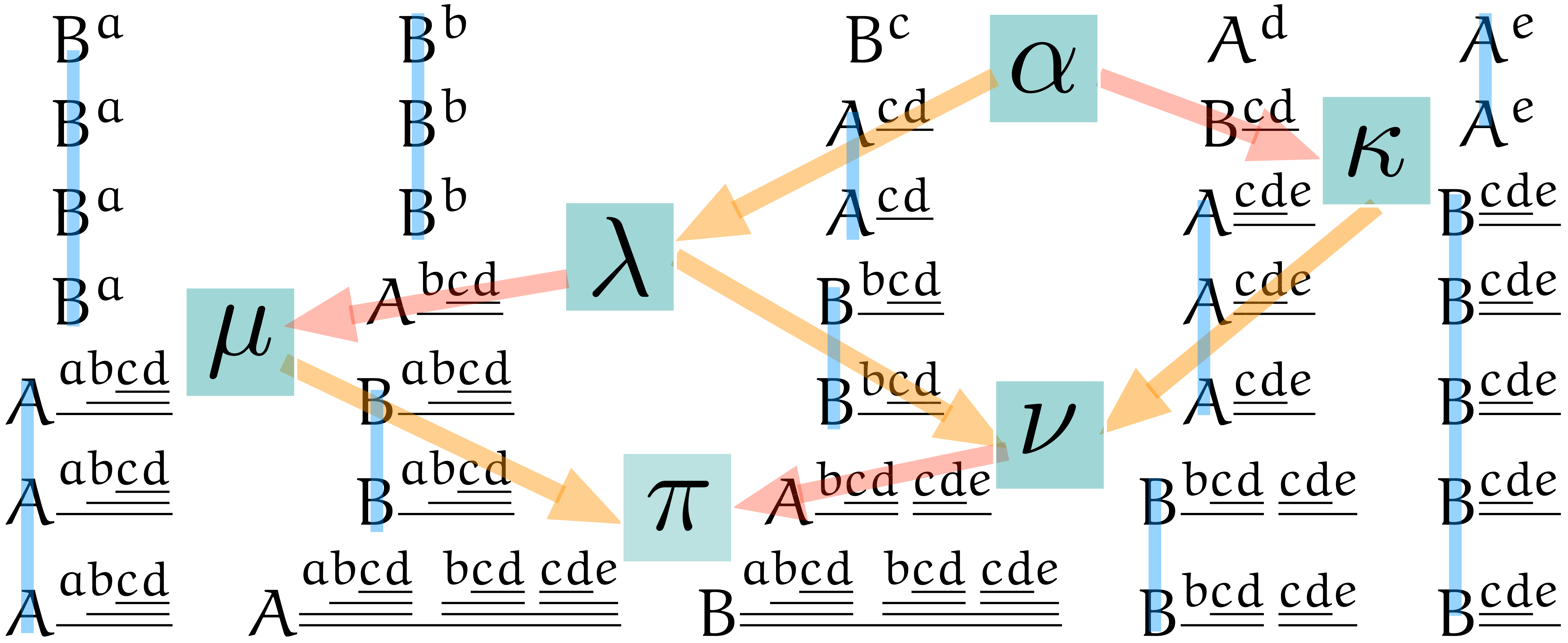


Figure 14: *Lattice of contracted redexes in an alternative reduction from* BBBAA *to its normal form* AABBB. *The causal lattice is the same. The isomorphism is given by the reduction diagram establishing the Lévy equivalence of the two reductions; see Figure xx. Note that creation (transparent red) is not invariant, but causation (transparent yellow or red) is.*

A term rewrite example

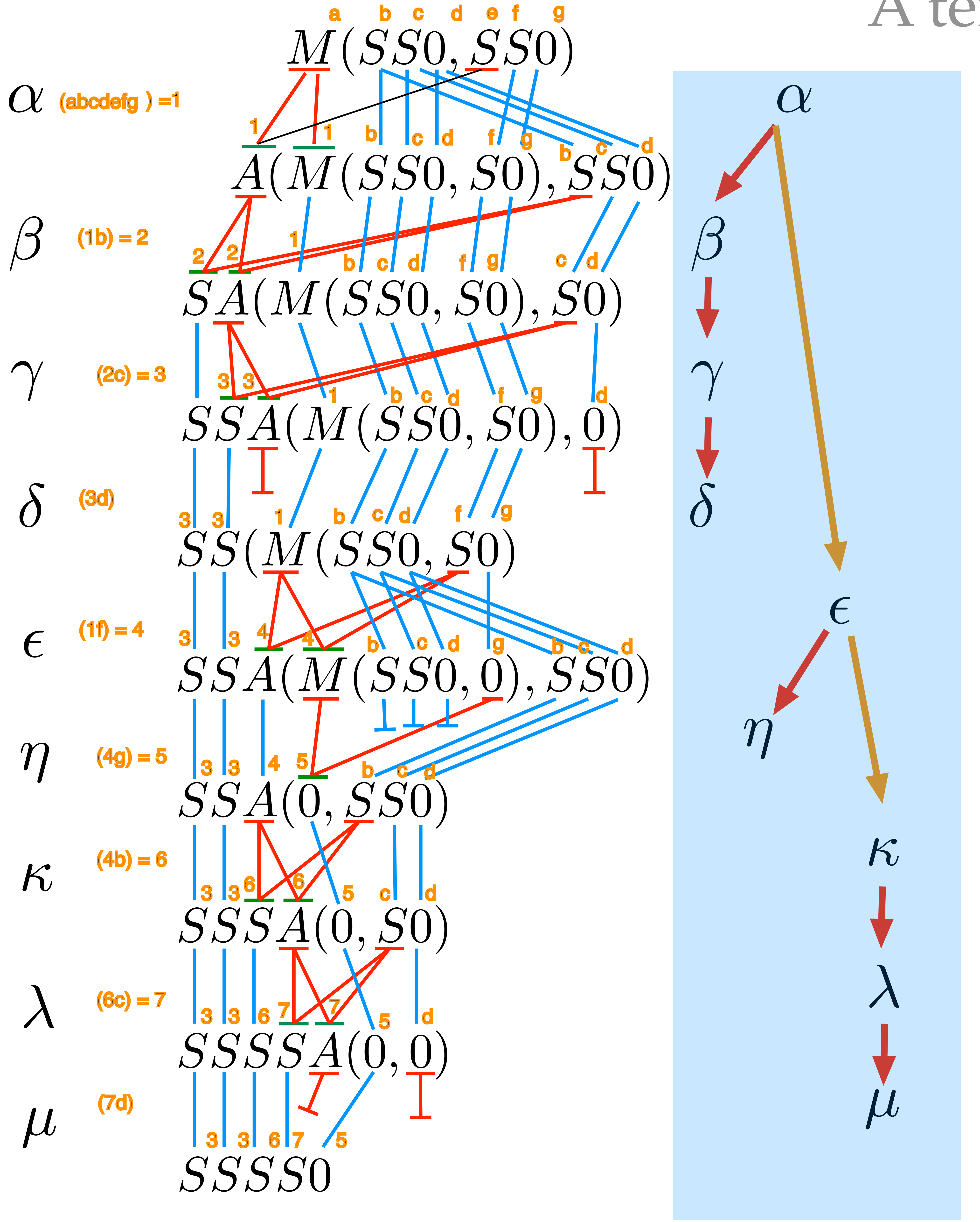


Figure 18: *2x2causalstructure*

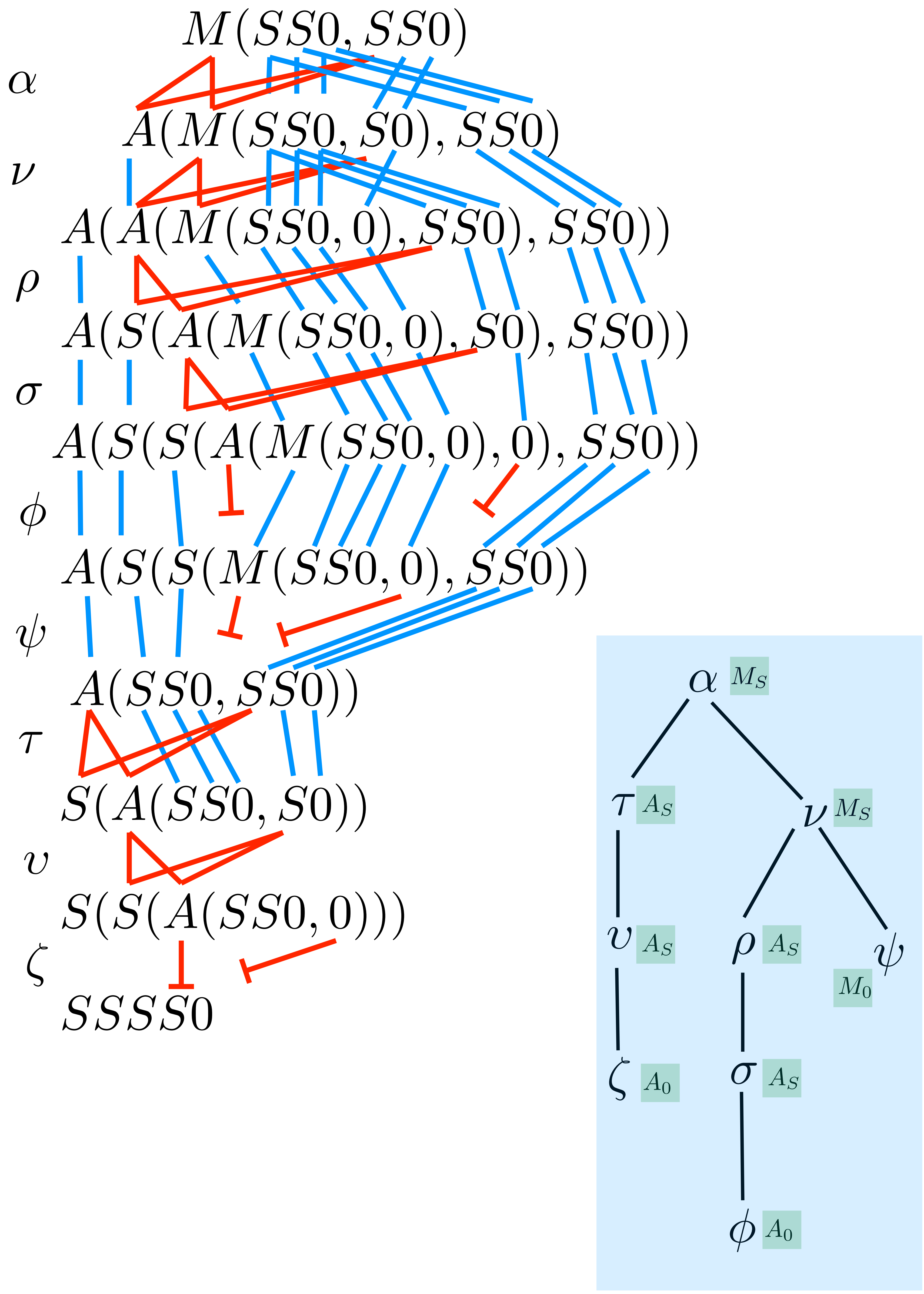
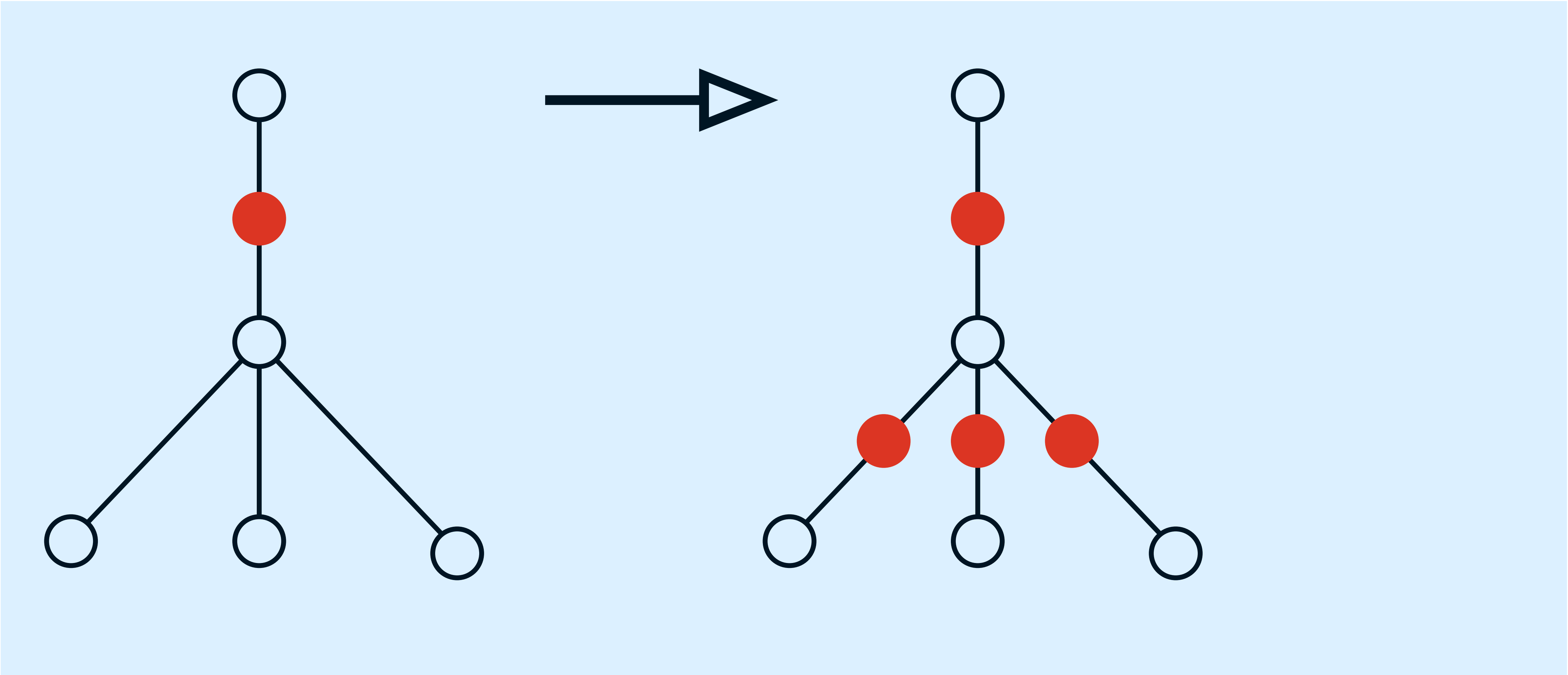


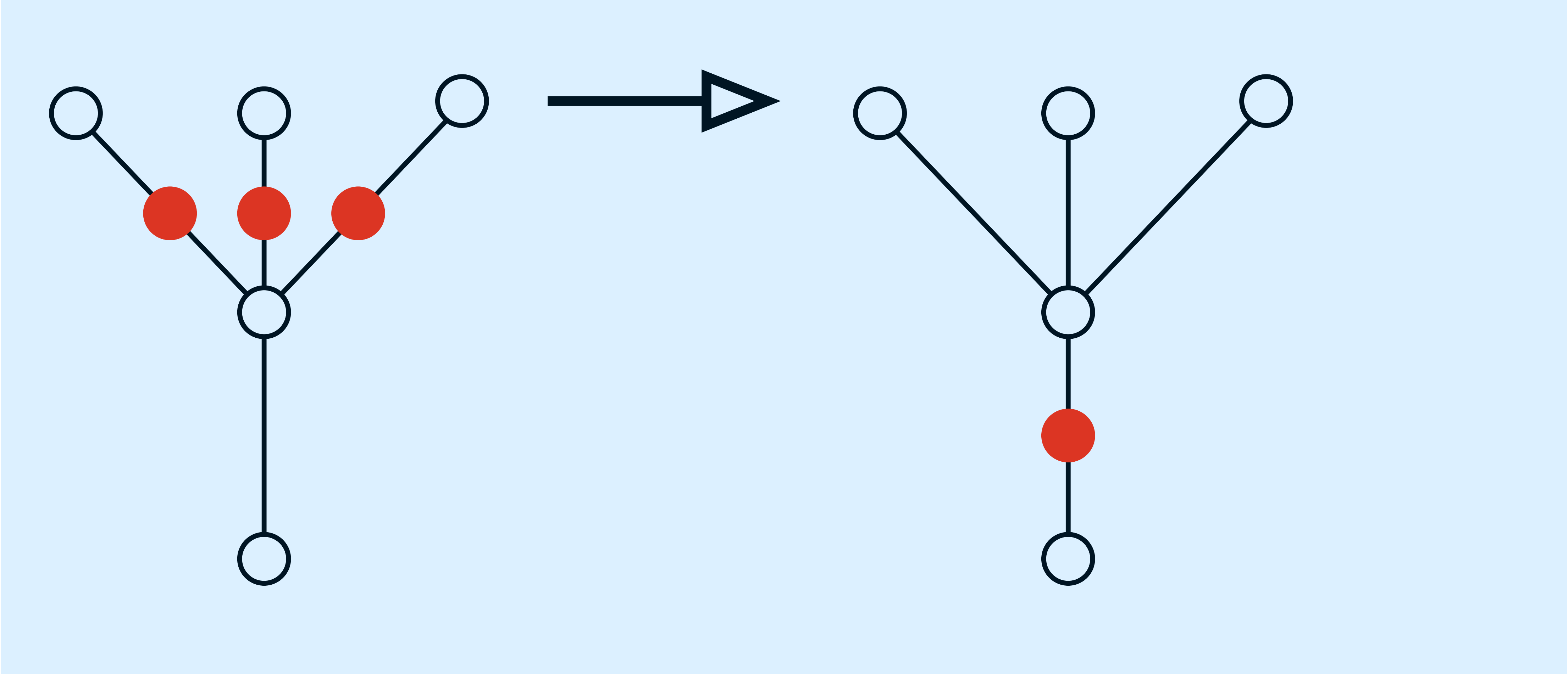
Figure 19: *2x2bis*

Petrinet-like pebble game on partial orders



a tentative observation, still in puzzle stage:

from the causal lattice we can reconstruct the original reduction graph via a petrinet-like pebble game



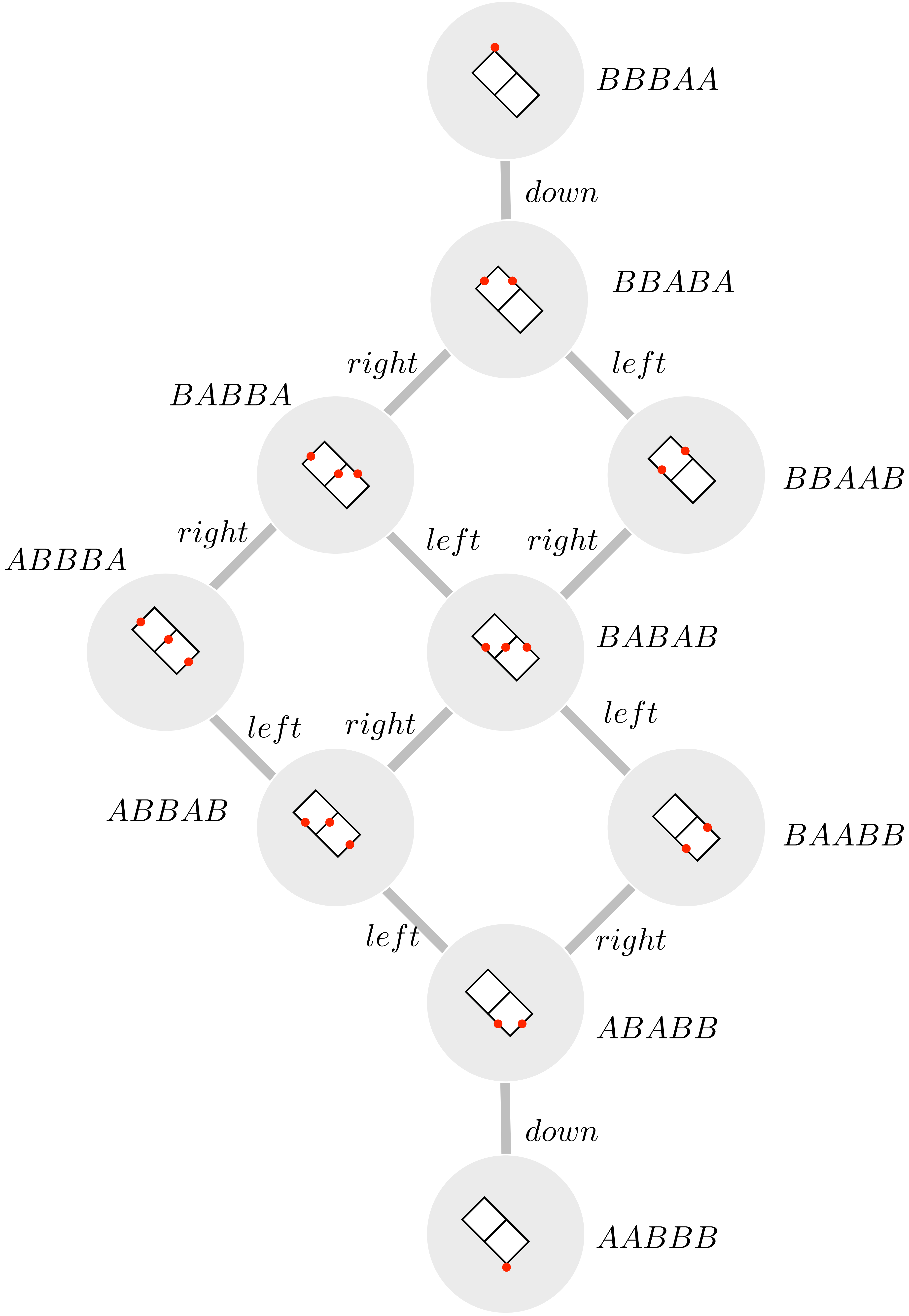


Figure 20: *petri-BBBAA*.

Lévy's tracing labels

Lévy-labeled β-reduction and Substitution

$$(\lambda x.M)^{\alpha} N \quad \to_{\beta_L} \quad M^{\underline{\alpha}}[x := N^{\underline{\alpha}}]$$

$$x^{\beta}[x := N^{\underline{\alpha}}] = N^{\underline{\alpha}\beta}$$

$$y^{\beta}[x := N^{\underline{\alpha}}] = y^{\beta} \qquad (y \not\equiv x)$$

$$(AB)^{\beta}[x := N^{\underline{\alpha}}] = (A[x := N^{\underline{\alpha}}]\ B[x := N^{\underline{\alpha}}])^{\beta}$$

$$(\lambda y.A)^{\beta}[x := N^{\underline{\alpha}}] = (\lambda y.A[x := N^{\underline{\alpha}}])^{\beta} \qquad (y \not\equiv x)$$

$$(\lambda x.A)^{\beta}[x := N^{\underline{\alpha}}] = (\lambda x.A)^{\beta}$$

Figure 79: *Lévy-labeled* β*-reduction*

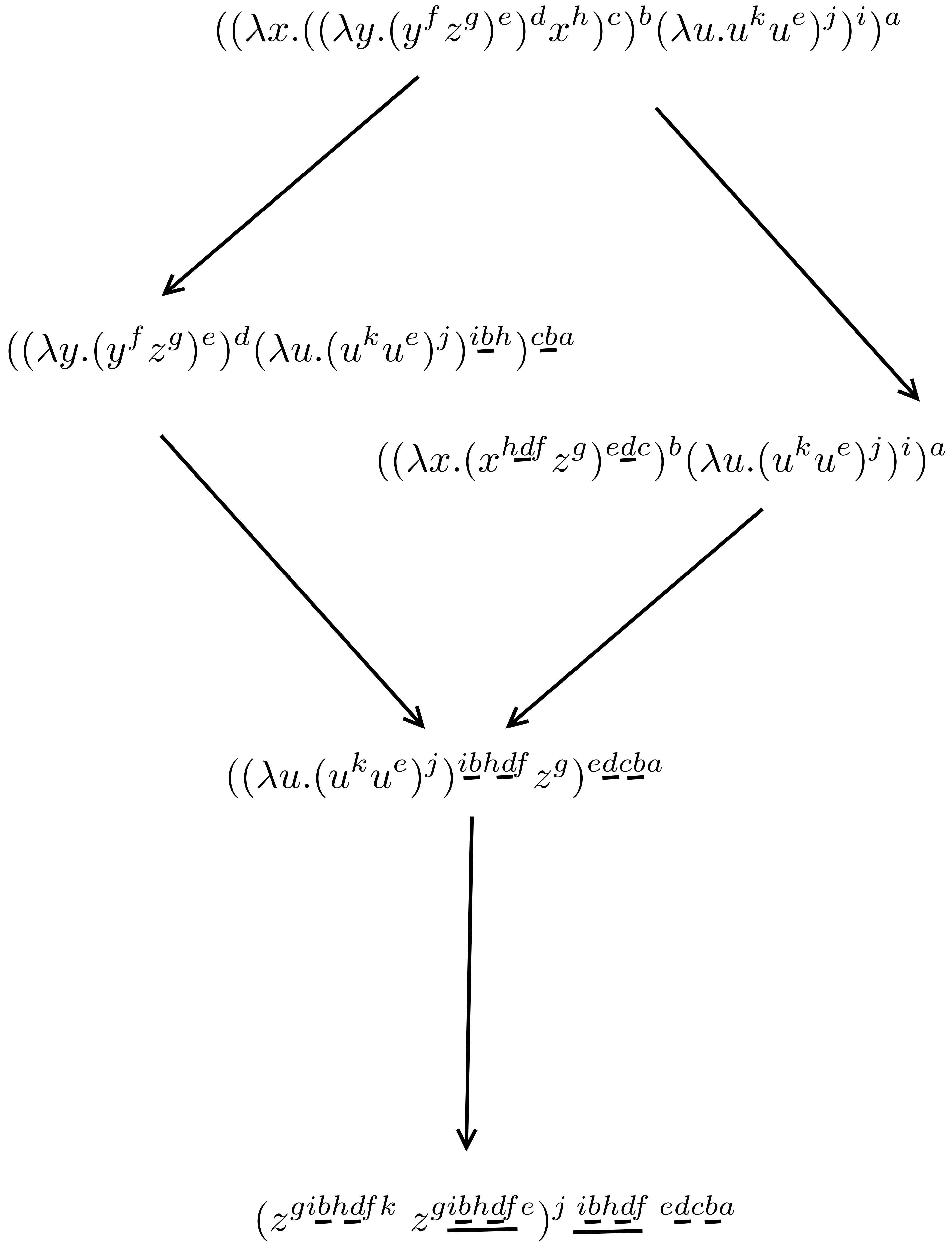
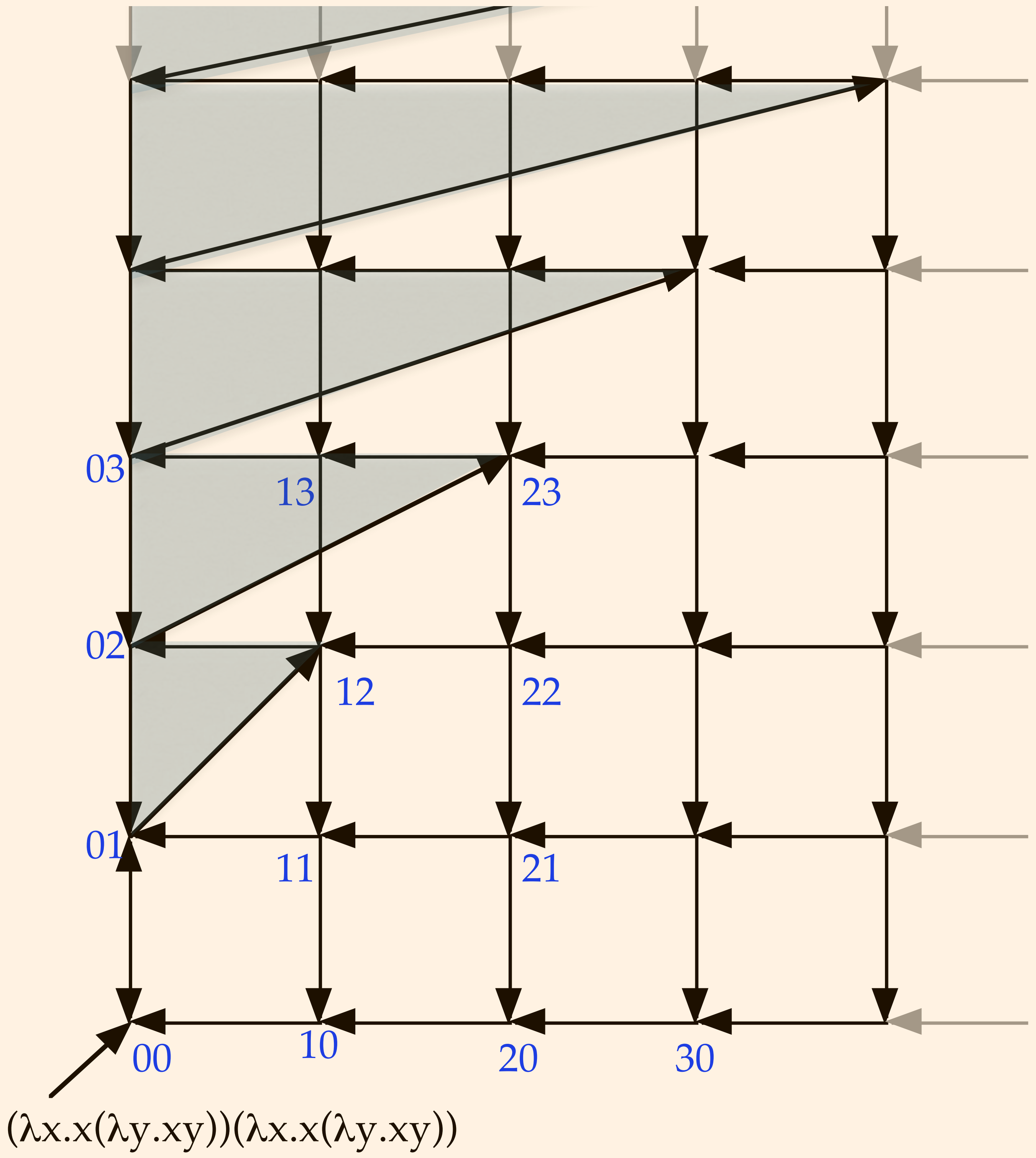


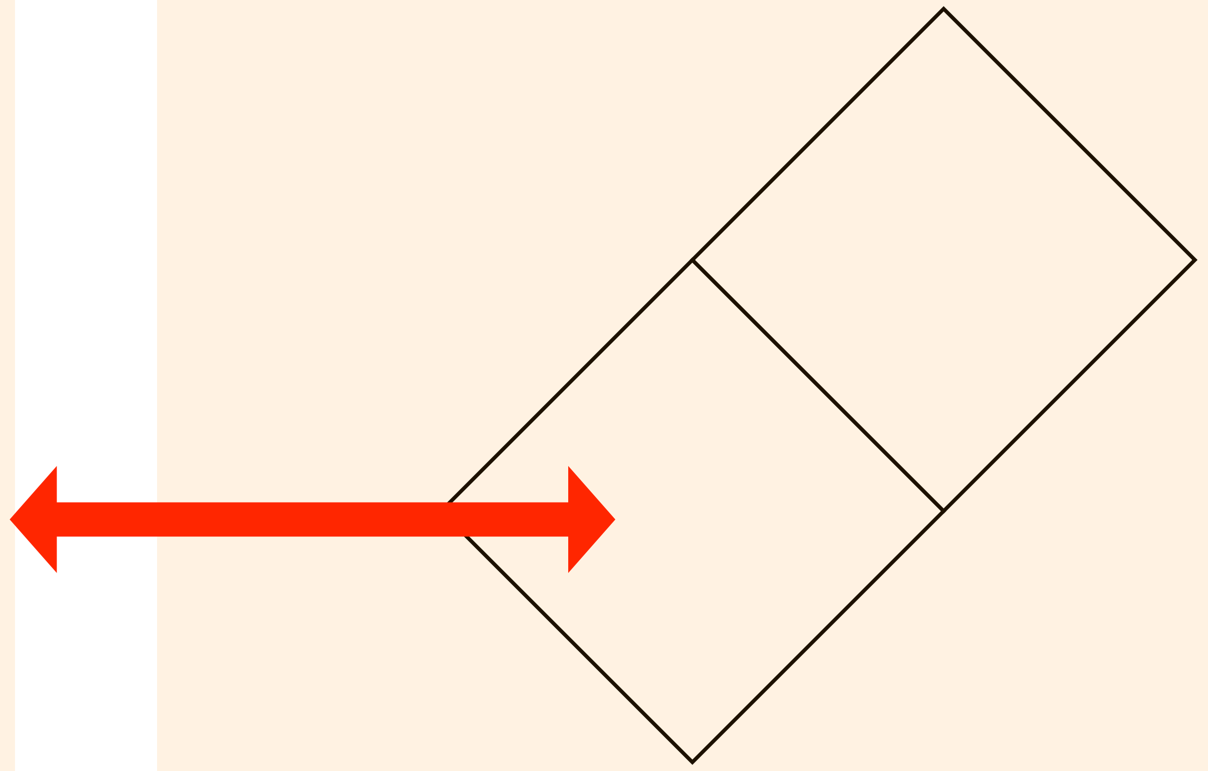
Figure 80: *Lévy's labeled* λ*-calculus, adapted example from Klop [80].*

Joining black holes, causality, graphs

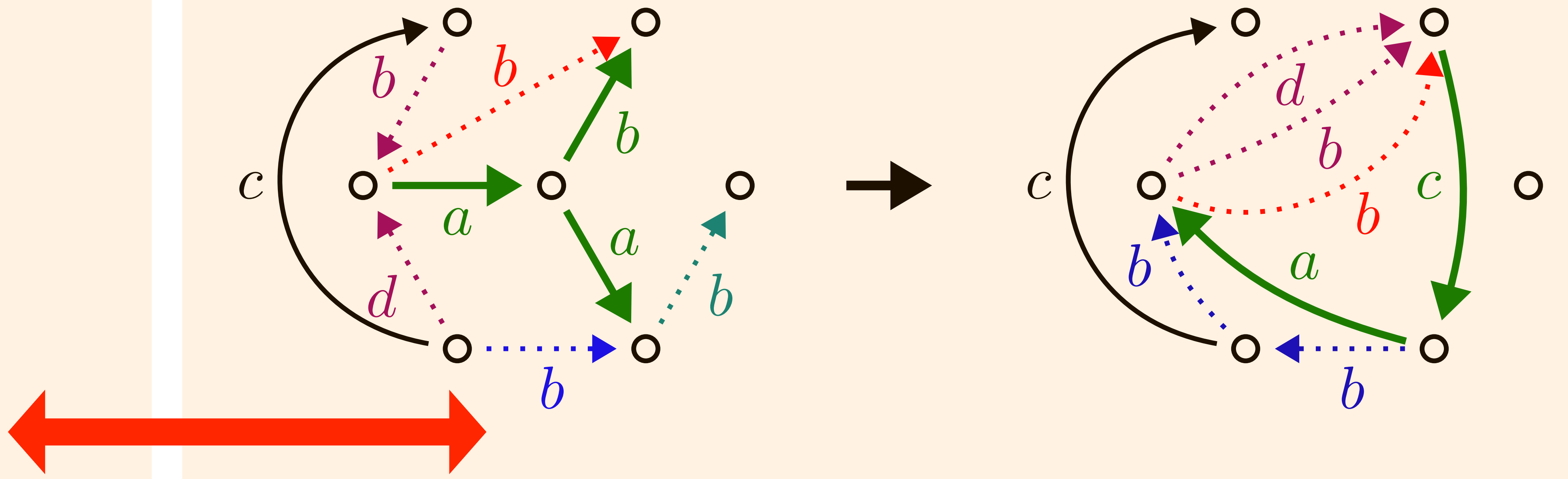
black hole lambda term



causality lattice



graph rewrite step



We need insight in the causality for lambda terms modulo unsolvables

We need insight in the causality for (orthogonal) graph rewriting

Further questions

Question 5.1. Wolfram's Causal Invariance theorem invites a number of questions that are interesting for term rewriting.

1. What is the nature of the causal partial ordering that are present in various kinds of rewrite families, orthgonal string rewrite systems, orthogonal term rewrite systems, CL, λ-calculus, orthogonal higher-order rewrite systems, CRSs, HRSs; Infinitary rewrite systems? Are they lattices, of what kind?

2. And how about the nature of these partial orders in (term) graph rewrite systems?

3. Can we relax orthogonality to weak orthogonality?

4. Important question: does Causal Invariance also hold when working modulo unsolvables, that is adopting Ω-reduction in any of the three canonical main formats? That would require an extension of Lévy's labels to include the arising of Ω's, as in the $\lambda^{\infty}\beta\Omega$-calculus.