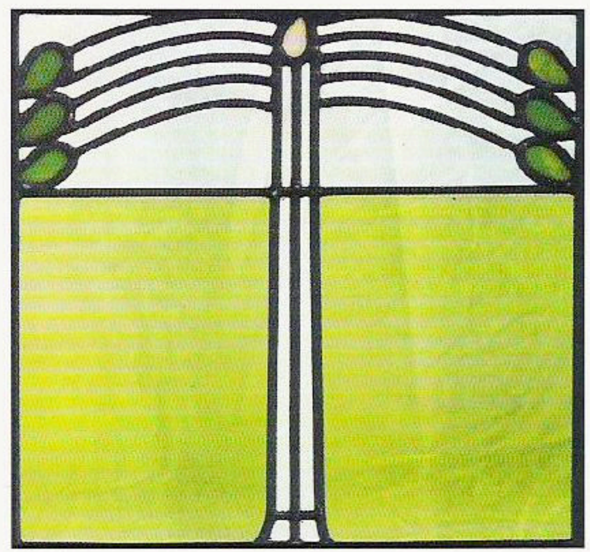
Ordinal Arithmetic via Transfinite Rewriting

PAM



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Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können.

David Hilbert (1926)

to answer the many objections, ...
magnitudes.

In mathematical circles however these activities of Cantor were considered suspect or worse. The following extract from the letter of 17–10–1887 from H. A. SCHWARZ to C. WEIERSTRASZ (in [85] p. 255) may illustrate this:

> "After I found an opportunity to contemplate this note[15]) I cannot conceal that it seems me to be a sickly confusion. What in the world have the churchfathers to do with the irrational numbers?! . . . If only one could succeed to accupy the unhappy young man with concrete problems, . . ., otherwise he will certainly come to a bad end."

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ordinal number

(Definition)

An ordinal number is a well ordered set S such that, for every $x \in S$,

$$x = \{z \in S \mid z < x\}$$

(where $<$ is the ordering relation on S).

There is a theory of ordinal arithmetic which allows construction of various ordinals; for example, all the numbers 0, 1, 2, ...have natural interpretations as ordinals, as does the set of natural numbers itself (often denoted ω in this context).

Ordinal arithmetic is the extension of normal arithmetic to the transfinite ordinal numbers. The successor operation Sx (sometimes written $x+1$, although this notation risks confusion with the general definition of addition) is part of the definition of the ordinals, and addition is naturally defined by recursion over this:

- $x+0=x$
- $x+Sy=S(x+y)$
- $x+\alpha=\sup_{\gamma<\alpha} x+\gamma$ for limit α

If x and y are finite then $x+y$ under this definition is just the usual sum, however when x and y become infinite, there are differences. In particular, ordinal addition is not commutative. For example,

$$\omega+1=\omega+S0=S(\omega+0)=S\omega$$

but

$$1+\omega=\sup_{n<\omega} 1+n=\omega$$

Multiplication in turn is defined by iterated addition:

- $x \cdot 0 = 0$
- $x \cdot Sy = x \cdot y + x$
- $x \cdot \alpha = \sup_{\gamma < \alpha} x \cdot \gamma$ for limit α

Once again this definition is equivalent to normal multiplication when x and y are finite, but is not commutative:

$$\omega \cdot 2 = \omega \cdot 1 + \omega = \omega + \omega$$

but

$$2 \cdot \omega = \sup_{n < \omega} 2 \cdot n = \omega$$

Both these functions are strongly increasing in the second argument and weakly increasing in the first argument. That is, if $\alpha < \beta$ then

- $\gamma + \alpha < \gamma + \beta$
- $\gamma \cdot \alpha < \gamma \cdot \beta$
- $\alpha + \gamma \leq \beta + \gamma$
- $\alpha \cdot \gamma \leq \beta \cdot \gamma$

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system of counting doesn't work for counting zero objects, since there isn't a last number that you used.

Cantor's Ordinal Numbers

The great German mathematician Georg Cantor was the earliest person to construct a coherent theory of counting collections that may be infinite. For this he extended the ordinary series of numbers used for counting, as follows:

> 0, 1, 2, . . . as usual,
> then ω, $\omega+1$, $\omega+2$, . . . then $\omega+\omega$, $\omega+\omega+1$, . . .

and so on.

The important point about these numbers (and, in essence, their definition) is that, no matter how many of them you've used, there's always a (uniquely determined) earliest one that you haven't. Cantor's opening infinite number,

$$\omega = \{0, 1, 2, \ldots \mid \}$$

is defined to be the earliest number greater than all the finite counting numbers. We'll use

$$\{a, b, c, \ldots \mid \}$$

for the earliest ordinal number after $a, b, c, \ldots$. The vertical bar signals the place where we've cut off the number sequence $a, b, c, \ldots$, for example,

$$\{0, 1, 2|\} = 3,\ \{0, 1, 2, \ldots|\} = \omega,\ \{0|\} = 1,$$
$$\{|\} = 0,\ \{0, 1, 2, \ldots \omega|\} = \omega + 1.$$

To avoid inventing lots of new words, the symbols $\omega + 1, \omega + 2, \ldots$ are used as proper names for the ordinary numbers following ω, just as "hundred and one" is the proper name of the number you get by adding "a hundred" and "one."

When you count things, you are really ordering them in a special way:

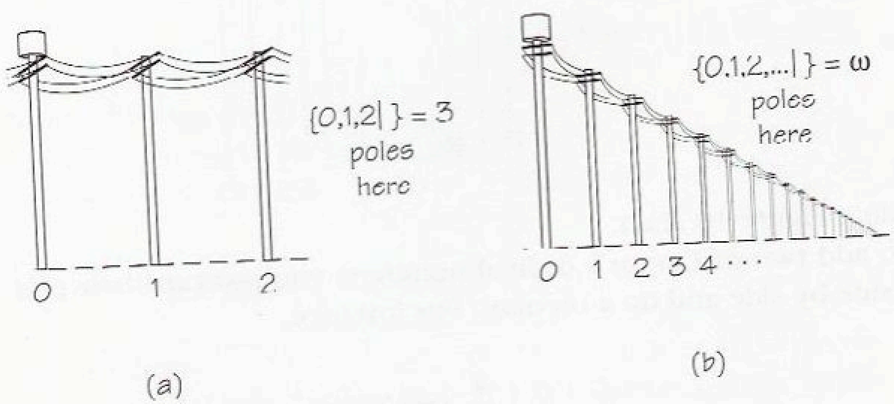
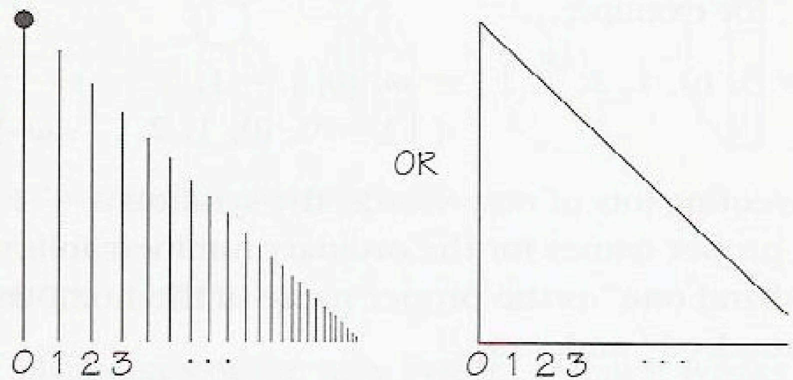


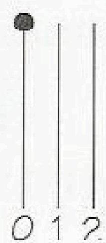
FIGURE 10.2 *Various numbers of poles.*

To count the poles in Figure 10.2(a), you'd say, "0, 1, 2, so there are $\{0, 1, 2|\} = 3$ poles here." But now look at Figure 10.2(b), where we imagine that the road is infinite, with a pole for each of the ordinary integers $0, 1, 2, \ldots$. Obviously, we should now say:

In the future, we'll represent such an infinite sequence of objects by

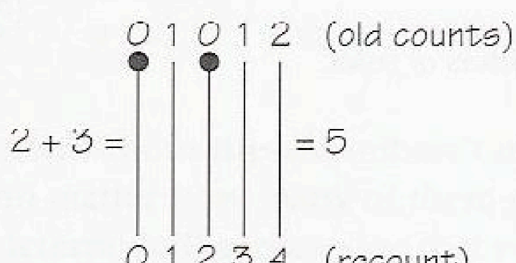


(recalling Figure 10.2(b)), but we'll represent a finite sequence by poles of equal height:

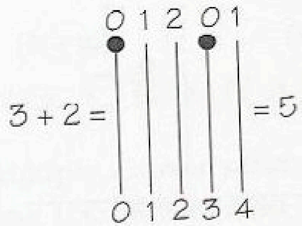


(recalling Figure 10.2(a)).

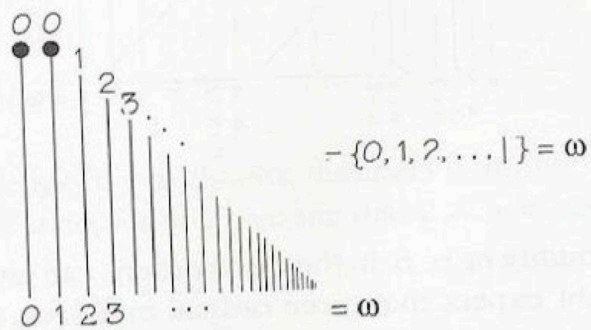
To add two of Cantor's ordinal numbers, you just put their pictures side by side and do a recount: For instance,



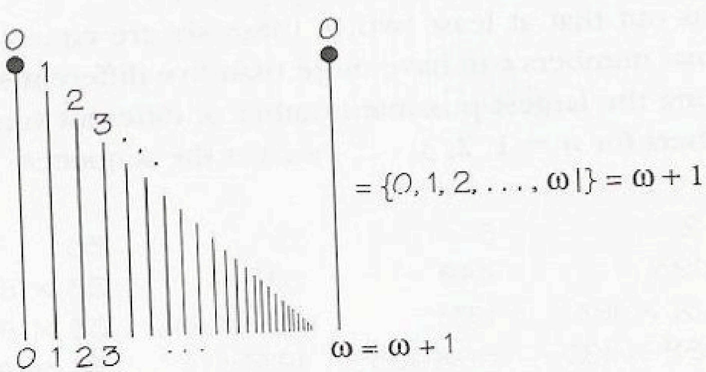
Of course, we get the same answer from 3 + 2, although the re-counting's in a different order:



But infinite numbers give some surprises! We find that $1 + \omega$ is the same as ω.

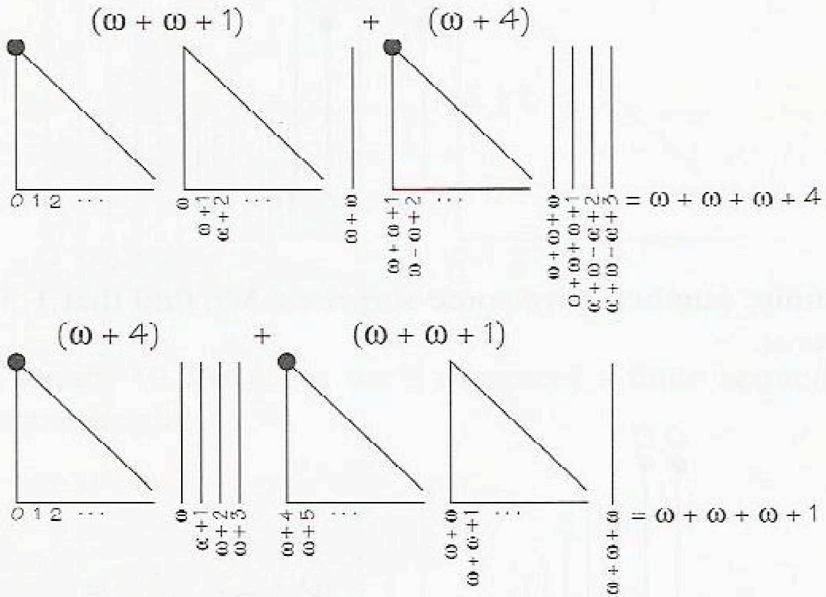


but $\omega + 1$ is bigger:



In other words, this kind of addition usually *fails* to satisfy the commutative law; $\beta + \alpha$ may be larger or smaller than $\alpha + \beta$.

As a bigger example, we'll add $\alpha = \omega+\omega+1$ to $\beta = \omega+4$, both ways around,



Since two numbers, α, β, in their two orders, can give two distinct sums, you might expect that three ordinal numbers, α, β, γ, could give six different sums,

$$\alpha+\beta+\gamma,\ \alpha+\gamma+\beta,\ \beta+\gamma+\alpha,\ \beta+\alpha+\gamma,\ \gamma+\alpha+\beta,\ \gamma+\beta+\alpha,$$

but it turns out that at least two of these six are equal, so that no three ordinal numbers can have more than five different sums.

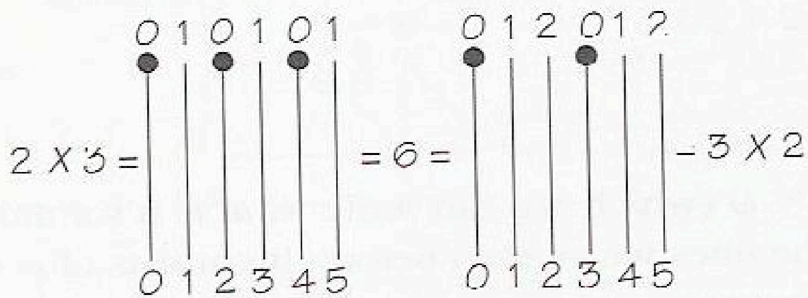
By taking the largest possible number of different sums of n ordinal numbers for $n = 1, 2, 3, \ldots$, we get the sequence

1	2	5	13	33
81	193	449	33^2	33×81
81^2	81×193	193^2	$33^2 \times 81$	33×81^2
81^3	$81^2 \times 193$	81×193^2	193^3	33×81^3
and	from here on	you multiply	the previous	row by 81:
81^4	$81^3 \times 193$	$81^2 \times 193^2$	81×193^3	33×81^4 ...

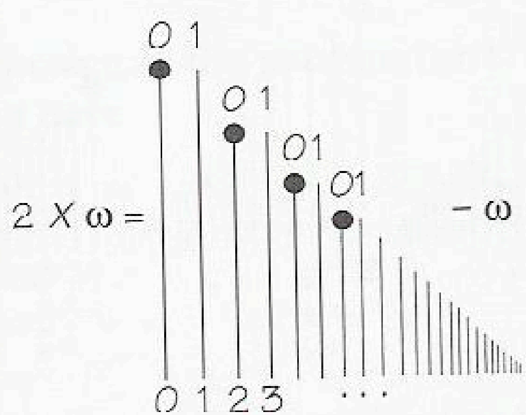
So the largest number of different sums that n ordinals can have behaves rather strangely. For 15 or more numbers, it will be either a power of 193 times a power of 81, or 33 times a power of 81.

MULTIPLYING ORDINAL NUMBERS

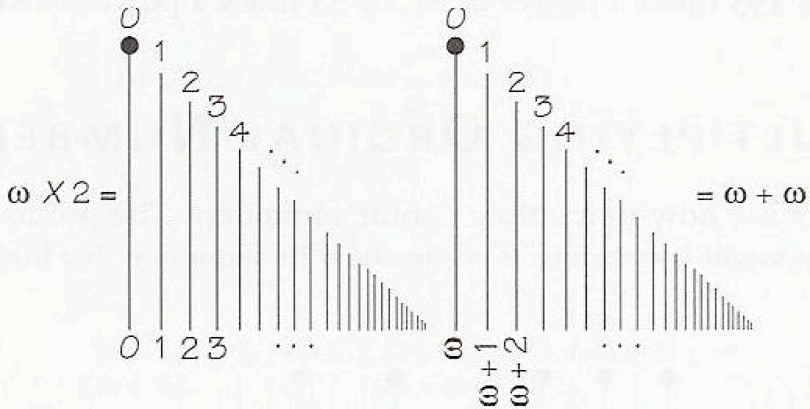
Now let's see how to multiply Cantor's numbers. The product $\alpha \times \beta$ is what you get by placing β copies of α in sequence: for instance,



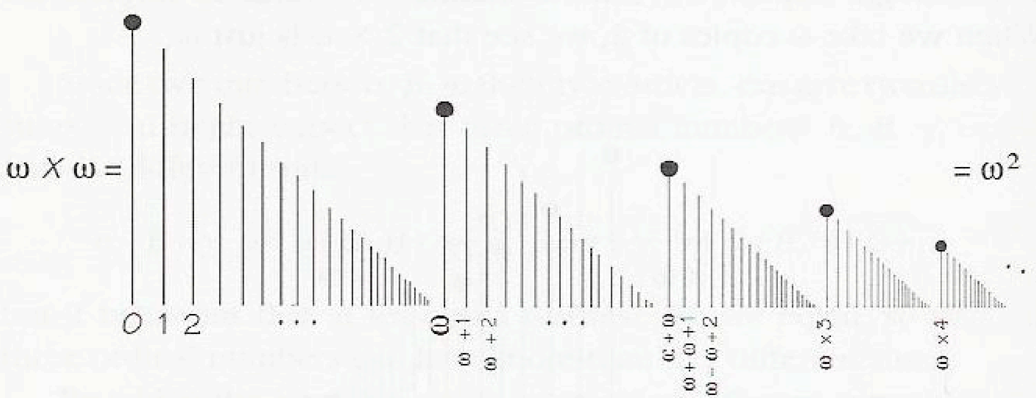
as you might expect, but infinite numbers continue to surprise us. When we take ω copies of 2, we see that $2 \times \omega$ is just ω:



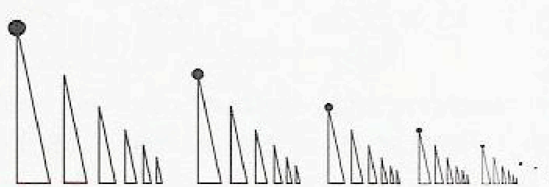
but $\omega \times 2$ (2 copies of ω) is the same as $\omega + \omega$:



What is $\omega \times \omega$ (which we can write as ω^2)? It's a much larger number than the ones we've seen before. It consists of ω copies of ω, placed in sequence:



What about ω^3, ω^4, . . . ? Well, of course, $\omega^3 = \omega^2 \times \omega$. We can get it by having ω copies of a pattern of ω^2:



Then you get ω^4 from ω copies of this; then ω^5 from ω copies of that, and so on—we won't draw the pictures for $\omega^4, \omega^5, \ldots$—and there are lots of other numbers. For instance,

$$\omega^6 \times 49 + \omega^3 \times 8 + \omega^2 \times 3 + \omega \times 57 + 1001$$

lies between ω^6 and ω^7. Figure 10.3 shows a pattern for the number $\omega^2 \times 2 + \omega \times 3 + 7$.

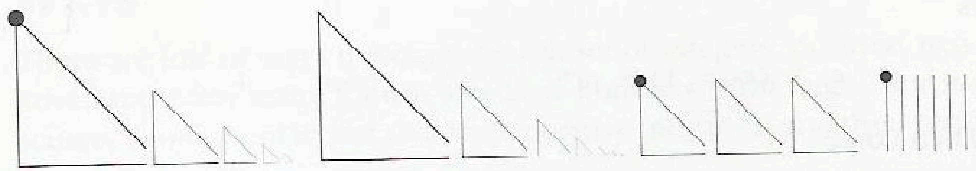


FIGURE **10.3** $(\omega^2 \times 2) + (\omega \times 3) + 7$.

Can we go further? Yes! In Cantor's system you can *always* go further! The number

$$\omega^\omega = 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \cdots$$

is obtained by juxtaposing all the patterns for $1, \omega, \omega^2, \omega^3, \omega^4, \ldots$, in that order. Then you have

$\omega^\omega + 1, \omega^\omega + 2, \ldots \omega^\omega + \omega, \ldots \omega^\omega + \omega \times 2, \ldots \omega^\omega + \omega \times 3, \ldots$
$\omega^\omega + \omega^2, \omega^\omega + \omega^2 + 1, \ldots \omega^\omega + \omega^2 + \omega, \ldots, \omega^\omega + \omega^3, \ldots$
$\omega^\omega + \omega^\omega = \omega^\omega \times 2, \omega^\omega \times 2 + 1, \ldots \omega^\omega \times 3, \ldots, \omega^\omega \times 4, \ldots$
$\omega^\omega \times \omega = \omega^{\omega+1}, \ldots \omega^{\omega+1} + \omega, \ldots \omega^{\omega+1} + \omega^2, \ldots$
$\omega^{\omega+1} + \omega^\omega, \ldots \omega^{\omega+2}, \ldots, \omega^{\omega+3}, \ldots \omega^{\omega\times 2}, \ldots \omega^{\omega\times 3}, \ldots$
$\omega^{\omega^2}, \ldots \omega^{\omega^3}, \ldots \omega^{\omega^4}, \ldots \omega^{\omega^\omega}, \ldots \omega^{\omega^{\omega+1}}, \ldots \omega^{\omega^{\omega^\omega}}, \ldots \omega^{\omega^{\omega^{\omega^\omega}}}, \ldots$

The "limit" of all these is a number that it is natural to write as

$$\omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}$$

where there are ω omegas. This famous number was called ϵ_0 by Cantor. It's the first ordinal number that you can't get from smaller ones by a finite number of additions $\alpha+\beta$, multiplications $\alpha\times\beta$, and exponentiations α^β. Another formula for it is

On the surface this axiom sounds quite innocuous. It says that if you have any collection of nonempty sets of things, you can make a new set by choosing just one from each set of the given collections. On the other hand, Zermelo's result was so astonishing that many mathematicians, from his day to ours, have had grave doubts about it.

COUNTING THE SAME SET IN DIFFERENT WAYS

There are lots of ways to count the full set of integers, positive, negative, and zero, using Cantor's ordinal numbers. You might, for instance, count them in just that way, positive, negative, and then zero:

1	2	3	4	...	−1	−2	−3	−4	...	0
0	1	2	3		ω	$\omega+1$	$\omega+2$	$\omega+3$		$\omega \times 2$

The answer this way (the first number missing) is $\omega \times 2 + 1$. It's a bit more economical to include zero with the positive numbers:

0	1	2	3	...	−1	−2	−3	−4
0	1	2	3		ω	$\omega+1$	$\omega+2$	$\omega+3$

That way you get the answer $\omega \times 2$. More economically still:

0	1	−1	2	−2	3	−3	
0	1	2	3	4	5	6	...

and we get just ω.

You can see that the answer you get depends not only on the *objects* you count, but also the *order* you count them in. The positive integers can be counted in lots and lots of different ways. The simplest is just to put them in order of size:

1 2 3 4 5 . . . ans.: ω

Or we might prefer odd numbers first:

1 3 5 7 . . . 2 4 6 8 . . . ans.: $\omega \times 2$

We might even discriminate further, classifying numbers according to exactly which power of 2 divides them. This gives

	1	3	5	7	9 . . . .
then	2	6	10	14	18 . . . .
then	4	12	20	28	36 . . . .
then	8	24	40	56	72 . . . .
		. . . . . .			

ans.: ω^2

Equally we could classify them by the odd factor, i.e., reading this by columns, getting the order

	1	2	4	8	16 . . . .
then	3	6	12	24	48 . . . .
then	5	10	20	40	80 . . . .
then	7	14	28	56	112 . . . .
		. . . . . .			

ans.: again ω^2

We can be even more profligate. Let's first have the powers of 2:

1 2 4 8 16 . . . . (ω, so far)

Then 3 times these, 3^2 times them, 3^3 times, etc.

3 6 12 24 48 . . . 9 18 36 72 . . . 27 54 108 . . . 81 . . . 243 . . . (ω^2 more)

Then 5 times all the numbers so far, 25× them, 125× . . . and so on:

5	10	20	40 . . .	15	30	60 . . .	45	90	180 . . .
25	50	100 . . .		75	150		225	450	. . .
125	250	500 . . .		375 . . .			1125 . . .		
625	1250 . . .			. . .			. . .		
. . . . . .									

(ω^3 here)

Then all these times successive powers of 7, (ω^4 more)
and all those times successive powers of 11, (ω^5 more)
and so on using the primes in order.

This way we get in all

$$\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \cdots = \omega^\omega$$

for the final answer.

$$\epsilon_0 = 1 + \omega + \omega^{\omega} + \omega^{\omega^{\omega}} + \omega^{\omega^{\omega^{\omega}}} + \cdots .$$

It is also the first number that satisfies Cantor's famous equation $\omega^{\epsilon} = \epsilon$. You'd think that this couldn't happen, because

ω^1 is much bigger than 1,
ω^2 even more so than 2,
ω^{ω} still more so than ω,

but Cantor showed that his equation has lots of solutions. The next is

$$\epsilon_1 = (\epsilon_0 + 1) + \omega^{\epsilon_0+1} + \omega^{\omega^{\epsilon_0+1}} + \omega^{\omega^{\omega^{\epsilon_0+1}}} + \cdots .$$

Then come

$$\epsilon_2, \epsilon_3, \ldots \quad \epsilon_{\omega}, \epsilon_{\omega+1}, \ldots \quad \epsilon_{\omega\times 2}, \ldots \quad \epsilon_{\omega^2}, \ldots \quad \epsilon_{\omega^{\omega}}, \ldots$$

$$\epsilon_{\epsilon_0}, \epsilon_{\epsilon_0+1}, \ldots \quad \epsilon_{\epsilon_0+\omega}, \ldots \quad \epsilon_{\epsilon_0+\omega^{\omega}}, \ldots \quad \epsilon_{\epsilon_0\times 2}, \ldots \quad \epsilon_{\epsilon_1}, \ldots$$

$$\epsilon_{\epsilon_2}, \ldots \quad \epsilon_{\epsilon_{\omega}}, \ldots \quad \epsilon_{\epsilon_{\epsilon_0}}, \ldots \quad \epsilon_{\epsilon_{\epsilon_1}}, \ldots \quad \epsilon_{\epsilon_{\epsilon_{\omega}}}, \ldots \quad \epsilon_{\epsilon_{\epsilon_{\epsilon_0}}}, \ldots$$

and eventually

$$\epsilon_{\epsilon_{\epsilon_{\epsilon_{\cdot}}}},$$

which is the first solution of $\epsilon_{\alpha} = \alpha$.

How Far Can We Go?

The ordinal numbers go on for an awfully long time! No matter how big the set of them you've already got, there's always another one, and another, and another, and . . . . The precise situation was guessed by Cantor and proved a quarter of a century later by his student Zermelo in 1904: there are enough ordinals to count the members of any set of objects, no matter how big it is. Zermelo's proof showed that this depends on a hitherto unrecognized principle in mathematics: the so-called **axiom of choice**.

How to Count Up to the First Epsilon

Note: Several boring passages in the following sequence are omitted. The way it should proceed at these spots is, however, quite clear. Similar cuts would be necessary even if presenting only the series up to one billion or less.

```
0
 1
 2
 3
 4
 ...
omega
   omega + 1
   omega + 2
   omega + 3
   omega + 4
   ...
  omega * 2
   omega * 2 + 1
   omega * 2 + 2
   omega * 2 + 3
   omega * 2 + 4
   ...
  omega * 3
   omega * 3 + 1
   omega * 3 + 2
   omega * 3 + 3
   omega * 3 + 4
   ...
  omega * 4, ...
  ...
 omega^2
    omega^2 + 1
    omega^2 + 2
    omega^2 + 3
```

THE GROWTH

$0, 1, 2, 3, \ldots, \omega, \omega+1, \omega+2, \omega+3, \ldots, \omega+\omega = \omega2, \omega2+1, \omega2+2, \omega2+3, \ldots, \omega3, \omega3+1, \omega3+2, \omega3+3,$

$\omega\omega = \omega^2, \omega^2+1, \omega^2+2, \ldots, \omega^2+\omega, \ldots, \omega^2+\omega2, \ldots, \omega^2+\omega3, \ldots, \omega^2 2, \ldots, \omega^2 3, \ldots, \omega^3, \ldots, \ldots, \omega$

$\ldots, \omega^{\omega^2}, \ldots, \omega^{\omega^3}, \ldots, \omega^{\omega^\omega} = \omega^{\omega 2}, \ldots, \omega^{\omega 3}, \ldots, \ldots, \omega^{\omega^2}, \ldots, \omega^{\omega^\omega}, \ldots, \omega^{\omega^{\cdot^{\cdot}}} = \varepsilon_0, \varepsilon_0+1, \ldots,$

$\varepsilon_0+\omega2, \ldots, \varepsilon_0+\omega^2, \ldots, \varepsilon_0+\omega^\omega, \ldots, \varepsilon_0+\varepsilon_0 = \varepsilon_0 2, \ldots, \varepsilon_0 3, \ldots, \varepsilon_0\omega, \ldots, \varepsilon_0\omega2, \ldots, \varepsilon_0\omega^2, \ldots,$

$\varepsilon_0\varepsilon_0 = \varepsilon_0^2, \ldots, \varepsilon_0^3, \ldots, \varepsilon_0^\omega, \ldots, \varepsilon_0^{\omega^\omega}, \ldots, \varepsilon_0^{\varepsilon_0}, \ldots, \varepsilon_0^{\varepsilon_0^{\varepsilon_0}} = \varepsilon_1, \ldots, \varepsilon_2, \ldots, \varepsilon_3, \ldots, \varepsilon_\omega$

$\varepsilon_{\omega^\omega}, \ldots, \varepsilon_{\varepsilon_0}, \ldots, \varepsilon_{\varepsilon_1}, \ldots, \varepsilon_{\varepsilon_\omega}, \ldots, \varepsilon_{\varepsilon_{\omega^\omega}}, \ldots, \varepsilon_{\varepsilon_{\omega^{\omega}}}, \ldots, \varepsilon_{\varepsilon_{\varepsilon_0}}, \ldots, \varepsilon_{\varepsilon_{\cdot_{\cdot}}} = \eta_0, \ldots, \eta_1, \ldots$

$\eta_{\varepsilon_0}, \ldots, \eta_{\varepsilon_\omega}, \ldots, \eta_{\varepsilon_{\varepsilon_0}}, \ldots, \eta_{\varepsilon_{\varepsilon_{\cdot_{\cdot}}}} = \eta_{\eta_0}, \ldots, \eta_{\eta_{\cdot_{\cdot}}} = \zeta_0, \ldots, \ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

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Chapter 9: Ordinal Number Theory

9.1. NORMAL FORM

It is easily shown in elementary number theory that if n is any natural number, then there exists a natural number m and natural numbers $a_i < 10$, $i = 0, 1, \ldots, m$, such that

$$n = a_m \cdot 10^m + a_{m-1} \cdot 10^{m-1} + \cdots + a_1 \cdot 10 + a_0$$

and the representation is unique. In fact, it can be shown that 10 can be replaced by any natural number $b > 1$, in which case the condition for the coefficients a_i, $i = 0, 1, \ldots, m$, is that they be natural numbers less than b.

We shall prove a similar theorem for ordinal numbers, taking $b = \omega$, but any other ordinal number larger than 1 would serve as well. First, we shall prove a preliminary lemma.

LEMMA 9.1.1. $[n \in \omega \;\&\; (\forall m)(m \leqq n \rightarrow$
(a) $a_m < \omega \;\&\; a_m \neq 0 \;\&$
(b) $m + 1 \leqq n \rightarrow \alpha_m < \alpha_{m+1} \;\&$
(c) $\alpha = \omega^{\alpha_n} a_n + \omega^{\alpha_{n-1}} a_{n-1} + \cdots + \omega^{\alpha_0} a_0)] \rightarrow$

$$\alpha < \omega^{\alpha_n^+}.$$

Proof: By (c),

$$\begin{aligned} \alpha &= \omega^{\alpha_n} a_n + \omega^{\alpha_{n-1}} a_{n-1} + \cdots + \omega^{\alpha_0} a_0 && \\ &\leqq \omega^{\alpha_n} a_n + \omega^{\alpha_n} a_{n-1} + \cdots + \omega^{\alpha_n} a_0 && \text{(b)} \\ &= \omega^{\alpha_n}(a_n + a_{n-1} + \cdots + a_0) && (8.5.14) \\ &< \omega^{\alpha_n} \omega && (a, 8.5.11) \\ &= \omega^{\alpha_n^+}. && (8.5.22b) \; \blacksquare \end{aligned}$$

THEOREM 9.1.2.

$$\begin{aligned} \alpha \neq 0 \rightarrow (\exists 1 n)[n \in \omega \;\&\; (\forall m)(m \leqq n \rightarrow \\ (\exists 1 a_m)(\exists 1 \alpha_m)(a_m < \omega \;\&\; a_m \neq 0 \;\&\; (m + 1 \leqq n \rightarrow \alpha_m < \alpha_{m+1}) \;\& \\ \alpha = \omega^{\alpha_n} a_n + \omega^{\alpha_{n-1}} a_{n-1} + \cdots + \omega^{\alpha_0} a_0))]. \end{aligned}$$

Proof: The proof of existence is by transfinite induction. Suppose that the theorem is true for all $\beta < \alpha$. It follows from exercise 8.5.8(f)

If $\alpha \neq 0$, the representation

$$\alpha = \omega^{\alpha_n}a_n + \omega^{\alpha_{n-1}}a_{n-1} + \cdots + \omega^{\alpha_0}a_0$$

where $\alpha_n > \alpha_{n-1} > \cdots > \alpha_0$, $a_n, a_{n-1}, \ldots, a_0 \in \omega$, and $a_n, a_{n-1}, \ldots, a_0 \neq 0$ is called the *normal form* of α. It follows from theorem 9.1.2 that every non-zero ordinal number has a unique normal form.

It is clear from the proof of 9.1.2 that any ordinal number $\beta > 1$ could have been chosen instead of ω. Therefore, the following more general theorem holds.

THEOREM 9.1.3.

$$(\alpha \neq 0 \;\&\; \beta > 1) \to (\exists 1 n)[n \in \omega \;\&\; (\forall m)(m \leqq n \to (\exists 1 a_m)(\exists 1 \alpha_m)(a_m < \beta \;\&\; a_m \neq 0 \;\&\; (m+1 \leqq n \to \alpha_m < \alpha_{m+1}) \;\&\; \alpha = \beta^{\alpha_n}a_n + \beta^{\alpha_{n-1}}a_{n-1} + \cdots + \beta^{\alpha_0}a_0))].$$

Suppose that $\alpha \neq 0$ and that α is in normal form. If $\alpha_0 \neq 0$, then

$$\alpha = \omega(\omega^{\alpha_n - 1}a_n + \omega^{\alpha_{n-1}-1}a_{n-1} + \cdots + \omega^{\alpha_0 - 1}a_0).$$

Therefore, since ω is a limit ordinal, it follows from theorem 8.5.9 that α is a limit ordinal. On the other hand, if $\alpha_0 = 0$ then, since $a_0 \neq 0$, there is a b_0 such that $a_0 = b_0^+$. Thus,

$$\begin{aligned} \alpha &= \omega^{\alpha_n}a_n + \omega^{\alpha_{n-1}}a_{n-1} + \cdots + \omega^{\alpha_1}a_1 + b_0^+ \\ &= (\omega^{\alpha_n}a_n + \omega^{\alpha_{n-1}}a_{n-1} + \cdots + \omega^{\alpha_1}a_1 + b_0)^+. \end{aligned}$$

Therefore, in this case α is a non-limit ordinal. Consequently, we have the following theorem.

THEOREM 9.1.4.

$$(\alpha \neq 0 \;\&\; \alpha = \omega^{\alpha_n}a_n + \omega^{\alpha_{n-1}}a_{n-1} + \cdots + \omega^{\alpha_0}a_0 \text{ is the normal form of } \alpha) \to (\alpha \text{ is a limit ordinal} \leftrightarrow \alpha_0 \neq 0).$$

Hence, the ordinal number

$$\omega^5 2 + \omega^2 + \omega^3$$

is a limit ordinal, but

$$\omega^\omega 7 + \omega^5 9 + \omega 4 + 7$$

is not.

The remaining part of this section is devoted to calculating the normal form of $\alpha + \beta$, $\alpha\beta$, and α^β.

THEOREM 9.1.5. $(a,b \in \omega \; \& \; a,b \neq 0) \rightarrow$

(a) $\omega^\alpha a + \omega^\alpha b = \omega^\alpha(a + b)$

(b) $\alpha < \beta \rightarrow \omega^\alpha a + \omega^\beta b = \omega^\beta b.$

Proof: Part (a) is true because multiplication from the left is distributive over addition (8.5.14).

To prove (b) we note that if $\alpha < \beta$, then there exists a $\gamma \neq 0$ such that $\beta = \alpha + \gamma$. Thus

$$\begin{aligned} \omega^\alpha a + \omega^\beta b &= \omega^\alpha a + \omega^{\alpha+\gamma} b \\ &= \omega^\alpha a + \omega^\alpha \omega^\gamma b && (8.5.33) \\ &= \omega^\alpha(a + \omega^\gamma b) && (8.5.14) \\ &= \omega^\alpha \omega^\gamma b && (\text{ex. } 8.4.1\text{b}) \\ &= \omega^{\alpha+\gamma} b && (8.5.33) \\ &= \omega^\beta b. \; \blacksquare \end{aligned}$$

If α and β are given in normal form, we can calculate the normal form of $\alpha + \beta$ using 9.1.5. For example, if

$$\alpha = \omega^\omega 2 + \omega^3 4 + \omega^2$$

and

$$\beta = \omega^3 3 + \omega^2 2 + 1,$$

then

$$\begin{aligned} \alpha + \beta &= \omega^\omega 2 + \omega^3 4 + \omega^2 + \omega^3 3 + \omega^2 2 + 1 \\ &= \omega^\omega 2 + \omega^3 4 + \omega^3 3 + \omega^2 2 + 1 && (9.1.4\text{b}) \\ &= \omega^\omega 2 + \omega^3 7 + \omega^2 2 + 1. && (9.1.4\text{a}) \end{aligned}$$

If

$$\alpha = \omega^6 3 + \omega^3 4 + 2$$

and

$$\beta = \omega^4 5 + \omega^2,$$

then it follows from 9.1.4(b) that

$$\alpha + \beta = \omega^6 3 + \omega^4 5 + \omega^2.$$

If the largest exponent in α is smaller than the largest exponent in β, then $\alpha + \beta = \beta$. Thus, if

$$\alpha = \omega^\omega 4 + \omega^{10} 5 + \omega^3$$

and

$$\beta = \omega^{\omega+1} 3 + \omega 2 + 5,$$

$\alpha = \omega^{\omega^2 3} \cdot 3 + \underbrace{\omega^{\omega} + 5}_{\gamma}$

$\alpha \cdot 7 = \omega^{\omega^2 3} \cdot 21 + \quad + \gamma$

$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$

niet $(\beta+\gamma)\alpha = \beta\alpha + \gamma\alpha$

$\alpha\beta$!

just the reverse as in process algebra!

p. 223 Rudin.

niet: $\alpha < \beta \Rightarrow \alpha + \beta = \beta$

$\omega < \omega + 1 \not\Rightarrow \omega + \omega + 1 = \omega + 1$

wel $\omega^2 + \omega^3 = \omega^3$

$\omega^2(1+\omega) =$

$\omega^2 \cdot \omega$

Rekenen met ordinalen in een termherschrijfsysteem

bachelorscriptie

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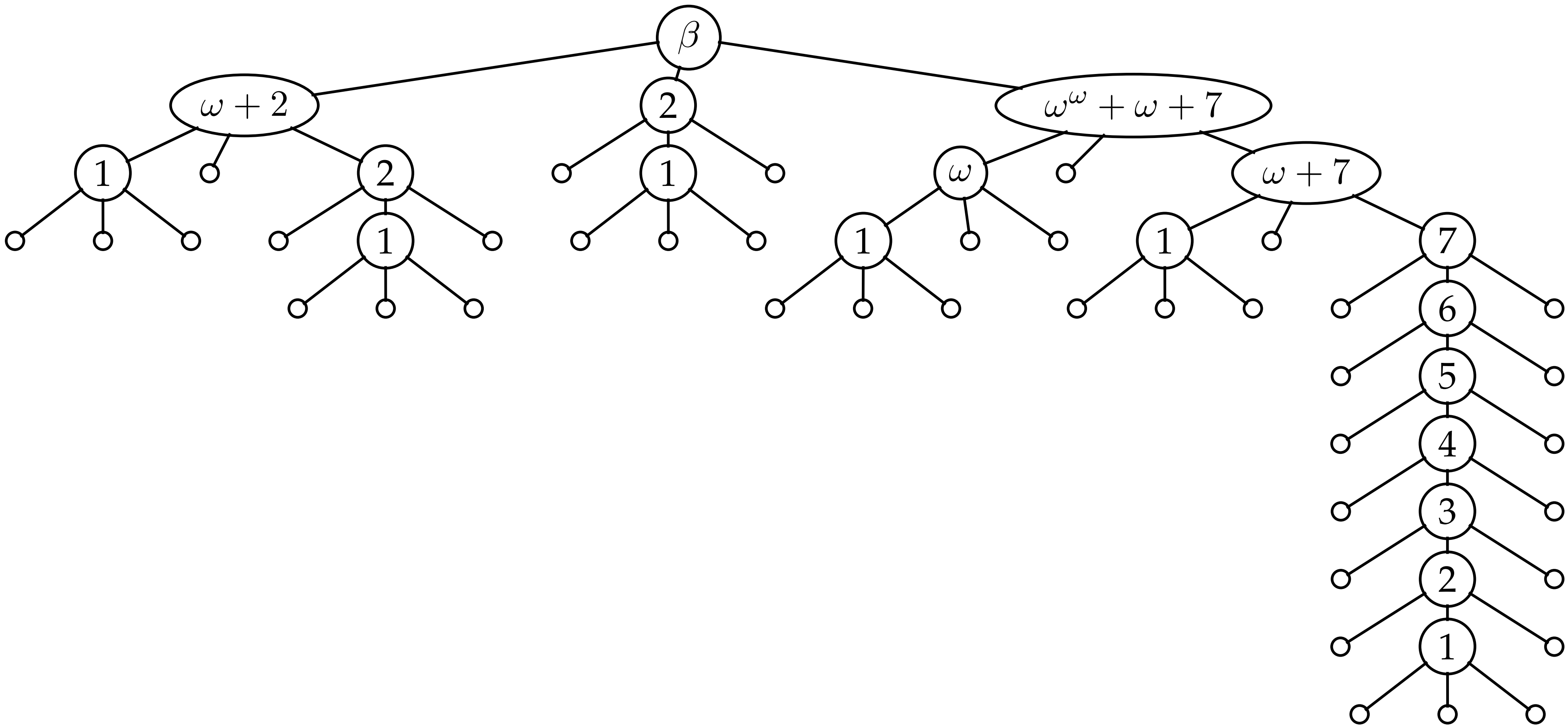
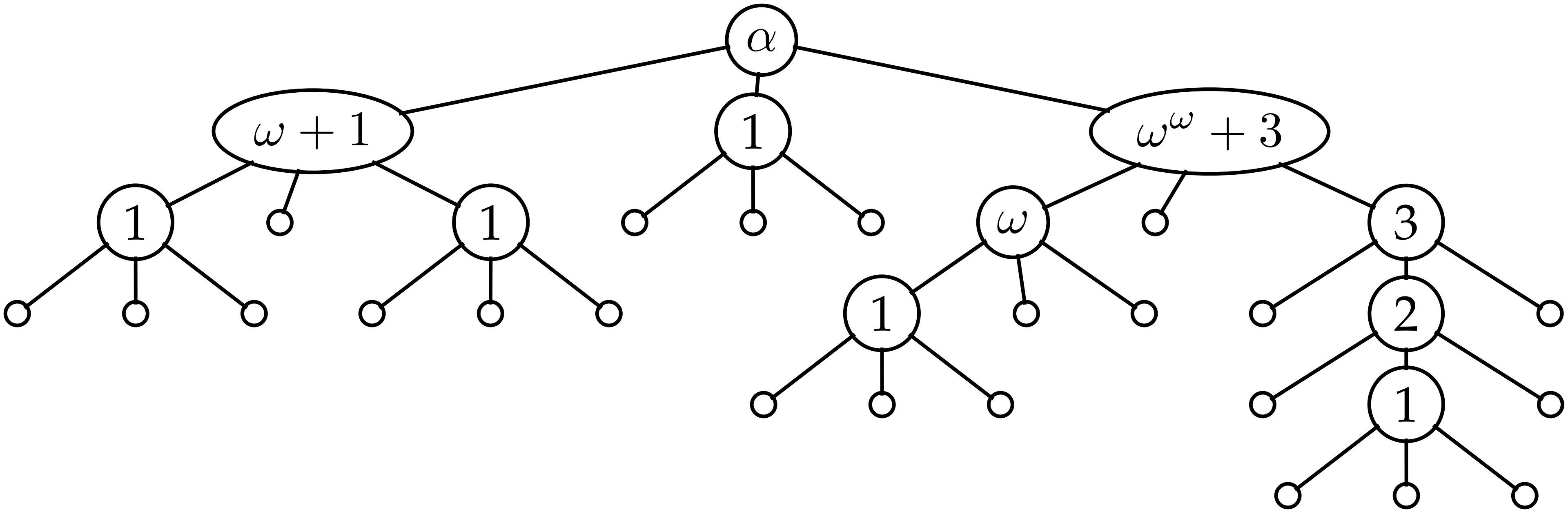
Samenvatting

Deze scriptie beschrijft een eindig termherschrijfsysteem (TRS) voor het rekenen met ordinalen tot ϵ_0. Voor het representeren van ordinalen wordt gebruik gemaakt van de `T1`-representatie van Pierre Castéran. Van het verkregen termherschijfsysteem wordt bewezen dat deze confluent (CR) en sterk terminerend (SN) is.

This paper describes a finite term rewriting system (TRS) to calculate with ordinals up to ϵ_0*. The ordinals will be represented like the* `T1` *representation of Pierre Castéran [Cas05]. It will be proven that the implemented TRS is confluent (CR) and strongly normalizing (SN).*



Figuur 3.1: $\mathcal{G}(\beta = \omega^{\omega+2}3 + \omega^\omega + \omega + 7)$



Figuur 3.2: $\mathcal{G}(\alpha = \omega^{\omega+1}2 + \omega^{\omega} + 3)$

If α and β are given in normal form, we can calculate the normal form of $\alpha + \beta$ using 9.1.5. For example, if

$$\alpha = \omega^{\omega}2 + \omega^{3}4 + \omega^{2}$$

and

$$\beta = \omega^{3}3 + \omega^{2}2 + 1,$$

then

$$\begin{aligned} \alpha + \beta &= \omega^{\omega}2 + \omega^{3}4 + \omega^{2} + \omega^{3}3 + \omega^{2}2 + 1 \\ &= \omega^{\omega}2 + \omega^{3}4 + \omega^{3}3 + \omega^{2}2 + 1 && (9.1.4b) \\ &= \omega^{\omega}2 + \omega^{3}7 + \omega^{2}2 + 1. && (9.1.4a) \end{aligned}$$

If

$$
\begin{aligned}
\alpha &= \omega^{10}3 + \omega^{2}7 + 3 \\
\beta_1 &= \omega^{5}4 + \omega^{4}9 \\
\beta_2 &= \omega^{5}4 + 9,
\end{aligned}
$$

then

$$
\begin{aligned}
\alpha\beta_1 &= \alpha\omega^{5}4 + \alpha\omega^{4}9 && (8.5.14) \\
&= \omega^{10+5}4 + \omega^{10+4}9 && (9.1.6b) \\
&= \omega^{15}4 + \omega^{14}9 && (8.5.33)
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha\beta_2 &= \alpha\omega^{5}4 + \alpha 9 && (8.5.14) \\
&= \omega^{15}4 + \alpha 9 && (9.1.6b, 8.5.33) \\
&= \omega^{15} + \omega^{10}(27) + \omega^{2}7 + 3. && (9.1.6a)
\end{aligned}
$$

… in een leesbaar ordinaal om te zetten.

Voorbeeld

$$(\omega^\omega 2 + \omega^3 4 + \omega^2) + (\omega^3 3 + \omega^2 2 + 1) = \omega^\omega 2 + \omega^3 7 + \omega^2 2 + 1$$

Dit is een voorbeeld uit Rubin, pagina 223, met $\alpha = \omega^\omega 2 + \omega^3 4 + \omega^2$ en $\beta = \omega^3 3 + \omega^2 2 + 1$. Hieronder volgen de herschrijfstappen voor de berekening.

$$
\begin{array}{rl}
A(\alpha, \beta) \equiv & A(C(\omega, 1, \omega^3 4 + \omega^2), C(3, 2, \omega^2 2 + 1)) \\
\to_{\text{a3}} & A^\star(C(\omega, 1, \omega^3 4 + \omega^2), C(3, 2, \omega^2 2 + 1), \underline{R(\omega, 3)}) \\
\to_{\text{r4}} & A^\star(C(\omega, 1, \omega^3 4 + \omega^2), C(3, 2, \omega^2 2 + 1), \\
 & R^\star(C(1, 0, 0), C(0, 2, 0), \underline{Eq(1, 0)}, \underline{Eq(0, 2)})) \\
\to_{\text{eq3|eq2}} & A^\star(C(\omega, 1, \omega^3 4 + \omega^2), C(3, 2, \omega^2 2 + 1), \underline{R^\star(C(1, 0, 0), C(0, 2, 0), \text{false}, \text{false})}) \\
\to_{\text{r7}} & A^\star(C(\omega, 1, \omega^3 4 + \omega^2), C(3, 2, \omega^2 2 + 1), \underline{R(1, 0)}) \\
\to_{\text{r3}} & \underline{A^\star(C(\omega, 1, \omega^3 4 + \omega^2), C(3, 2, \omega^2 2 + 1), \text{gt})} \\
\to_{\text{a5}} & C(\omega, 1, \underline{A(\omega^3 4 + \omega^2, C(3, 2, \omega^2 2 + 1))}) \\
\to_{\text{a3}} & C(\omega, 1, A(C(3, 3, \omega^2), C(3, 2, \omega^2 2 + 1), \underline{R(3, 3)})) \\
\twoheadrightarrow_{\text{lemma 3.3}} & C(\omega, 1, \underline{A^\star(C(3, 3, \omega^2), C(3, 2, \omega^2 2 + 1), \text{eq})}) \\
\to_{\text{a6}} & C(\omega, 1, C(3, C(0, \underline{A(3, 2)}, 0), \omega^2 2 + 1)) \\
\twoheadrightarrow_{\text{a3,a6,...}} & C(\omega, 1, C(3, \underline{C(0, 5, 0)}, \omega^2 2 + 1)) \\
\equiv & C(\omega, 1, \underline{C(3, 6, \omega^2 2 + 1)}) \\
\equiv & C(\omega, 1, \omega^3 7 + \omega^2 2 + 1) \\
\equiv & \omega^\omega 2 + \omega^3 7 + \omega^2 2 + 1
\end{array}
$$

Voorbeeld

$$(\omega^6 3 + \omega^2 4 + 2) + (\omega^4 5 + \omega^2) = \omega^6 3 + \omega^4 5 + \omega^2$$

Dit is een ander voorbeeld uit Rubin, pagina 223, met $\alpha = \omega^6 3 + \omega^2 4 +^2$ en $\beta = \omega^4 5 + \omega^2$. Hieronder volgen de herschrijfstappen voor de berekening.

$$\begin{aligned}
A(\alpha, \beta) \equiv\;& A(C(6, 2, \omega^2 4 + 2), C(4, 4, \omega^2)) \\
\to_{\text{a3}}\;& A^\star(C(6, 2, \omega^2 4 + 2), \beta, \underline{R(6, 4)}) \\
\twoheadrightarrow_{\text{lemma 3.7}}\;& A^\star(C(6, 2, \omega^2 4 + 2), \beta, \text{gt}) \\
\to_{\text{a5}}\;& C(6, 2, \underline{A(C(2, 3, 2), \beta)}) \\
\to_{\text{a3}}\;& C(6, 2, \overline{A^\star(C(2, 3, 2), \beta, \underline{R(2, 4)}})) \\
\twoheadrightarrow_{\text{lemma 3.5}}\;& C(6, 2, \underline{\overline{A^\star(C(2, 3, 2), \beta, \text{lt})}}) \\
\to_{\text{a5}}\;& C(6, 2, \beta) \equiv \omega^6 3 + \omega^4 5 + \omega^2
\end{aligned}$$

Voorbeeld

$$(\omega^{\omega+2}3 + \omega^{\omega} + \omega + 7) \times (\omega^{\omega+1}2 + \omega^{\omega} + 3) = \omega^{\omega 2+1}2 + \omega^{\omega 2} + \omega^{\omega+2}9 + \omega^{\omega} + \omega + 7$$

Dit is voorbeeld 8.6 uit *Derivation and Computation* [Sim00] en wordt door *Harald Simmons* bewezen middels het interlacing principle. Neem $\beta = \omega^{\omega+2}3 + \omega^{\omega} + \omega + 7$ en $\alpha = \omega^{\omega+1}2 + \omega^{\omega} + 3$. Figuur 3.1 en 3.2 tonen respectievelijk de bomen $\mathcal{G}(\beta)$ en $\mathcal{G}(\alpha)$. Hieronder volgen de herschrijfstappen voor de berekening.

$$
\begin{aligned}
M(\beta,\alpha) \equiv\ & M(C(\omega+2,2,\omega^{\omega}+\omega+7), C(\omega+1,1,\omega^{\omega}+3)) \\
\to_{\mathrm{m5}}\ & C(A(\omega+2,\omega+1),1,M(\beta,C(\omega,0,3))) \\
\to_{\mathrm{m5}}\ & C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,M(\beta,3))) \\
\to_{\mathrm{m4}}\ & C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2, \\
& A(M(C(0,2,0),2),2),\omega^{\omega}+\omega+7))) \\
\to_{\mathrm{m3}}\ & C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2, \\
& A(A(M(C(0,2,0),1),C(0,2,0)),2),\omega^{\omega}+\omega+7))) \\
\to_{\mathrm{m3}}\ & C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2, \\
& A(A(A(M(C(0,2,0),0),C(0,2,0)),C(0,2,0)),2),\omega^{\omega}+\omega+7))) \\
\to_{\mathrm{m1}}\ & C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2, \\
& A(A(A(0,C(0,2,0)),C(0,2,0)),2),\omega^{\omega}+\omega+7))) \\
\to_{\mathrm{a2}}\ & C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2, \\
& A(A(C(0,2,0),C(0,2,0)),2),\omega^{\omega}+\omega+7))) \\
\to_{\mathrm{a3}}\ & C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2, \\
& A(A^{\star}(C(0,2,0),C(0,2,0),R(0,0)),2),\omega^{\omega}+\omega+7))) \\
\to_{\mathrm{r2}}\ & C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2, \\
& A(A^{\star}(C(0,2,0),C(0,2,0),\mathrm{eq}),2),\omega^{\omega}+\omega+7))) \\
\to_{\mathrm{a6}}\ & C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2, \\
& A(C(0,C(0,A(2,2),0),0),C(0,1,0)),\omega^{\omega}+\omega+7))) \\
\twoheadrightarrow_{\mathrm{a3,a6,\ldots,a1}}\ & C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2, \\
& A(C(0,C(0,4,0),0),C(0,1,0)),\omega^{\omega}+\omega+7))) \\
\twoheadrightarrow_{\mathrm{a3,a6,\ldots,a1}}\ & C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2,8,\omega^{\omega}+\omega+7))) \\
\to_{\mathrm{a3}}\ & C(A(\omega+2,\omega+1),1,C(A^{\star}(C(1,0,2),C(1,0,0),R(1,1)),0,C(\omega+2,8,\omega^{\omega}+\omega+7))) \\
\twoheadrightarrow_{\mathrm{lemma\ 3.3}}\ & C(A(\omega+2,\omega+1),1,C(A^{\star}(C(1,0,2),C(1,0,0),\mathrm{eq}),0,C(\omega+2,8,\omega^{\omega}+\omega+7))) \\
\to_{\mathrm{a6}}\ & C(A(\omega+2,\omega+1),1,C(C(1,C(0,A(0,0),0),0),0,C(\omega+2,8,\omega^{\omega}+\omega+7))) \\
\to_{\mathrm{a1}}\ & C(A(\omega+2,\omega+1),1,C(C(1,C(0,0,0),0),0,C(\omega+2,8,\omega^{\omega}+\omega+7))) \\
\to_{\mathrm{a3}}\ & C(A^{\star}(C(1,0,2),C(1,0,1),R(1,1)),1,C(C(1,C(0,0,0),0),0,C(\omega+2,8,\omega^{\omega}+\omega+7))) \\
\twoheadrightarrow_{\mathrm{lemma\ 3.3}}\ & C(A^{\star}(C(1,0,2),C(1,0,1),\mathrm{eq}),1,C(C(1,C(0,0,0),0),0,C(\omega+2,8,\omega^{\omega}+\omega+7))) \\
\twoheadrightarrow_{\mathrm{a6,a1}}\ & C(C(1,C(0,0,0),1),1,C(C(1,C(0,0,0),0),0,C(\omega+2,8,\omega^{\omega}+\omega+7))) \\
\equiv\ & C(C(1,1,1),1,C(C(1,1,0),0,\omega^{\omega+2}9+\omega^{\omega}+\omega+7)) \\
\equiv\ & C(\omega 2+1,1,C(\omega 2,0,\omega^{\omega+2}9+\omega^{\omega}+\omega+7)) \\
\equiv\ & C(\omega 2+1,1,\omega^{\omega 2}+\omega^{\omega+2}9+\omega^{\omega}+\omega+7) \\
\equiv\ & \omega^{\omega 2+1}2+\omega^{\omega 2}+\omega^{\omega+2}9+\omega^{\omega}+\omega+7
\end{aligned}
$$

Voorbeeld

$$(\omega^{\omega+2}3 + \omega^{\omega} + \omega + 7) \times (\omega^{\omega+1}2 + \omega^{\omega} + 3) = \omega^{\omega 2+1}2 + \omega^{\omega 2} + \omega^{\omega+2}9 + \omega^{\omega} + \omega + 7$$

Dit is voorbeeld 8.6 uit *Derivation and Computation* [Sim00] en wordt door *Harald Simmons* bewezen middels het interlacing principle. Neem $\beta = \omega^{\omega+2}3 + \omega^{\omega} + \omega + 7$ en $\alpha = \omega^{\omega+1}2 + \omega^{\omega} + 3$. Figuur 3.1 en 3.2 tonen respectievelijk de bomen $\mathcal{G}(\beta)$ en $\mathcal{G}(\alpha)$. Hieronder volgen de herschrijfstappen voor de berekening.

$$
\begin{aligned}
M(\beta,\alpha) \equiv\;& M(C(\omega+2,2,\omega^\omega+\omega+7),C(\omega+1,1,\omega^\omega+3))\\
\to_{\mathrm{m5}}\;& C(A(\omega+2,\omega+1),1,M(\beta,C(\omega,0,3)))\\
\to_{\mathrm{m5}}\;& C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,M(\beta,3)))\\
\to_{\mathrm{m4}}\;& C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2,\\
& A(M(C(0,2,0),2),2),\omega^\omega+\omega+7)))\\
\to_{\mathrm{m3}}\;& C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2,\\
& A(A(M(C(0,2,0),1),C(0,2,0)),2),\omega^\omega+\omega+7)))\\
\to_{\mathrm{m3}}\;& C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2,\\
& A(A(A(M(C(0,2,0),0),C(0,2,0)),C(0,2,0)),2),\omega^\omega+\omega+7)))\\
\to_{\mathrm{m1}}\;& C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2,\\
& A(A(A(0,C(0,2,0)),C(0,2,0)),2),\omega^\omega+\omega+7)))\\
\to_{\mathrm{a2}}\;& C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2,\\
& A(A(C(0,2,0),C(0,2,0)),2),\omega^\omega+\omega+7)))\\
\to_{\mathrm{a3}}\;& C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2,\\
& A(A^\star(C(0,2,0),C(0,2,0),R(0,0)),2),\omega^\omega+\omega+7)))\\
\to_{\mathrm{r2}}\;& C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2,\\
& A(A^\star(C(0,2,0),C(0,2,0),\mathrm{eq}),2),\omega^\omega+\omega+7)))\\
\to_{\mathrm{a6}}\;& C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2,\\
& A(C(0,C(0,A(2,2),0),0),C(0,1,0)),\omega^\omega+\omega+7)))\\
\twoheadrightarrow_{\mathrm{a3,a6,\ldots,a1}}\;& C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2,\\
& A(C(0,C(0,4,0),0),C(0,1,0)),\omega^\omega+\omega+7)))\\
\twoheadrightarrow_{\mathrm{a3,a6,\ldots,a1}}\;& C(A(\omega+2,\omega+1),1,C(A(\omega+2,\omega),0,C(\omega+2,8,\omega^\omega+\omega+7)))\\
\to_{\mathrm{a3}}\;& C(A(\omega+2,\omega+1),1,C(A^\star(C(1,0,2),C(1,0,0),R(1,1)),0,C(\omega+2,8,\omega^\omega+\omega+7)))\\
\twoheadrightarrow_{\text{lemma 3.3}}\;& C(A(\omega+2,\omega+1),1,C(A^\star(C(1,0,2),C(1,0,0),\mathrm{eq}),0,C(\omega+2,8,\omega^\omega+\omega+7)))\\
\to_{\mathrm{a6}}\;& C(A(\omega+2,\omega+1),1,C(C(1,C(0,A(0,0),0),0),0,C(\omega+2,8,\omega^\omega+\omega+7)))\\
\to_{\mathrm{a1}}\;& C(A(\omega+2,\omega+1),1,C(C(1,C(0,0,0),0),0,C(\omega+2,8,\omega^\omega+\omega+7)))\\
\to_{\mathrm{a3}}\;& C(A^\star(C(1,0,2),C(1,0,1),R(1,1)),1,C(C(1,C(0,0,0),0),0,C(\omega+2,8,\omega^\omega+\omega+7)))\\
\twoheadrightarrow_{\text{lemma 3.3}}\;& C(A^\star(C(1,0,2),C(1,0,1),\mathrm{eq}),1,C(C(1,C(0,0,0),0),0,C(\omega+2,8,\omega^\omega+\omega+7)))\\
\twoheadrightarrow_{\mathrm{a6,a1}}\;& C(C(1,C(0,0,0),1),1,C(C(1,C(0,0,0),0),0,C(\omega+2,8,\omega^\omega+\omega+7)))\\
\equiv\;& C(C(1,1,1),1,C(C(1,1,0),0,\omega^{\omega+2}9+\omega^\omega+\omega+7))\\
\equiv\;& C(\omega2+1,1,C(\omega2,0,\omega^{\omega+2}9+\omega^\omega+\omega+7))\\
\equiv\;& C(\omega2+1,1,\omega^{\omega2}+\omega^{\omega+2}9+\omega^\omega+\omega+7)\\
\equiv\;& \omega^{\omega2+1}2+\omega^{\omega2}+\omega^{\omega+2}9+\omega^\omega+\omega+7
\end{aligned}
$$

8.6 Using

$$\beta = \omega^{\omega+2}3 + \omega^{\omega} + \omega + 7 \qquad \alpha = \omega^{\omega+1}2 + \omega^{\omega} + 3$$

calculate $\beta \times \alpha$ in a canonical form. Use this example to discover some useful rules of manipulation for ordinal arithmetic. Zie p. 345, 346 by another use of the interlacing principle.

Finally

$$\beta\omega^{\omega+1} = \beta\omega^{\omega}\omega = \omega^{\omega 2+1}$$

and hence we obtain

$$\beta\alpha = \omega^{\omega 2+1}2 + \omega^{\omega 2} + \omega^{\omega+2}9 + \omega^{\omega} + \omega + 7$$

as the required result.

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Terminatiebewijs middels AProVE

In deze bijlage treft u de input en output aan voor het terminatiebewijs middels de AProVEtool (`http://aprove.informatik.rwth-aachen.de/`).

C.1 Input

```
[x, u, d, e, n, k, l, j, y, v, f, g, b]
a(x,nul)                     ->  x
a(nul,x)                     ->  x
a(c(x,n,y),c(u,k,v))         ->  as(c(x,n,y),c(u,k,v),r(x,u))
as(c(x,n,y),c(u,k,v),lt)     ->  c(u,v,k)
as(c(x,n,y),c(u,k,v),gt)     ->  c(x,n,a(y,c(u,k,v)))
as(c(x,n,y),c(u,k,v),eq)     ->  c(x,c(nul,a(n,k),nul),v)
m(x,nul)                     ->  nul
m(nul,x)                     ->  nul
m(c(nul,n,y),c(nul,k,v))     ->  a(m(c(nul,n,nul),k),c(0,n,0))
m(c(c(d,l,e),n,y),c(nul,k,v))        -> c(c(d,l,e),a(m(c(nul,n,nul),k),n),v)
m(c(c(d,l,e),n,y),c(c(f,j,g),k,v)) -> c(a(c(d,l,e),c(f,j,g)),k,m(c(c(d,l,e),n,y),v))
r(nul,c(x,n,y))              ->  lt
r(nul,nul)                   ->  eq
r(c(x,n,y),nul)              ->  gt
r(c(x,n,y),c(u,k,v))         ->  rs(c(x,n,y),c(u,k,v),equal(x,u),equal(n,k))
rs(c(x,n,y),c(u,k,v), true, true)      ->  r(y,v)
rs(c(x,n,y),c(u,k,v), true, false)     ->  r(n,k)
rs(c(x,n,y),c(u,k,v), false, b)        ->  r(x,y)
equal(nul,nul)               ->  true
equal(nul,c(x,n,y))          ->  false
equal(c(x,n,y),nul)          ->  false
equal(c(x,n,y),c(u,k,v))     ->  and(equal(x,u),and(equal(n,k),equal(y,v)))
and(false,x)                 ->  false
and(true,x)                  ->  x
```

Terminatiebewijs middels Tyrolean Termination Tool

In deze bijlage treft u de input en output aan voor het terminatiebewijs middels de Tyrolean Termination Tool (`http://cl2-informatik.uibk.ac.at/ttt/`).

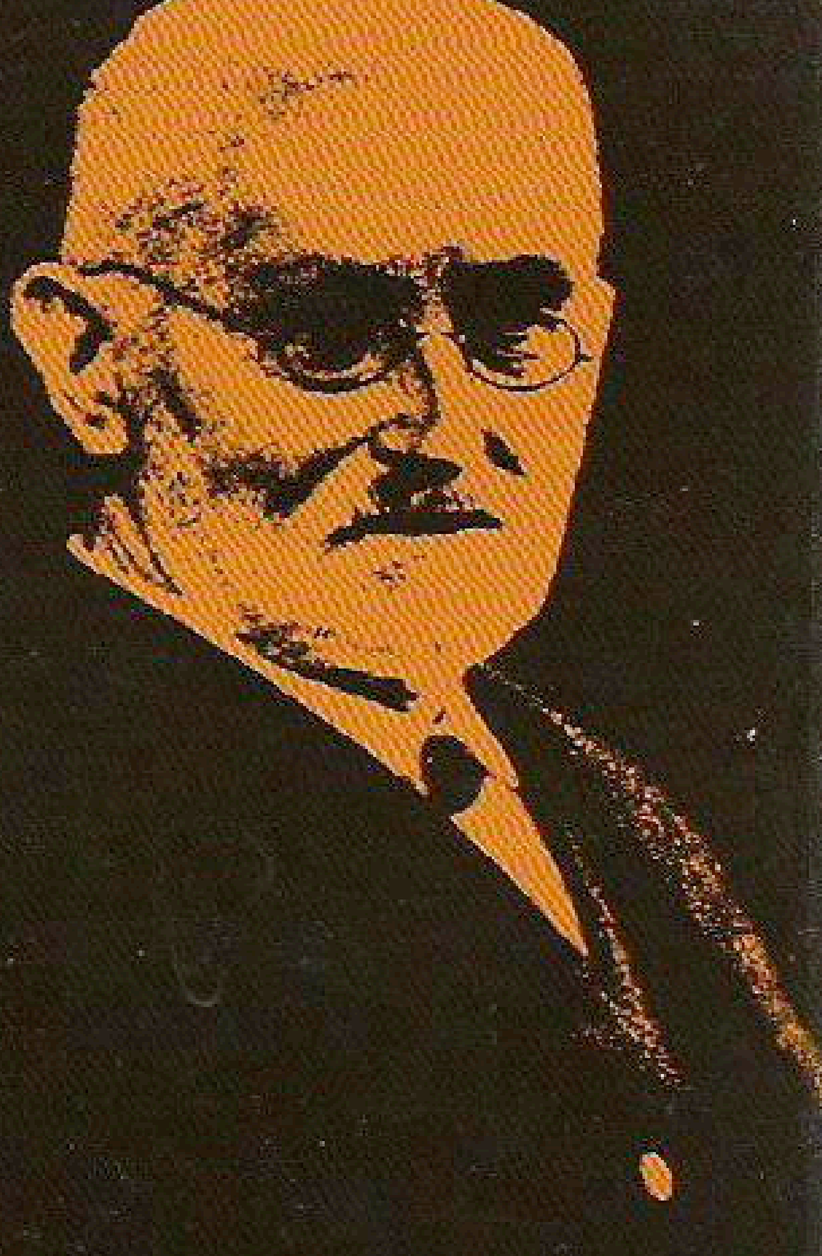
B.1 Input

```
a(x,nul())                    ->  x;
a(nul(),x)                    ->  x;
a(c(x,n,y),c(x',n',y'))       ->  a#(c(x,n,y),c(x',n',y'),r(x,x'));
a#(c(x,n,y),c(x',n',y'),lt()) ->  c(x',n',y');
a#(c(x,n,y),c(x',n',y'),gt()) ->  c(x,n,a(y,c(x',n',y')));
a#(c(x,n,y),c(x',n',y'),eq()) ->  c(x,c(nul(),a(n,n'),nul()),y');
m(x,nul())                    ->  nul();
m(nul(),y)                    ->  nul();
m(c(nul(),n,y),c(nul(),n',y'))
              -> a(m(c(nul(),n,nul()),n'),c(nul(),n,nul()));
m(c(c(u,k,v),n,y),c(nul(),n',y'))
              -> c(c(u,k,v),a(m(c(nul(),n,nul()),n'),n),y);
m(c(c(u,k,v),n,y),c(c(u',k',v'),n',y'))
              -> c(a(c(u,k,v),c(u',k',v')), n', m(c(c(u,k,v),n,y),y'));
r(nul(),c(x,n,y))             ->  lt();
r(nul(),nul())                ->  eq();
r(c(x,n,y),nul())             ->  gt();
r(c(x,n,y),c(x',n',y'))
              ->  r#(c(x,n,y),c(x',n',y'),equal(x,x'),equal(n,n'));
r#(c(x,n,y),c(x',n',y'), true(), true())      ->  r(y,y');
r#(c(x,n,y),c(x',n',y'), true(), false())     ->  r(n,n');
r#(c(x,n,y),c(x',n',y'), false(), b)          ->  r(x,x');
equal(nul(),nul())            ->  true();
equal(nul(),c(x,n,y))         ->  false();
equal(c(x,n,y),nul())         ->  false();
equal(c(x,n,y),c(x',n',y'))
              ->  and(equal(x,x'),and(equal(n,n'),equal(y,y')));
and(false(),x)                ->  false();
and(true(),x)                 ->  x;
```

Contents

$1.75

123



ESSAYS ON THE THEORY OF NUMBERS

Continuity and Irrational Numbers
The Nature and Meaning of Numbers

RICHARD DEDEKIND

Therefore by (126) this sum is completely determined by the conditions*

$$\text{II.} \quad m + 1 = m',$$

$$\text{1II.} \quad m + n' = (m + n)'.$$

or in short the product of the numbers m, n. This therefore by (126) is completely determined by the conditions

$$\text{II.} \quad m \, . \, 1 = m$$

$$\text{III.} \quad m n' = m n + m,$$

Definition I.1.2 (Grassmann [1861])

A4 $x + 0 = x$

A5 $x + \mathcal{S}(y) - \mathcal{S}(x + y)$

A6 $x \cdot 0 - 0$

A7 $x \cdot \mathcal{S}(y) - x \cdot y + x.$

but for now we stick to the official notation. Each such operation is specified by a recursion over the second argument, α, with the first argument, β, held fixed. Addition is specified in terms of Suc, multiplication is specified in terms of addition, and exponentiation in terms of multiplication.

(base)	(step)	(leap)
$\beta + 0 = \beta$	$\beta + \alpha' = (\beta + \alpha)'$	$\beta + \mu = \bigvee\{\beta + \alpha \mid \alpha < \mu\}$
$\beta \times 0 = 0$	$\beta \times \alpha' = (\beta \times \alpha) + \beta$	$\beta \times \mu = \bigvee\{\beta \times \alpha \mid \alpha < \mu\}$
$\beta^0 = 1$	$\beta^{\alpha'} = (\beta^\alpha) \times \beta$	$\beta^\mu = \bigvee\{\beta^\alpha \mid \alpha < \mu\}$

Here, as usual, α is an arbitrary ordinal and μ is a limit ordinal. Also β is an arbitrary ordinal.

for arbitrary α, β, γ to produce the ordinal stacking function. The (base, step, leap) specification of this is

$$\beth(\gamma,\beta,0) = \gamma \qquad \beth(\gamma,\beta,\alpha') = \beta^{\beth(\gamma,\beta,\alpha)} \qquad \beth(\gamma,\beta,\mu) = \bigvee\{\beth(\gamma,\beta,\alpha) \mid \alpha < \mu\}$$

for arbitrary ordinals α, β, γ and limit ordinals μ. For finite α, β, γ the ordinal value $\beth(\gamma,\beta,\alpha)$ is just the natural number value. You might like to worry about the value $\beth(\omega,\omega,\omega)$ for a while.

In this book we always calculate with a limit ordinal via a selected cofinal
of smaller ordinal. To do this efficiently we introduce an important idea.

8.3 DEFINITION. A fundamental sequence for a limit ordinal μ is a function

$$\mu[\cdot] : \mathbb{N} \longrightarrow \mathbb{O}\mathrm{rd}$$

which is monotone and with a range that is cofinal in μ, i.e. such that

- $(\forall r, s : \mathbb{N})[r \leq s \Rightarrow \mu[r] \leq \mu[s]]$
- $(\forall r : \mathbb{N})[\mu[r] < \mu]$
- $(\forall \gamma < \mu)(\exists r : \mathbb{N})[\gamma \leq \mu[r]]$

hold.

$$A(x,0) \to x$$

$$A(x,S(y)) \to S(A(x,y))$$

$$A(x,P(y,z)) \to P(A(x,y),A(x,z))$$

$$M(x,0) \to 0$$

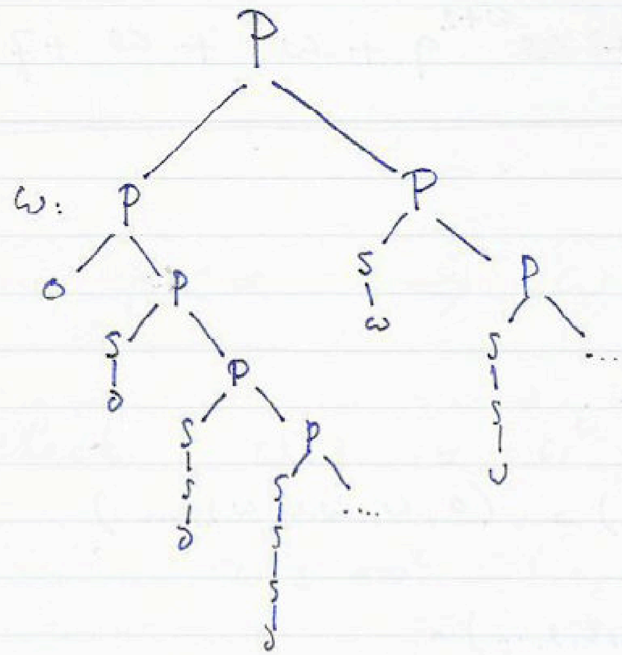
$$M(x,S(y)) \to A(M(x,y),x)$$

$$M(x,P(y,z)) \to P(M(x,y),M(x,z))$$

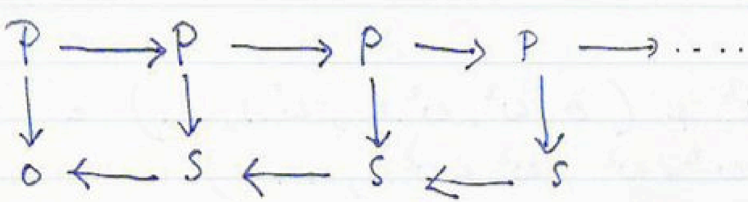
$$E(x,0) \Rightarrow S(0)$$

$$E(x,S(y)) \to M(E(x,y),x)$$

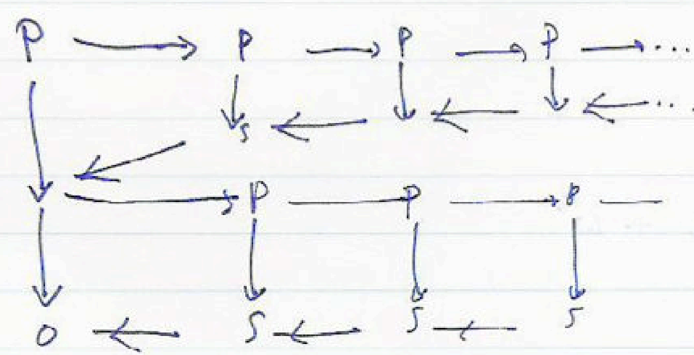
$$E(x,P(y,z)) \to P(E(x,y),E(x,z))$$



met sharing: ω:



$\omega + \omega$:



Dedekind TRS

$$x + 0 \to x$$
$$x + y' \to (x+y)'$$
$$x \cdot 0 \to 0$$
$$x \cdot y' \to (x \cdot y) + x$$

$$x^0 \to 1$$
$$x^{y'} \to x^y \cdot x$$

$$x + (y:z) \to (x+y) : (x+z)$$
$$x \cdot (y:z) \to (x \cdot y) : (x \cdot z)$$
$$x^{y:z} \to x^y : x^z$$

$$nats(x) \to x : nats(x')$$
$$\omega \to nats(0)$$

$$B(x,y,0) \to x$$
$$B(x,y,z') \to y^{B(x,y,z)}$$
$$B(x,y,u:z) \to B(x,y,u) : B(x,y,z)$$

Dedekind TRS

$$
\begin{aligned}
x+0 &\to x \\
x+y' &\to (x+y)' \\
x.0 &\to 0 \\
x.y' &\to (x.y)+x \\
x^{0} &\to 1 \\
x^{y'} &\to x^{y}.x \\
x+(y:z) &\to (x+y):(x+z) \\
x.(y:z) &\to (x.y):(x.z) \\
x^{y:z} &\to x^{y} : x^{z} \\
\mathrm{nats}(x) &\to x : \mathrm{nats}(x') \\
\omega &\to \mathrm{nats}(0) \\
B(x,y,0) &\to x \\
B(x,y,z') &\to y^{B(x,y,z)} \\
B(x,y,u:z) &\to B(x,y,u):B(x,y,z)
\end{aligned}
$$

for arbitrary α, β, γ to produce the ordinal stacking function. The (base, step, lea
specification of this is

$$\beth(\gamma,\beta,0) = \gamma \qquad \beth(\gamma,\beta,\alpha') = \beta^{\beth(\gamma,\beta,\alpha)} \qquad \beth(\gamma,\beta,\mu) = \bigvee\{\beth(\gamma,\beta,\alpha) \mid \alpha < \mu\}$$

for arbitrary ordinals α, β, γ and limit ordinals μ. For finite α, β, γ the ordin
value $\beth(\gamma,\beta,\alpha)$ is just the natural number value. You might like to worry abo
the value $\beth(\omega,\omega,\omega)$ for a while.

$$x + 0 \to x$$
$$x + S(y) \to S(x+y)$$

$$x . 0 \to 0$$
$$x . S(y) \to x.y + x$$

$$x^0 \to 1$$
$$x^{y+1} \to x^y . x$$

$$x + y:z \to (x+y) : x+z$$
$$x . y:z \to (x.y) : x.z$$
$$x^{y:z} \to x^y : x^z$$

$$nats(x) \to x : nats(S(x))$$
$$\omega \to nats(0)$$

$$\omega + 1 \to S(\omega + 0) \to S(\omega)$$
$$1 + \omega \to 1 : 2 : 3 : 4 : \ldots \sim \omega .$$

$$\omega + \omega \to \omega : S(\omega) : SS(\omega) : S^3\omega : \ldots$$
$$\omega . 2 \to \omega . 1 + \omega \to \omega + \omega$$
$$\omega^2 \to \omega^1 . \omega \to \omega^0 . \omega . \omega \to 1 . \omega . \omega \to \omega . \omega$$
$$\omega + \omega^2 \to$$

$$\omega : (\omega + \omega . 1) : (\omega + \omega . 2) : \ldots \sim 0 : \omega : \omega.2 : \omega.3 : \ldots$$

$$2 . \omega \to 0 : 2 : 4 : 6 : \ldots$$
$$\omega . 2 \to \omega + \omega \to \omega : S\omega : S^2\omega : S^3\omega : \ldots$$

$$\omega^\omega = \omega^0 : \omega^1 : \omega^2 : \omega^3 : \ldots$$
$$= 1 : \omega : \omega^2 : \omega^3 : \ldots$$

$$\omega^2 + \omega^3 = \omega^3$$

tre probem dere vay. te verifiere.

$$\omega^2 = \omega \cdot \omega = \omega \cdot (0, 1, 2, \ldots) = (0, \omega, \omega \cdot 2, \omega \cdot 3, \ldots)$$

$$\omega^3 = \omega^2 \cdot \omega = \omega^2 \cdot (0, 1, 2, 3, \ldots) =$$

$$(\omega^2 \cdot 0, \omega^2 \cdot 1, \omega^2 \cdot 2, \ldots) = (0, \underline{\omega^2}, \underline{\omega^2 \cdot 2}, \ldots)$$

$$\omega^2 + \omega^3 = \omega^2 + (0, \omega^2, \omega^2 \cdot 2, \omega^2 \cdot 3, \ldots) =$$

$$(\omega^2 + 0, \omega^2 + \omega^2, \omega^2 + \omega^2 \cdot 2, \ldots) =$$

$$(\underline{\omega^2}, \underline{\omega^2 + \omega^2}, \omega^2 + \omega^2 \cdot 2, \ldots)$$

Rogers p.221.

Er is een ordinal α met $\alpha = \omega . \alpha$

Is dit ω^ω? geldt $\omega^\omega = \omega . \omega^\omega$?

$\omega^\omega = (\omega^0, \omega^1, \omega^2, \omega^3, \ldots)$

$\omega . \omega^\omega = \omega . (\omega^0, \omega^1, \omega^2, \omega^3, \ldots) =$

$(\omega . 1 , \omega . \omega^2, \omega . \omega^3, \ldots) =$

$(\omega , \omega^3, \omega^4, \ldots) = \omega^\omega.$

Check $2^\omega = 3^\omega = \omega$ (p. 374 Hinman)

$2^\omega = 2^{(0,1,2,\ldots)} = (2^0, 2^1, 2^2, 2^3, \ldots) =$

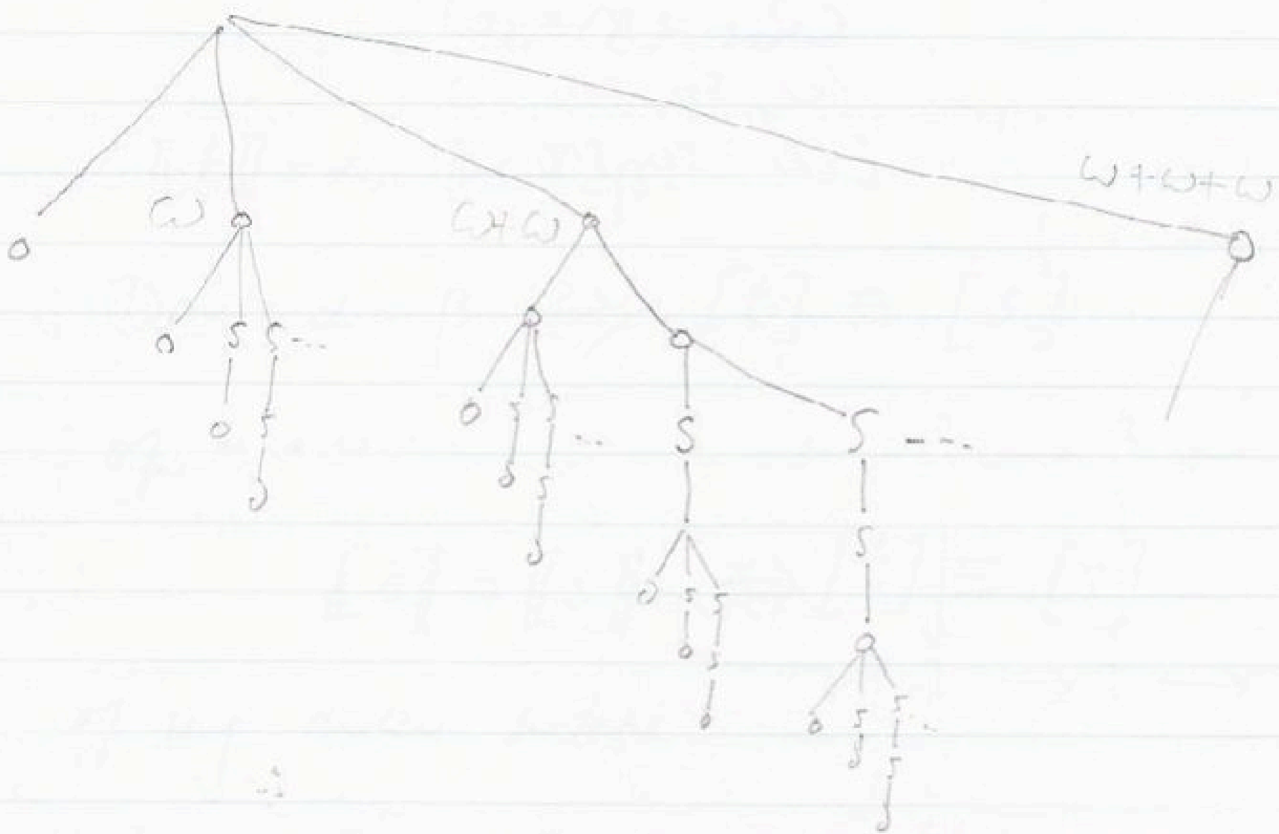
$(1, 2, 4, 8, \ldots) = \omega$

en $3^\omega = (3^0, 3^1, 3^2, \ldots) = (1, 3, 9, 27, \ldots) = \omega.$

|| Dit is mooi voorbeeld dat je rijtjes via het 'interlacing principle' (Simmons) moet identificeren.

nl $\omega = (0, 1, 2, 3, 4, \ldots)$ en
$(1, 2, 4, 8, 16, \ldots)$
are interlacing.

$$\omega^2 \to \omega . \omega \to (\omega.0, \omega.1, \omega.2, \ldots) \to$$
$$(0, \omega, \omega+\omega, \omega+\omega+\omega, \ldots)$$

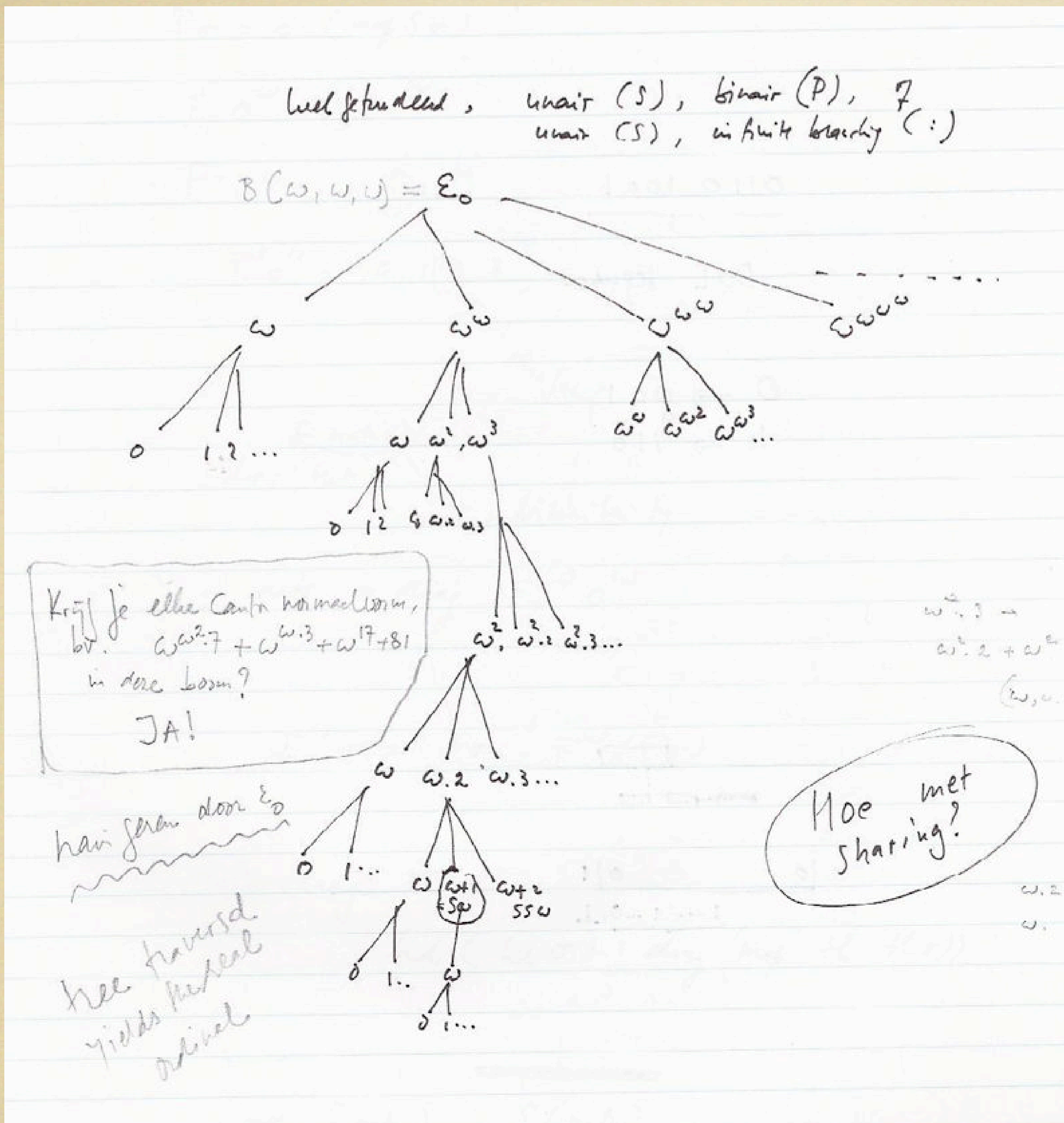


$\varepsilon_0 = B(\omega, \omega, \omega)$, dan $\omega^{\varepsilon_0} = \varepsilon_0$.

Nl. $B(\omega, \omega, \omega) = B(\omega, \omega, 0:1:2:3:\dots) =$

$$(B(\omega,\omega,0), B(\omega,\omega,1), B(\omega,\omega,2), \dots) =$$
$$(\omega, \omega^{\omega}, \omega^{(\omega^{\omega})}, \omega^{\omega^{\omega^{\omega}}}, \dots) \equiv$$

en $$\omega^{(\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \dots)} = (\omega^{\omega}, \omega^{\omega^{\omega}}, \omega^{\omega^{\omega^{\omega}}}, \dots)$$


welgefundeerd, unair (S), binair (P), ?
unair (S), infinite branching (:)
B(ω, ω, ω) = ε0
Krijg je elke Cantor normaalvorm, bv. ω^(ω^2·7) + ω^(ω·3) + ω^17 + 81 in deze boom?
JA!
Hoe met sharing?
tree traversed yields the real ordinal

En dan hebben we de <u>STELLING</u>

Laten α, β ordinalen zijn en $t, s \in Ter^{\circ}(\Sigma)$ zodat

$[\![t]\!] = \alpha$, $[\![s]\!] = \beta$.

Dan $\alpha = \beta \iff [t] \equiv [s]$.

of m.a.w.:

$$[\![t]\!] = [\![s]\!] \iff [t] \equiv [s]$$

of nog andere notatie:

(i) $[\![t]\!] = [\![s]\!] \iff nf(t) \equiv nf(s)$.

Bovendien is (ii) $nf(t)$ uniek.

en (iii) De TRS is SN^{∞}.

<u>Opm.</u> (ii) vereist (iii) <u>niet</u>.

Contents

epsilon-zero ε_0

Gamma-zero Γ_0

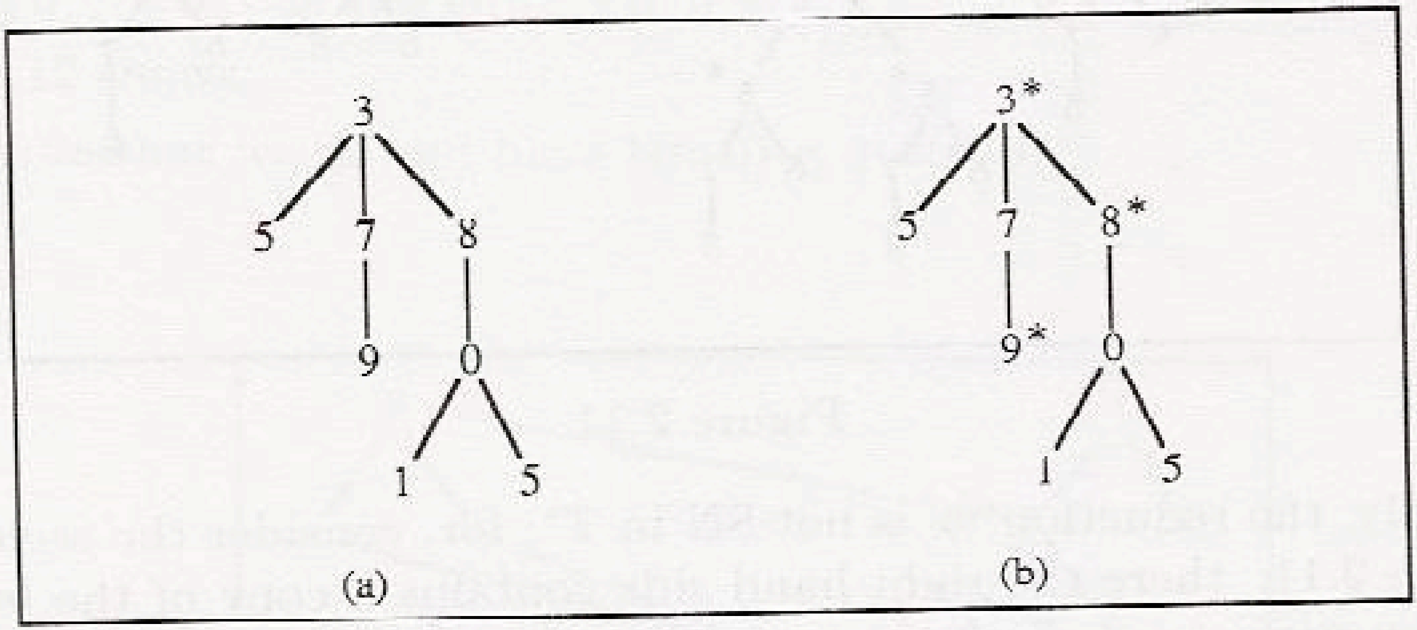


Figure 2.10

Definition 2.3.3. On $\mathbb{T}^*$ we define a reduction relation $\Rightarrow$ as follows.

1. *place marker at the top:*
   $n(\vec{t}) \Rightarrow n^*(\vec{t}) \qquad (\vec{t} = t_1, \ldots, t_k, k \geq 0)$
2. *make copies below lesser top:*
   if $n > m$, then $n^*(\vec{t}) \Rightarrow m(n^*(\vec{t}), \ldots, n^*(\vec{t})) \quad (j \geq 0$ copies of $n^*(\vec{t}))$
3. *push marker down:*
   $n^*(s, \vec{t}) \Rightarrow n(s^*, \ldots, s^*, \vec{t}) \qquad (j \geq 0$ copies of $s^*)$
4. *select argument:*
   $n^*(t_1, \ldots, t_k) \Rightarrow t_i \qquad (i \in \{1, \ldots, k\}, k \geq 1)$

Example 2.3.4. Figure 2.11 displays a reduction in $\mathbb{T}^*$.

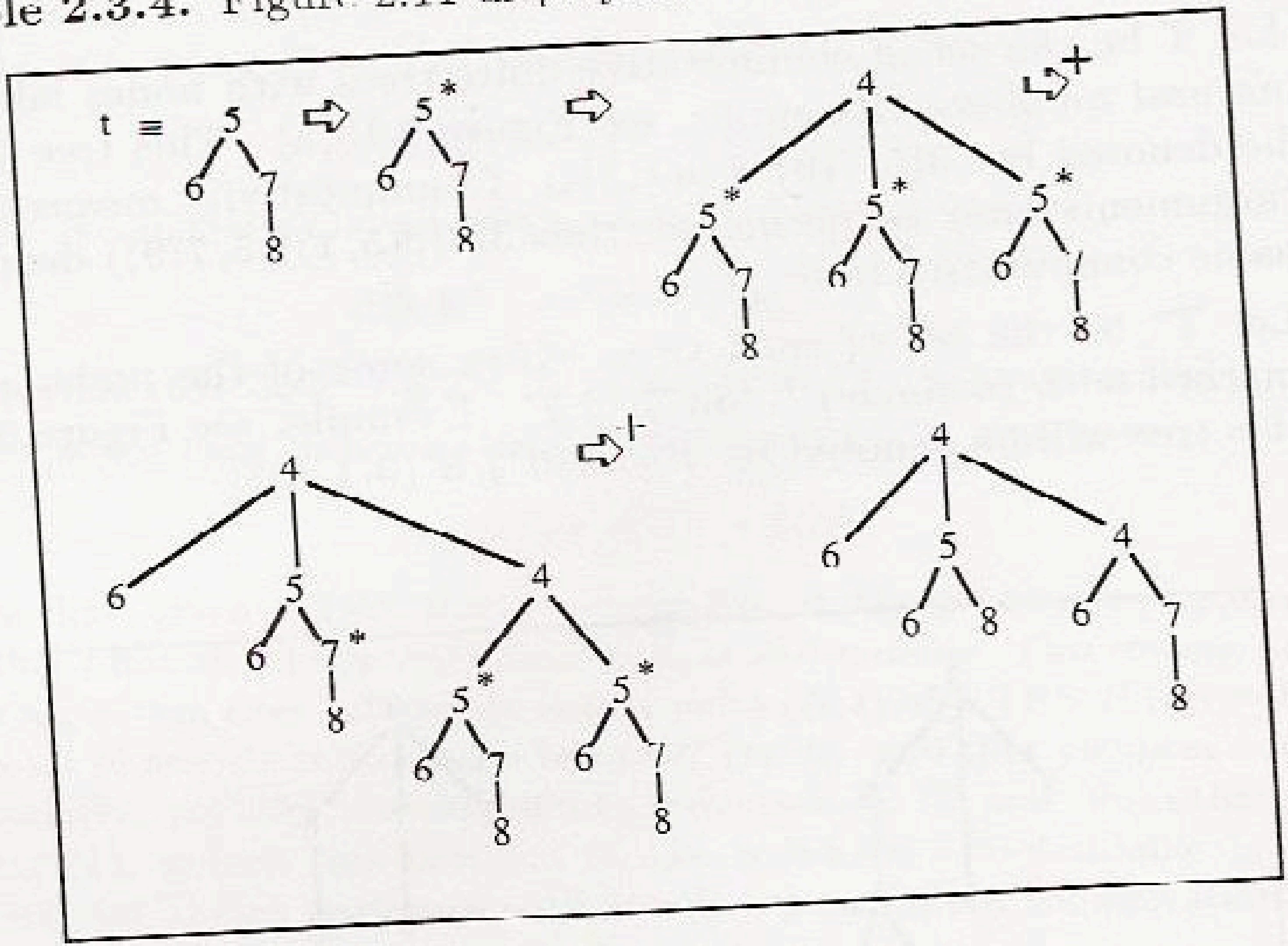
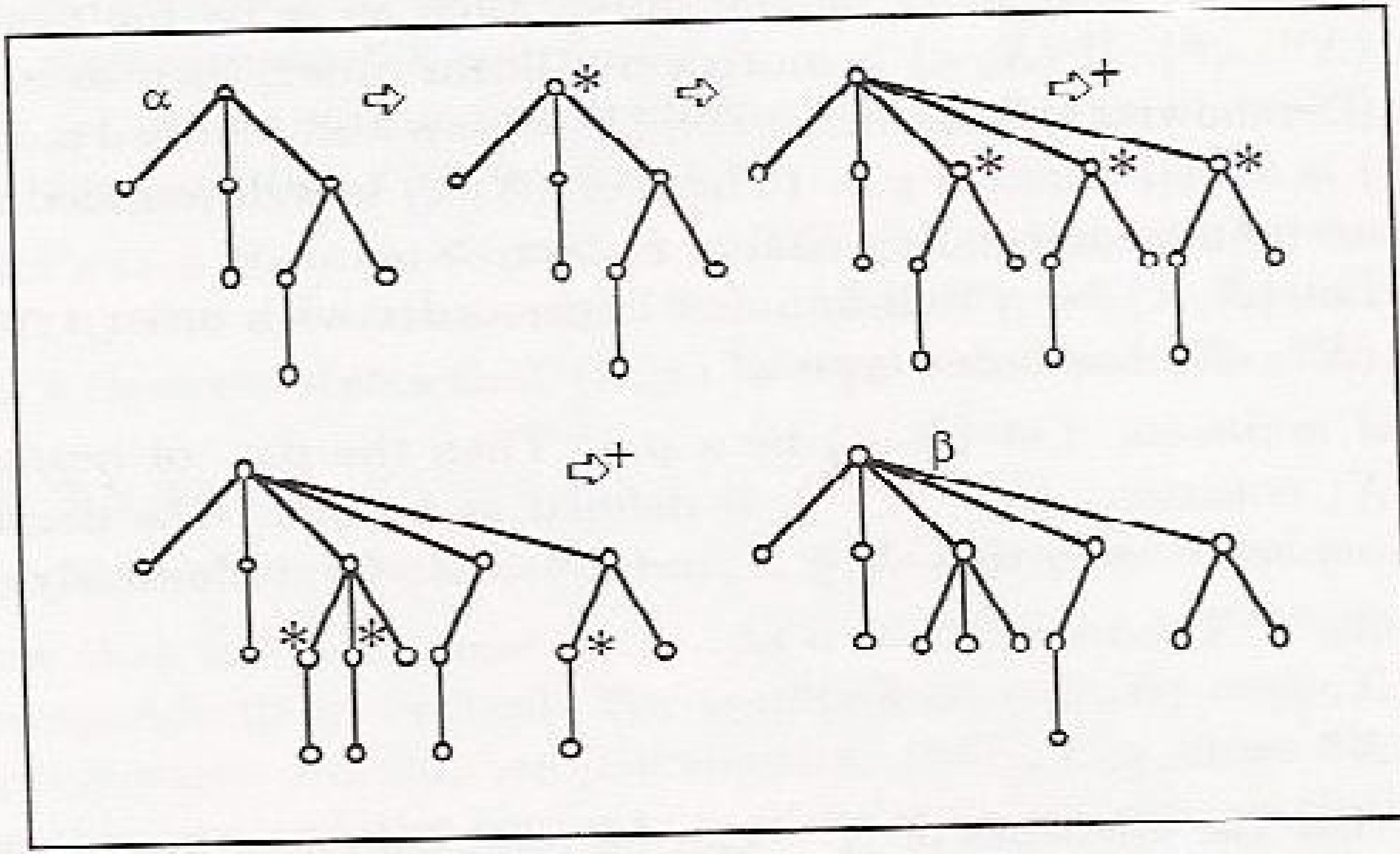


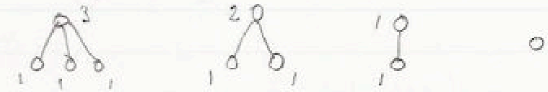
Figure 2.11



α
β

$\omega > 4 > 3 > 2 > 1 > 0$

$\{1\} \to \{0,0,0,0\} \to \{0,0,0\} \to \{0,0\} \to \{0\} \to \{\}(=\phi)$

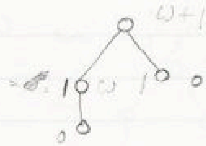


$\uparrow$

$\{1,0\}$ $\omega+1$

$\uparrow$

$\{1,0,0\}$ $\omega+2$



$\uparrow$

$\vdots$

$\{1,1\}$ $\omega+\omega = \omega\cdot 2$

$\uparrow$

$\{1,1,1\} = \omega\cdot 3$

$\uparrow$

$\{1,1,1,1\} = \omega\cdot 4$

$\uparrow$

$\vdots$

$\{2\} = \omega\cdot\omega = \omega^2$

$\{2,0\} = \omega^2+1$

$\{2,1\} = \omega^2+\omega$

$\{2,2\} = \omega^2+\omega^2$

$\{3\} = \omega^2\cdot\omega = \omega^3$

$\{3,3\} = \omega^3\cdot 2$

$\{3,3,3,3,\ 2,2,\ 5,5,5,\ 7,\ 1,1,\ 0,0\}$

$\omega^3\cdot 4 + \omega^2\cdot 2 + \omega^5\cdot 3 + \omega^7 + \omega\cdot 2 + 2$

(=φ) **Multisets van level 2 (gewone multisets)**

$\{ \{1,1\}, \{2,2,2\} \}$

$\{1\} = \omega$

$\{\{1\}\} \rightarrow \{ \{0,0,0\}^3, \{0,0,0\}^3, \{0,0,0\}^3 \}$

$\{2,2,2,2,2, 3,3\}$

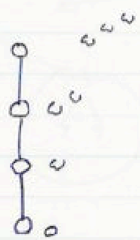
$\omega^2 \cdot 5 + \omega^3 \cdot 2$

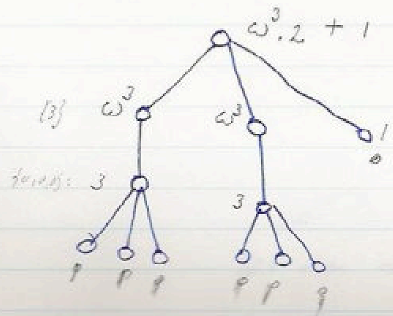
ω^ω

$\{ \{1\}, \{1\}\ 0, 0 \} = \omega^\omega \cdot 2 + 2$

$\{\{\{\{1\}\}\}\} = \omega^{\omega^\omega}$

$1 = \{0\} =$ 





put	$F(\mathbf{x}) \to F^*(\mathbf{x})$	
copy	$F^*(\mathbf{x}) \to G(F^*(\mathbf{x}),\ldots,F^*(\mathbf{x}))$	$(F > G)$
select	$F^*(x_1,\ldots,x_n) \to x_i$	$(1 \leq i \leq n)$
down	$F^*(\mathbf{x}, G(\mathbf{y}), \mathbf{z}) \to F(\mathbf{x}, G^*(\mathbf{y}), \mathbf{z})$	

Table 1.1. Transformation rules for IPO with stars.

1.2. EXAMPLE. Let Σ contain binary symbols F, H, a unary G and constants A, B with the ordering F > H and B > A. Then we have in T^* the reduction:

$$
\begin{array}{ll}
F(A, G(B)) & \to_{put} \\
F^*(A, G(B)) & \to_{copy} \\
H(F^*(A, G(B)), F^*(A, G(B))) & \to_{select} \\
H(G(B), F^*(A, G(B))) & \to_{put} \\
H(G(B), F^*(A, G^*(B))) & \to_{select} \\
H(G(B), F^*(A, B)) & \to_{copy} \\
H(G(B), H(F^*(A, B), F^*(A, B))) & \to_{select} \\
H(G(B), H(A, F^*(A, B))) & \to_{select} \\
H(G(B), H(A, B)) &
\end{array}
$$

put	$F(\mathbf{x}) \to F^k(\mathbf{x})$	$(k \in \mathbb{N})$
copy	$F^{k+1}(\mathbf{x}) \to G(F^k(\mathbf{x}),\ldots,F^k(\mathbf{x}))$	$(F > G, k \in \mathbb{N})$
select	$F^k(x_1,\ldots,x_n) \to x_i$	$(1 \le i \le n, k \in \mathbb{N})$
down	$F^k(\mathbf{x}, G(\mathbf{y}), \mathbf{z}) \to F(\mathbf{x}, G^{k'}(\mathbf{y}), \mathbf{z})$	$(k, k' \in \mathbb{N})$

Table 3.1. Transformation rules for IPO with labels.

The Matrix

Part I

The Growth

$0, 1, 2, 3, \dots, \omega, \omega+1, \omega+2, \omega+3, \dots, \omega+\omega = \omega 2, \omega 2+1, \omega 2+2, \omega 2+3, \dots, \omega 3, \omega 3+1, \omega 3+2, \omega 3+3,$

$\omega\omega = \omega^2, \omega^2+1, \omega^2+2, \dots, \omega^2+\omega, \dots, \omega^2+\omega 2, \dots, \omega^2+\omega 3, \dots, \omega^2 2, \dots, \omega^2 3, \dots, \omega^3, \dots, \dots,$

$\dots, \omega^{\omega^2}, \dots, \omega^{\omega^3}, \dots, \omega^{\omega^\omega} = \omega^{\omega 2}, \dots, \omega^{\omega 3}, \dots, \dots, \omega^{\omega^2}, \dots, \omega^{\omega^\omega}, \dots, \omega^{\omega^{\omega^{\cdots}}} = \varepsilon_0, \varepsilon_0+1, \dots,$

$\varepsilon_0+\omega 2, \dots, \varepsilon_0+\omega^2, \dots, \varepsilon_0+\omega^\omega, \dots, \varepsilon_0+\varepsilon_0 = \varepsilon_0 2, \dots, \varepsilon_0 3, \dots, \varepsilon_0\omega, \dots, \varepsilon_0\omega 2, \dots, \varepsilon_0\omega^2, \dots,$

$\varepsilon_0\varepsilon_0 = \varepsilon_0^2, \dots, \varepsilon_0^3, \dots, \varepsilon_0^\omega, \dots, \varepsilon_0^{\omega^\omega}, \dots, \varepsilon_0^{\varepsilon_0}, \dots, \varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\cdots}}} = \varepsilon_1, \dots, \varepsilon_2, \dots, \varepsilon_3, \dots, \varepsilon_\omega$

$\varepsilon_{\omega^\omega}, \dots, \varepsilon_{\varepsilon_0}, \dots, \varepsilon_{\varepsilon_1}, \dots, \varepsilon_{\varepsilon_\omega}, \dots, \varepsilon_{\varepsilon_{\omega^\omega}}, \dots, \varepsilon_{\varepsilon_{\omega^\omega}}, \dots, \varepsilon_{\varepsilon_{\varepsilon_0}}, \dots, \varepsilon_{\varepsilon_{\varepsilon_{\cdots}}} = \eta_0, \dots, \eta_1, \dots$

$\eta_{\varepsilon_0}, \dots, \eta_{\varepsilon_\omega}, \dots, \eta_{\varepsilon_{\varepsilon_0}}, \dots, \eta_{\varepsilon_{\varepsilon_{\cdots}}} = \eta_{\eta_0}, \dots, \eta_{\eta_{\eta_{\cdots}}} = \zeta_0, \dots, \dots\dots\dots$

$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

Although ε_0 may seem like a "large" ordinal, it is in fact a very small countable ordinal. Γ_0 is a much larger countable ordinal.

Γ_0 is obtained by creating a large matrix whose rows and columns are labeled by the countable ordinals. The i-th entry in row 0 contains ω^i (where i ranges over the countable ordinals). The j-th row (for $j > 0$) consists of all ordinals that are fixed points in every previous row, in order, e.g., the first entry of row 1 is ε_0. Γ_0 is the least ordinal a such that the first entry of row a is a. Another way of thinking about this is that column 0 is a fast growing function, going from 0 to ε_0 in one step; Γ_0 is the first fixed point of this function.

classes of ordinals and enumerating functions

(Topic)

A class of ordinals is just a subclass of the class $\mathbf{On}$ of all ordinals. For every class of ordinals M there is an *enumerating function* f_M defined by transfinite recursion:

$$f_M(\alpha) = \min\{x \in M \mid f(\beta) < x \text{ for all } \beta < \alpha\},$$

and we define the *order type* of M by $\operatorname{otype}(M) = \operatorname{dom}(f)$. The possible values for this value are either $\mathbf{On}$ or some ordinal α. The above function simply lists the elements of M in order. Note that it is not necessarily defined for all ordinals, although it is defined for a segment of the ordinals. If $\alpha < \beta$ then $f_M(\alpha) < f_M(\beta)$, so f_M is an order isomorphism between $\operatorname{otype}(M)$ and M.

For an ordinal κ, we say M is *κ-closed* if for any $N \subseteq M$ such that $|N| < \kappa$, also $\sup N \in M$.

We say M is *κ-unbounded* if for any $\alpha < \kappa$ there is some $\beta \in M$ such that $\alpha < \beta$.

We say a function $f : M \to \mathbf{On}$ is *κ-continuous* if M is κ-closed and

$$f(\sup N) = \sup\{f(\alpha) \mid \alpha \in N\}$$

A function is *κ-normal* if it is order preserving ($\alpha < \beta$ implies $f(\alpha) < f(\beta)$) and continuous. In particular, the enumerating function of a κ-closed class is always κ-normal.

Veblen function

(Definition)

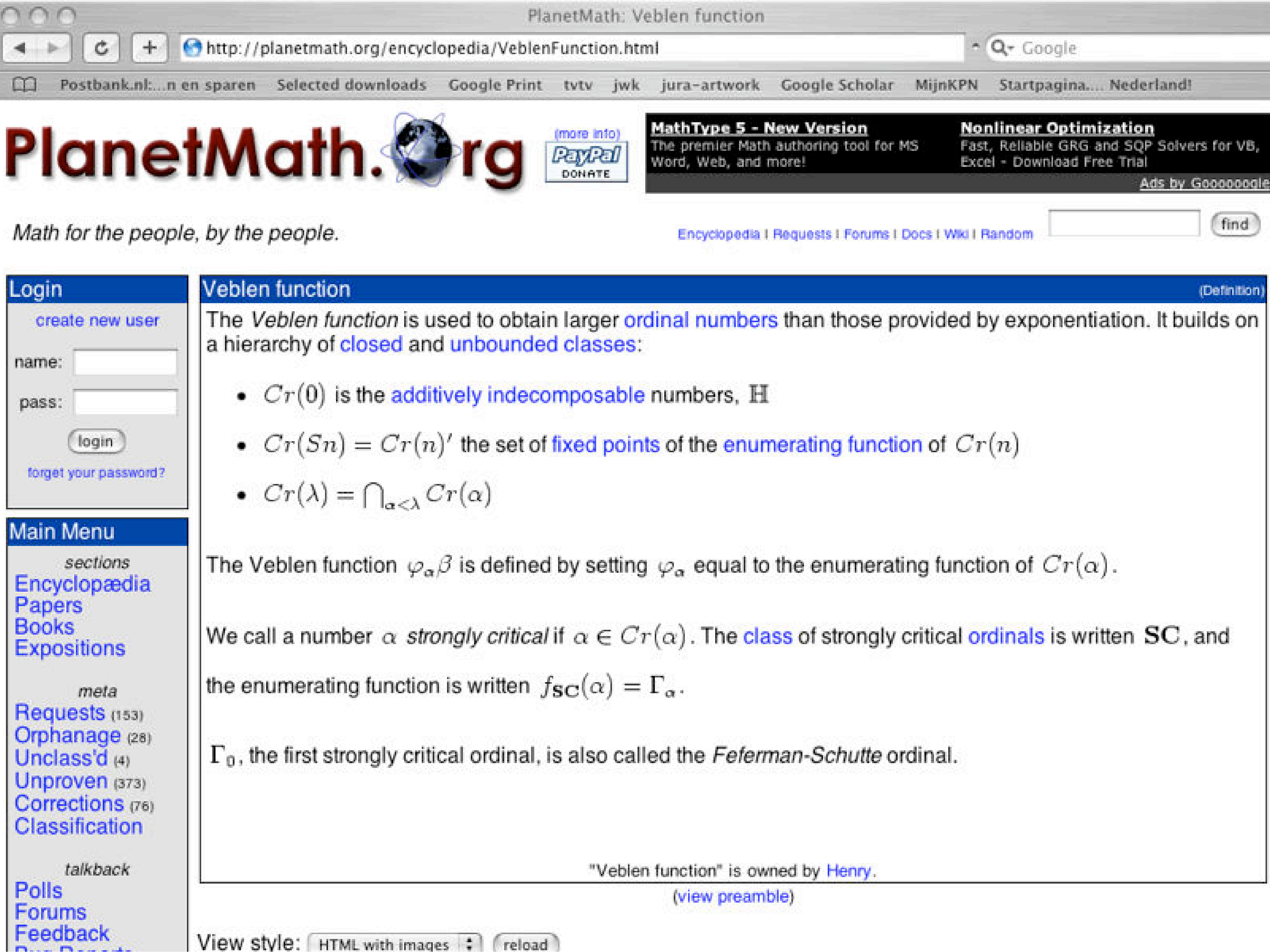
The *Veblen function* is used to obtain larger ordinal numbers than those provided by exponentiation. It builds on a hierarchy of closed and unbounded classes:

- $Cr(0)$ is the additively indecomposable numbers, $\mathbb{H}$
- $Cr(Sn) = Cr(n)'$ the set of fixed points of the enumerating function of $Cr(n)$
- $Cr(\lambda) = \bigcap_{\alpha<\lambda} Cr(\alpha)$

The Veblen function $\varphi_\alpha \beta$ is defined by setting φ_α equal to the enumerating function of $Cr(\alpha)$.

We call a number α *strongly critical* if $\alpha \in Cr(\alpha)$. The class of strongly critical ordinals is written $\mathbf{SC}$, and the enumerating function is written $f_{\mathbf{SC}}(\alpha) = \Gamma_\alpha$.

Γ_0, the first strongly critical ordinal, is also called the *Feferman-Schutte* ordinal.



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"Veblen function" is owned by Henry.

(view preamble)

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$$f(\sup N) = \sup\{f(\alpha) \mid \alpha \in N\}$$

A function is *κ-normal* if it is order preserving ($\alpha < \beta$ implies $f(\alpha) < f(\beta)$) and continuous. In particular,

the enumerating function of a κ-closed class is always κ-normal.

All these definitions can be easily extended to all ordinals: a class is *closed* (resp. *unbounded*) if it is κ-closed (unbounded) for all κ. A function is *continuous* (resp. *normal*) if it is κ-continuous (normal) for all κ.

10. Prove ($F \in (On)^{On}$ & F is a continuous, strictly monotonically increasing function) $\rightarrow (\exists\alpha)(F(\alpha) = \alpha)$.

9.5. EPSILON NUMBERS

J W KLOP

dr. D. van Dalen
drs. H.C. Doets
drs. H.C.M. de Swart

Verzamelingen
naïef, axiomatisch en toegepast

Oosthoek, Scheltema & Holkema

Definitie 6.10: Enumeraties van gesloten collecties heten *normaal.*

Een normale operatie is dus een ordinaal-waardige strikt monotone operatie gedefinieerd op een ordinaalgetal of op OR die continu is in de zin van de interval-topologie (zie 6.6).

Als σ een enumeratie is en $\alpha \in \text{Dom}\,\sigma$ dan is $\alpha \leqslant \sigma(\alpha)$: anders bestond een *kleinste* $\alpha \in \text{Dom}\,\sigma$ met $\sigma(\alpha) < \alpha$, maar dan was $\sigma(\sigma(\alpha)) < \sigma(\alpha)$, in strijd met de minimaliteit van α. Een $\alpha \in \text{Dom}\,\sigma$ met $\sigma(\alpha) = \alpha$ heet *dekpunt* van σ. Enumeraties hoeven geen dekpunten te hebben: de enumeratie σ van alle opvolgers gedefinieerd door $\sigma(\alpha) := \alpha + 1$ heeft bij voorbeeld geen dekpunten. Deze enumeratie is ook niet normaal: $\sigma(\omega) = \omega + 1$ terwijl $\bigcup_{n<\omega} \sigma(n) = \bigcup_{n<\omega} (n + 1) = \omega$.

Voor het gemak houden we ons in het volgende uitsluitend bezig met normale operaties gedefinieerd op alle ordinaalgetallen; dit zijn dus de enumeraties van gesloten collecties die cofinaal zijn in OR. Hiervoor hebben we de volgende *dekpuntstelling*:

Stelling 6.11: De dekpunten van een normale operatie vormen een gesloten collectie cofinaal in OR.

Laat σ een normale operatie zijn; dan is het mogelijk het bestaan van een binaire operatie τ te bewijzen zó dat voor $\sigma_\alpha(\beta) := \tau(\alpha, \beta)$ aan de volgende voorwaarden is voldaan: (*i*) $\sigma_0 = \sigma$; (*ii*) $\sigma_{\alpha+1}$ is de enumeratie-operatie van de dekpunten van σ_α; (*iii*) als Lim α, dan enumereert σ_α de collectie $\{\delta : \forall \xi < \alpha \;\; \sigma_\xi(\delta) = \delta\}$. Merk op, dat het recursielemma *niet* van toepassing is op (*i*)–(*iii*) omdat de σ_α geen verzamelingen zijn. Uit het laatste lemma volgt dat alle σ_α normaal zijn en dat reguliere β die dekpunt zijn van σ ook dekpunt zijn van σ_α als $\alpha < \beta$. σ_α heet de *α-de afgeleide* van σ.

Lemma 6.14:

(*i*) Als $A_\alpha \subseteq$ OR gesloten is en cofinaal in OR voor $\alpha \in$ OR dan is $\{\beta : \beta \in \bigcap_{\alpha < \beta} A_\alpha\}$ gesloten en cofinaal in OR.

Met behulp van de voorgaande resultaten kunnen we sterker en sterker stijgende normale operaties construeren uitgaande van een normale operatie σ die van de identiteit verschilt: als σ_α de α-de afgeleide van σ is, dan stijgt σ_α sneller dan σ_β als $\beta < \alpha$. Als σ_α A_α enumereert, dan stijgt de enumeratie τ van $A := \{\beta : \beta \in \bigcap_{\alpha<\beta} A_\alpha\}$ uiteindelijk sneller dan iedere A_α omdat $A - \alpha \subseteq A_\alpha$.

We kunnen vervolgens het proces dat van σ naar τ voert itereren over OR en weer 'diagonalizeren' etc. etc.; een regulier dekpunt α van σ blijft echter dekpunt van iedere zo verkregen normale operatie als we alleen indices $< \alpha$ toelaten.

The Matrix

Part II

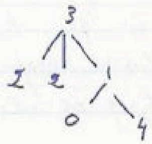
Zaterdag 26 nov. 2005

Idee I.

In de copy rule wordt het label kleiner. Dit kan zelf weer door een * te plaatsen. Dan is het ook mogelijk te werken met algemene label sets; en te itereren door boompjes zelf als label te gebruiken.

Idee II.

Noem gelabelde boompjes: labeled trees multi-sets.

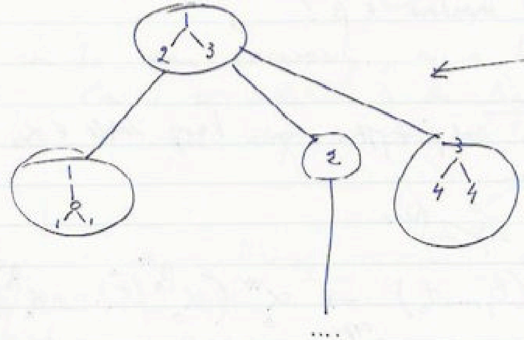
Bv.  $= \{ 2, 2, \{0, 4\}_1 \}_3$

eigenlijk $\{ \}_2$

Een kwestie van notatie: hoe teken je dit?

3 : labeled tree of depth 0

labeled tree of depth 1



labeled tree of depth 2.

in limiet: nested labeled trees

kan je weer als labels gebruiken, een soort meta-successor functie;

en ook meta-limiet;

en dit kan eindig of oneindig.

Dit doen van gewone nested multisets.

Gegeven een ordening α, dan kunnen we als nieuwe ordening maken:

I. α^*, dit is door labels $\in \alpha$ te gebruiken in de IPO regels op de bomen toe te passen.

Of $\alpha^\#$ of zo.

II. Maar je kunt ook Vincent's labelled IPO uitbreiding nemen. Dat is ipv $*$, labels $\in \beta$ nemen. (Verwarring is dat knoop-labels $\in \mathbb{N}$, en de $*$-vertaling van $\in \mathbb{N}$ is

(noem dat de command labels, of markers

Wat is de trade-off tussen knoop labels $\in \alpha$ en markers $\in \beta$?

Notatie: α^β is wat je krijgt door knoop labels $\in \alpha$ en markers $\in \beta$ te nemen.

trees

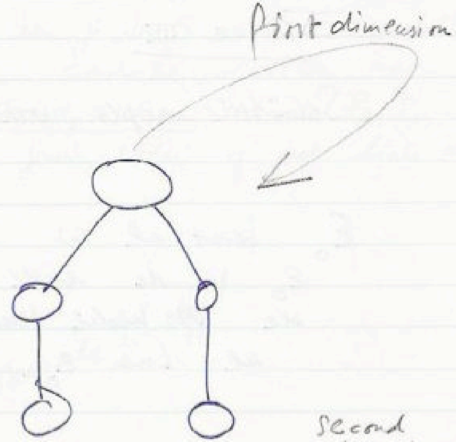
Dus $\alpha_0^{\beta_0}(t_1, \ldots, t_n) \underset{copy}{\rightarrow} \alpha_0^*(\alpha_0^{\beta_0^*}(\vec{t}) \ldots, \alpha_0^{\beta_0^*}(\vec{t}))$

of:

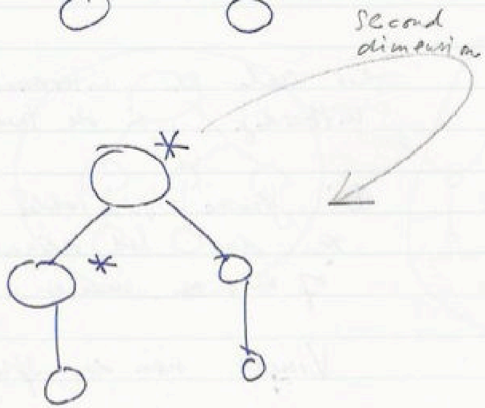
$f^{n+1}(t_1, \ldots, t_n) \rightarrow f_g^{n'}(f^{n*}(\vec{t}), \ldots, f^{n*}(\vec{t}))$ voor $g < f$?

Dimensions for refining / iteration:

(i) in the position NODES of the trees



(ii) in the control markers



Both can be done recursively, in a kind of successor operation. Can it be done in the limit? Then we can do it all along the ordinals!!

→ like the limit in nested multisets.

So in analogy with:

multisets → of deg. 1 → of deg. 2 → ... → of any finite degree

$\mathbb{N} \rightarrow \mathbb{N}_{\mu}$ $\quad \mathbb{N}_{\mu\mu}$ $\quad \mathbb{N}_{\mu\mu\mu}$ $\quad \mathbb{N}_{\mu^*}$

ω $\quad \omega^{\omega}$ $\quad \omega^{\omega}$ $\quad \dots$ $\quad \varepsilon_0$

Hier kun je die objecten in de limiet nog mooi tekenen, noteren.

Hoe te noteren in het algemene geval?

Problem of Skolem.

Nothing to do with logic really, but just about every one who's made a contribution works in logic..

Consider the set of formal terms S defined inductively below:

i) The symbols 1 and X are in S.

ii) If u and v are in S, then so are $(u+v)$, $(u \times v)$, and u^v .

Each term determines in a natural way a function of one variable on positive natural numbers.

Consider the ordering of the functions.

$f < g$ iff $f(x) < g(x)$ for sufficiently large x.

Questions:

1) Is it a linear ordering? (Yes, Richardson)

2) Is it a well ordering? (Yes, Ehrenfeucht using Kruskal's Theorem)

3) If yes to the above, what's the ordinal? (Unknown)

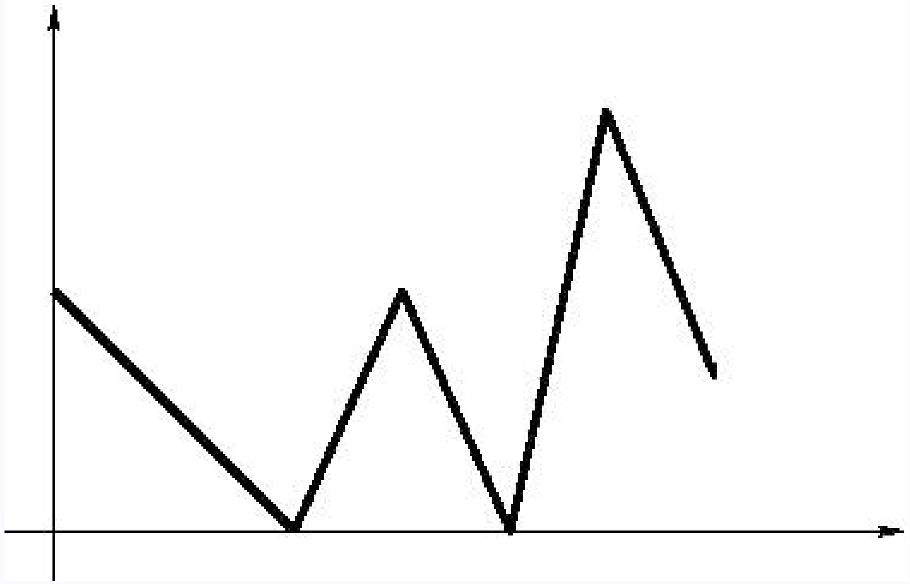
Skolem showed ϵ_0 is a lower bound. Levitz showed that the first critical epsilon number is an upper bound. The first critical epsilon number is defined as follows. Arrange the solutions of $\omega^x = x$ in order and call them ϵ_0, ϵ_1, ϵ_2, ... etc. Then the first critical epsilon number is the smallest member of the sequence equal to own subscript.

Levitz, Van den Dries, and Dahn have partial results supporting the conjecture that the actual ordinal is ϵ_0

note for aart:
the next 4 slides about the worm principle
are from a lecture of lev beklemishev,
utrecht university

4. The Worm Principle

Worm is a function $f : [0, n] \to \mathbb{N}$.
List: $w = (f(0), f(1), \ldots, f(n))$
Word: $w = 2102031$



4.1. Rules of the game

First define a function $\mathrm{next}(w, m)$:

1. If $f(n) = 0$ then
   $\mathrm{next}(w, m) := (f(0), \ldots, f(n-1))$.
2. If $f(n) > 0$ let $k := \max_{i<n} f(i) < f(n)$;

   $r := (f(0), \ldots, f(k))$;
   $s := (f(k+1), \ldots, f(n-1), f(n) - 1)$;
   $\mathrm{next}(w, m) := r * \underbrace{s * s * \cdots * s}_{m+1 \text{ times}}$.

$\mathrm{next}(2102031, 1) = 210203030$
$k = 4$; $r = 21020$; $s = 30$.

Now let $w_0 := w$ and $w_{n+1} := \mathrm{next}(w_n, n+1)$.

$$w_0 = 2102031$$
$$w_1 = 210203030$$
$$w_2 = 21020303$$
$$w_3 = 21020302222$$
$$w_4 = 210203022212221222122212221$$
$$w_5 = 2102030(2221222122212221222 0)^6$$
$$\ldots$$

Notice that w_n is an elementary function.

$$|w_n| \leq (n+2)! \cdot |w_0|$$

Every Worm Dies $\Leftrightarrow \forall w \exists n \, w_n = \varnothing$

Th. 5. *EWD is true but unprovable in* PA.

Th. 6. *EWD is* EA*-equivalent to* $1\text{-Con}(\text{PA})$.

$1\text{-Con}(T)$ means "$(T +$ all true $\Pi_1^0)$ *is consistent*"