On Defining The Undefined

Concepts of undefinedness in orthogonal rewriting

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BRA SEMAGRAPH.

- semantics of orthogonal rewriting
- attempt to generalize concepts of undefinedness from λ-calculus to term rewriting.

Ω-values	Ω-defined	Ω-undefined
$\cap \#$	$\cap \#$	$\cup \#$
hnf's	having a hnf	having no hnf

Intuition:

$$\begin{cases} F(A) \to A \\ B \to B \end{cases}$$

$F(B)$ hnf, not an Ω-value

Ω-values are very much in hnf, already on the basis of the LHSs of the rules (without knowing the RHSs).

Graph reduction and circular redexes

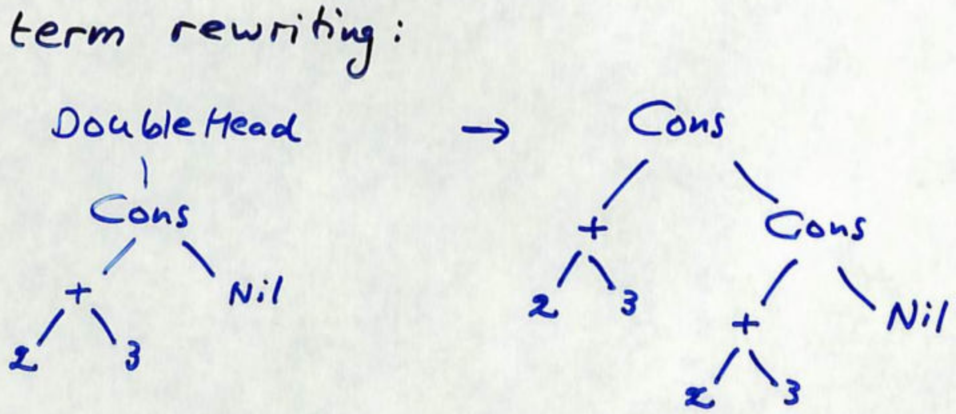
Graph reduction:

when the RHS of a rule contains a repeated variable, we don't make multiple copies of the corresponding subterm, but instead make multiple pointers to that subterm.

<u>Example</u>.

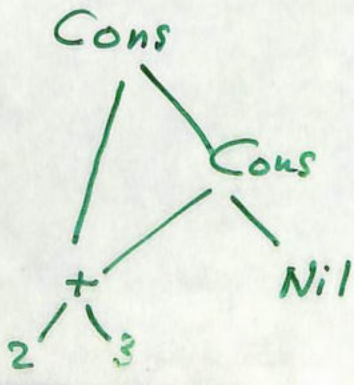
$$\text{DoubleHead}(\text{Cons}(x,y)) \to \text{Cons}(x, \text{Cons}(x,y))$$

by term rewriting:



by graph rewriting:

" →



Comparison with λ-calculus:

Define a closed term to be a transparent value if it is an abstraction, and an open term to be a transparent value if some closed instance is.

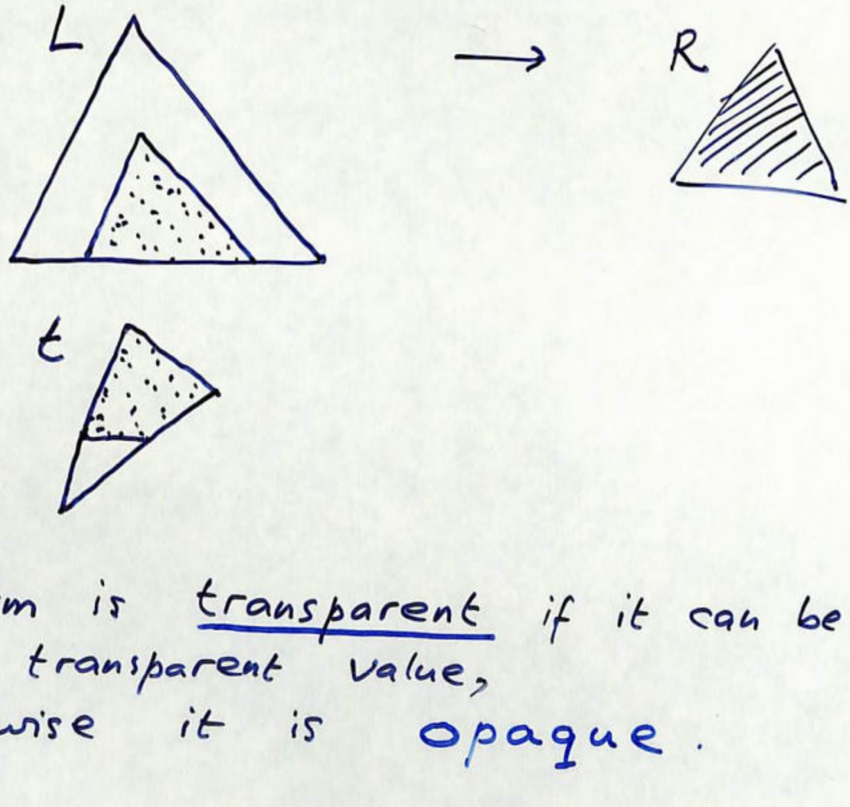
THEOREM. The transparent values are the whnf's.

The transparent terms are the terms having whnf.

transparent values	opaque
$\cap\#$	$\cup\#$
projective Ω-values	projectively Ω-undefined
$\cap\#$	$\cup\#$
Ω-values	Ω-undefined
$\cap\#$	$\cup\#$
hnf's	having no hnf.

Opaque and transparent terms

A term t is a transparent value if there is a rule $L \to R$, and a proper non-variable subterm L' of L s.t. t is unifiable with L'.

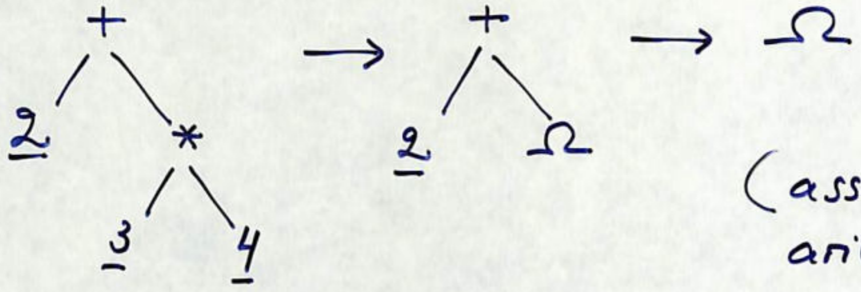


A term is <u>transparent</u> if it can be reduced to a transparent value, otherwise it is opaque.

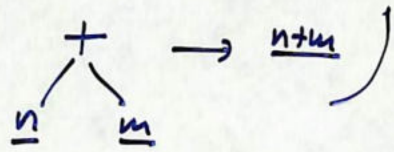
Intuition: opaque terms can be considered meaningless. No information can be extracted from them, by placing them in a context.

projective Ω-values	projectively Ω-undefined
$\cap$#	$\cup$#
Ω- values	Ω- undefined
$\cap$#	$\cup$#
hnf's	having no hnf

Examples.

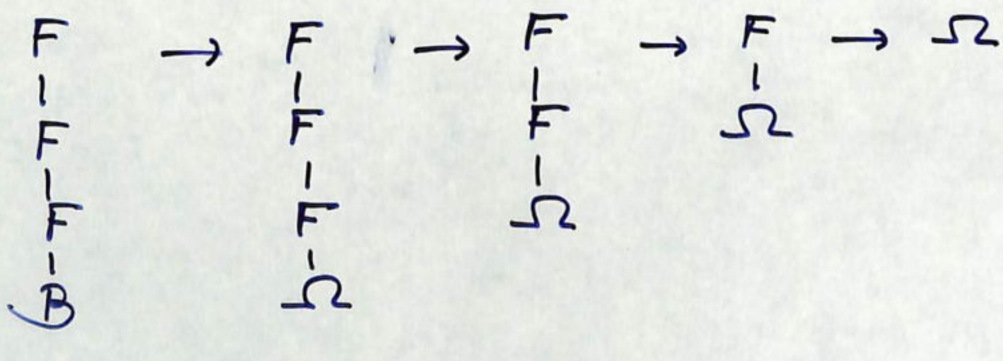


(assuming usual arithmetic rules



)

$$\begin{cases} F(A) \to A \\ B \to C \end{cases}$$



THEOREM. Every Ω-value is a hnf, but not conversely.

Every Ω-defined term has a hnf, but not conversely.

$F(A) \to A, \quad B \to B$

$F(B)$ is a hnf, but its Ω-n.f. is Ω.

THEOREM. Ω-reduction is confluent and strongly normalizing.

Define: for a term t,

$\Omega(t)$ is its normal form w.r.t. Ω-reduction.

t is an Ω-value, if $\Omega(t) \neq \Omega$.

t is Ω-defined if t is reducible (by ordinary reduction, not Ω-reduction) to an Ω-value, otherwise t is Ω-undefined.

Comparison with λ-calculus:

$(\beta:)$ $\quad Ap(\lambda x. Z_0(x), Z_1) \to Z_0(Z_1)$

Ω-reduction rules are:

$$Ap(\lambda x. Z_0(x), Z_1) \to \Omega$$

$$Ap(\Omega, Z_1) \to \Omega$$

THEOREM. The Ω-values in $\lambda\beta$-calculus are the whnf's.

Example:

$$\mathit{Append}(\mathit{Nil}, x) \to x$$

$$\mathit{Append}(\mathit{Cons}(x,y), z) \to \mathit{Cons}(x, \mathit{Append}(y,z))$$

$$\mathit{ListAnd}(\mathit{Cons}(\mathit{True}, x)) \to \mathit{ListAnd}(x)$$

$$\mathit{ListAnd}(\mathit{Cons}(\mathit{False}, x)) \to \mathit{False}$$

$$\mathit{ListAnd}(\mathit{Nil}) \to \mathit{True}$$

Pre-redexes:

all LHSs ;

$$\mathit{Append}(\Omega, x)$$

$$\mathit{ListAnd}(\mathit{Cons}(\Omega, x))$$

$$\mathit{ListAnd}(\Omega)$$

Ω-rules:

$$\text{pre-redex} \to \Omega$$

e.g. $\mathit{Append}(\Omega, x) \to \Omega$

Ω - values

(notion of Huet-Lévy)

Ω : a new constant symbol added to the TRS under consideration.

Ordering of terms :

$\Omega \leq t$

$s \leq t \Rightarrow$ for any $C[\ ]$, $C[s] \leq C[t]$.

A pre-redex is a term t $(\neq \Omega)$, obtained from some LHS s of a rule, by replacing zero or more non-variable subterms by Ω.

Example: $\quad Append(Nil, x) \to x$

$Append(Cons(x, y), z) \to Cons(x, Append(y, z))$

$ListAnd(Cons(True, x)) \to \ldots$

Head Normal Form

Define: A term is in hnf if it cannot be reduced to a redex.

A term has a hnf if it reduces to hnf.

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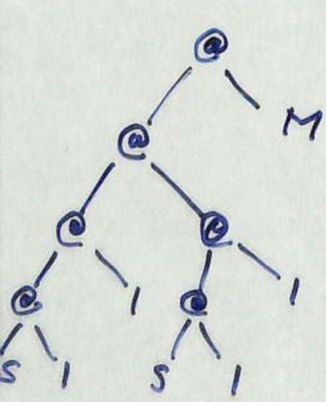
Define: In an applicative TRS, a term is in applicative hnf (ap-hnf) if it has the form

$$F T_1 T_2 \dots T_n \quad (n \geq 0) \quad \text{s.t.}$$

none of $F T_1 \dots T_i$ $(0 \leq i \leq n)$ is reducible to a redex.

Example: in CL:

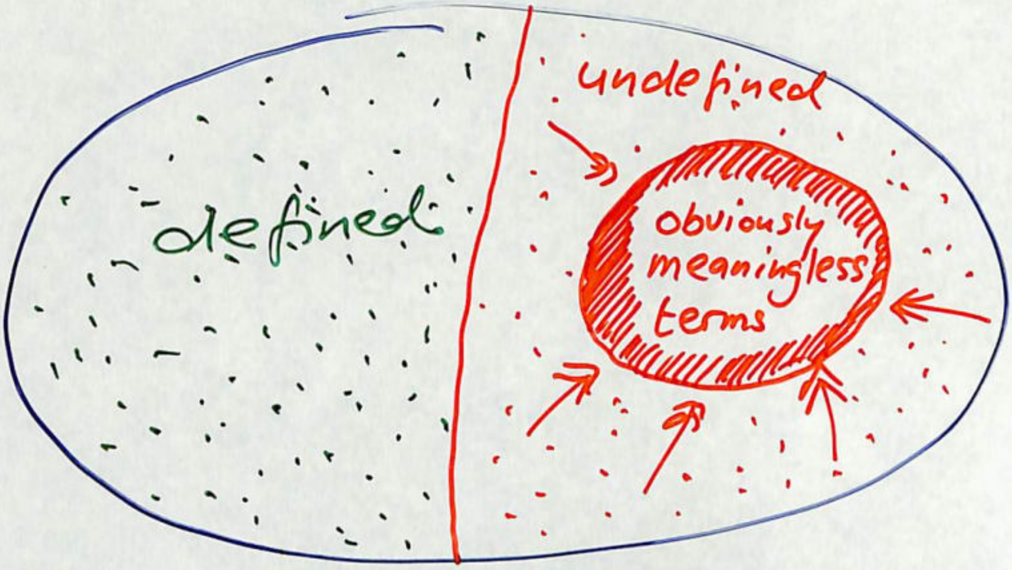
$$SII(SII) \twoheadrightarrow SII(SII)$$



$SII(SII)M$ is a hnf, but not an ap-hnf.

In λ-calculus: the ap-hnf's are precisely the whnf's.

Sometimes the dual approach:



(because  or  will always be closed under reduction $\twoheadrightarrow$, and because of confluence, we can replace 'reduces to' above by 'is convertible to' (=).)

M = N, P-value

$\twoheadrightarrow$ P-value

M = N, obv. undefined

$\twoheadrightarrow$ obv. und.

A scheme for defining meaningfulness.

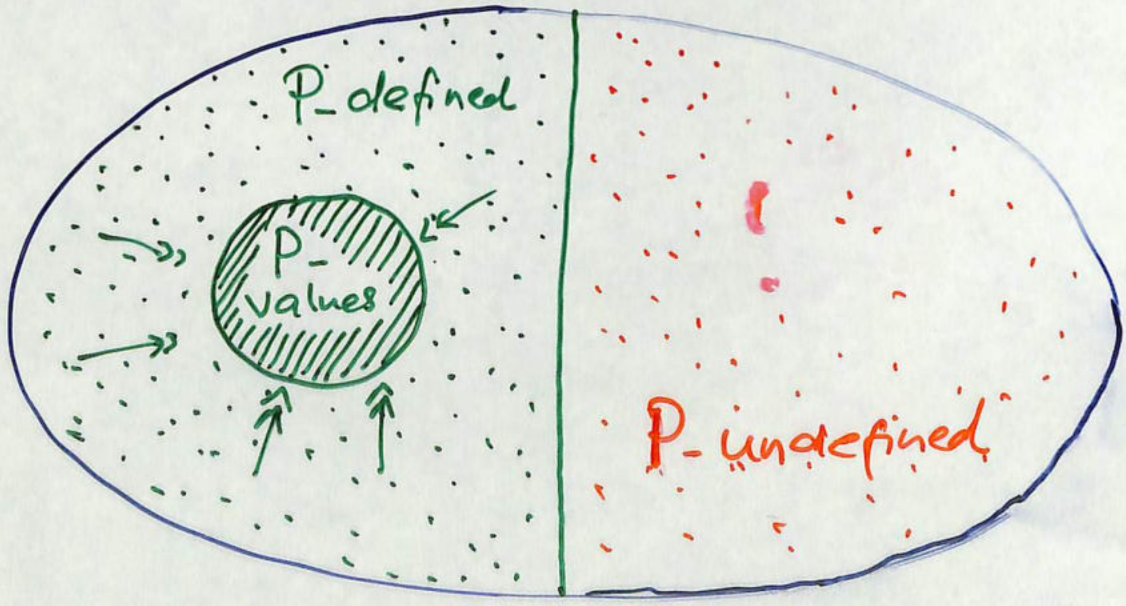
P is some set of "obviously meaningful" terms :

P-values

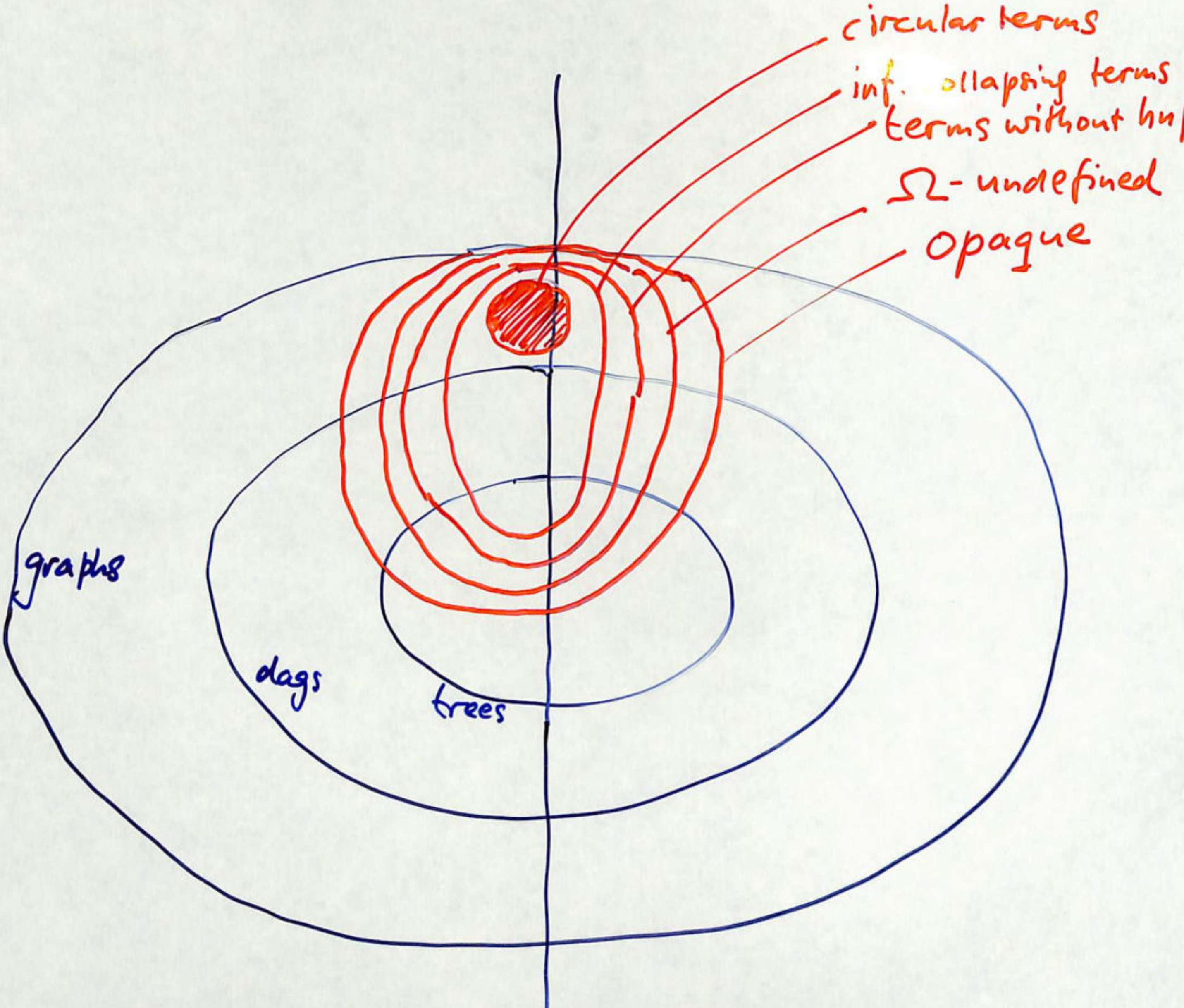
A term is P-defined if it reduces to a P-value.
A term is P-undefined if it is not P-defined.

$M \sim_P N$, M and N are P-equivalent,
if for any context $C[\ ]$,

$$C[M] \text{ is } P\text{-defined} \Leftrightarrow C[N] \text{ is } P\text{-defined}$$



e.g. in λ-calculus
P = HNF
P = WHNF


circular terms
inf. ollapsing terms
terms without hnf
Ω- undefined
opaque
graphs
dags
trees
finitary
infinitary

We define several notions of undefinedness.
They will turn out to be linearly ordered:

circular terms
(applies to cyclic graphs only)

infinitely collapsing terms

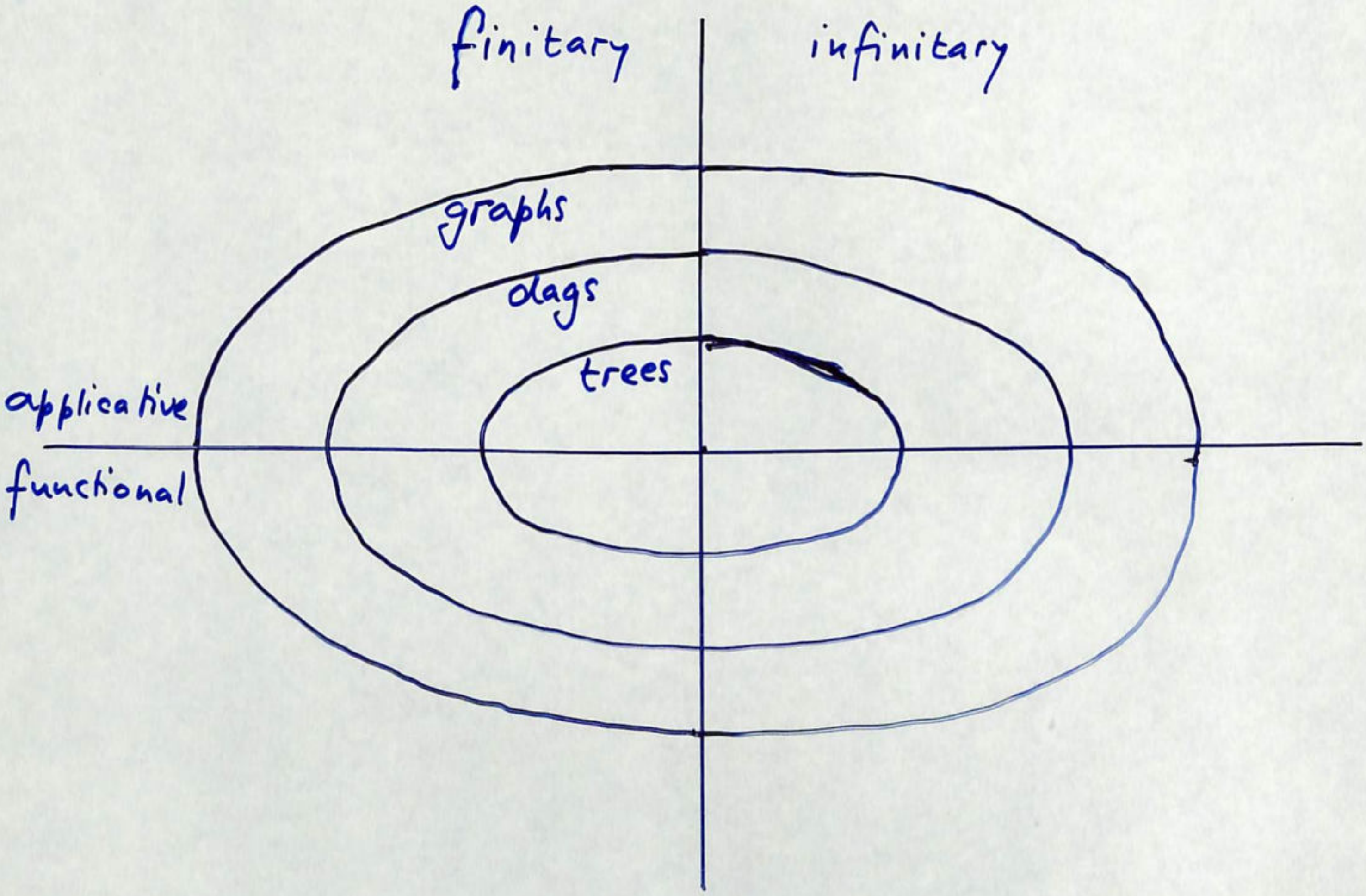
terms without hnf

Ω-undefined terms

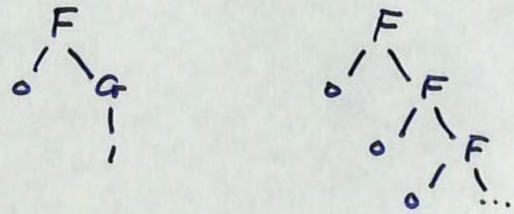
projectively Ω-undefined terms
(applies to infinitary reduction only)

opaque terms

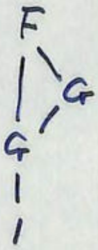
Orthogonal rewriting



trees :

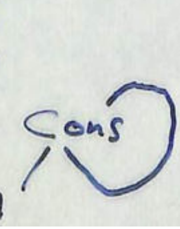


dags :



applicative:
$Sxyz$

graphs:



functional:
$F(x, y, z)$

every hnf is a whnf

$\lambda x. \Delta\Delta$ (where $\Delta = \lambda x. xx$)

has (is) a whnf, but has no hnf.

Plotkin (early)

Abramsky, Ong

Can we find analogues of the concepts of hnf and whnf for term rewriting ?

'Classical' lambda calculus:

beautiful interplay between syntax, semantics, proof theory;

based on notion of <u>HNF</u>.

'Modern' lambda calculus:

develops connections likewise,

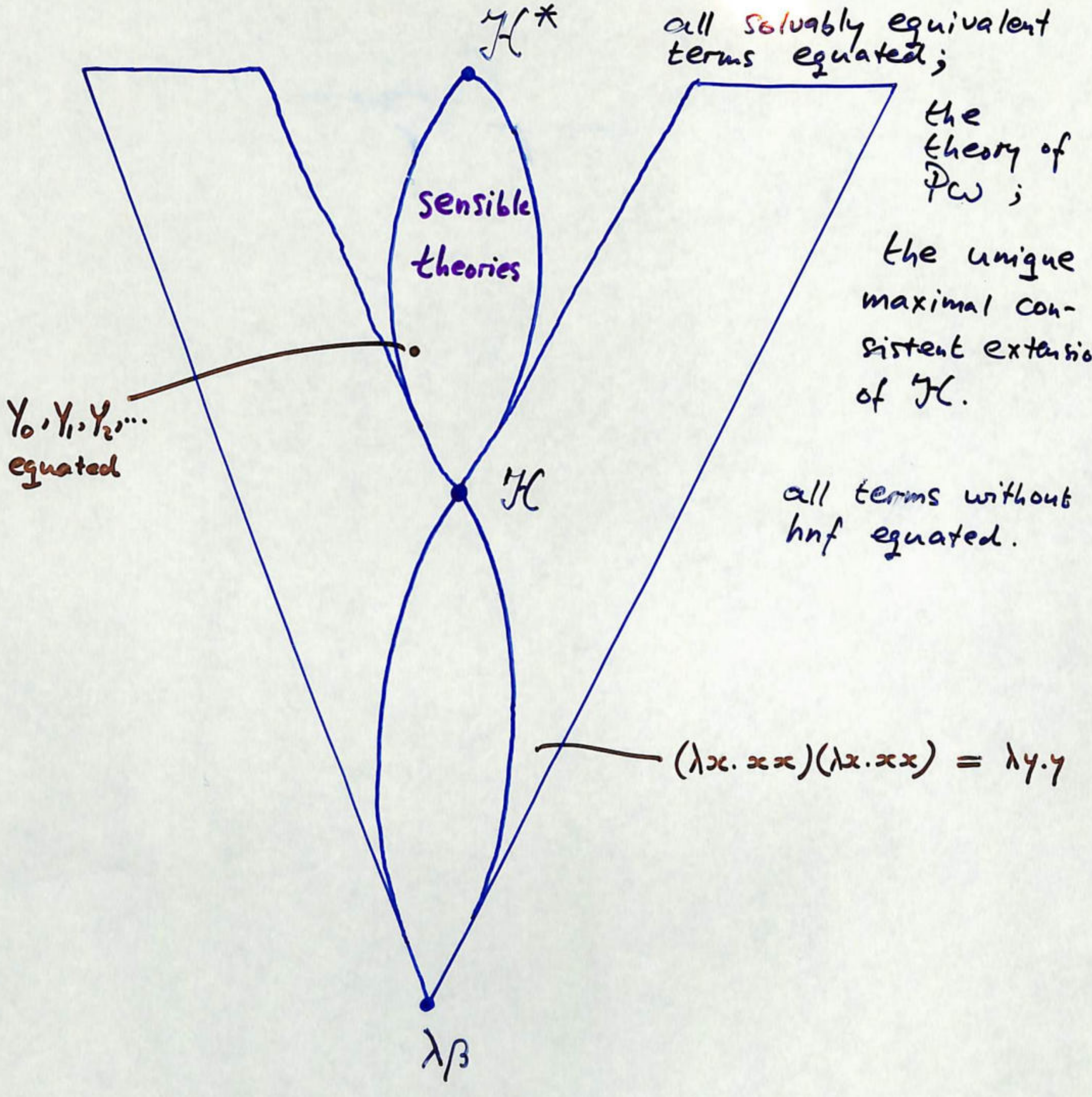
based on notion of <u>WHNF</u>

M is in <u>weak head normal form</u> (whnf) if M has one of the forms:

$$\lambda x . E$$

$$x M_1 \ldots M_n$$

M <u>has a whnf</u> if it can be reduced to one.



Partial order of λ-calculus theories

$M \sim N$, M and N are <u>solvably equivalent</u>

if for every context $C[\ ]$:

$$C[M] \text{ is solvable} \Leftrightarrow C[N] \text{ is solvable}.$$

THEOREM. M, N are solvably equivalent

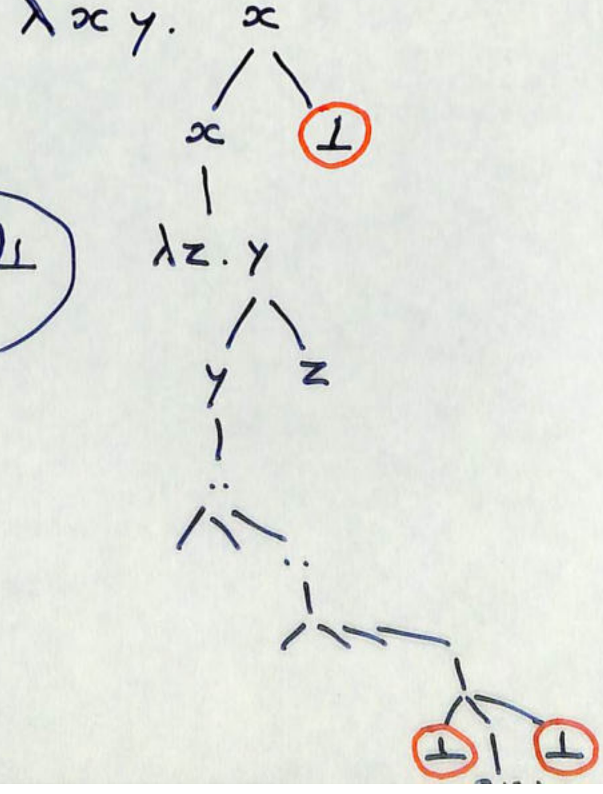
$$\Updownarrow$$

$$P\omega \models M = N$$

$$\Updownarrow$$

$$BT(M) = BT(N)$$

$BT(M)$, the <u>Böhm Tree</u> of M:



$\lambda xy.\, x(x(\lambda z.y \ldots))\perp$

possibly infinite λ-term, everywhere in hnf, possibly with ⊥'s

If M is not in hnf, its <u>head redex</u> is its leftmost - outermost redex :

$$M : \quad \lambda x_1 \ldots x_n . \; ((\underline{\lambda x . A) B}) M_1 \ldots M_k$$
$$(n \geq 0) \qquad\qquad (k \geq 0)$$

THEOREM. M has a hnf

$\Updownarrow$

$[\![M]\!]_{D^\infty} \neq \perp$

$\Updownarrow$

M is solvable

$\Updownarrow$

<u>Head reduction</u> of M terminates.

$\Updownarrow$

M can be reduced to a <u>stable</u> term

M is <u>stable</u> if it cannot be reduced to Ω by the rules

$$(\lambda x . A) B \to \Omega$$
$$\Omega A \to \Omega$$
$$\lambda x . \Omega \to \Omega$$

λ - calculus

Head normal form (hnf):

$$\lambda x_1 \dots x_n .\ x\ M_1 \dots M_k \qquad (n, k \geq 0)$$

A term has a hnf if it can be reduced to hnf.

THEOREM. M has a hnf $\Leftrightarrow$

$$[\![M]\!]_{D^\infty} \neq \perp$$

M is solvable if

there exist terms $N_1, \dots, N_k$ s.t. $M N_1 \dots N_k =_\beta I$

there exist terms $N_1 .. N_k$ s.t. $M N_1 \dots N_k$ has a normal form

$M \sim N$, M and N are solvably equivalent, if for every context $C[\ ]$:

$$C[M] \text{ is solvable} \Leftrightarrow C[N] \text{ is solvable}$$

$I(x) \to x$

Options for $I^{\circlearrowleft}$ to reduce to:

(all occurring in the literature):

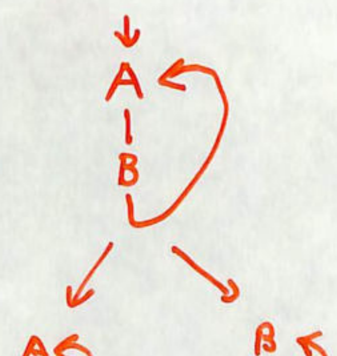
I. $I^{\circlearrowleft}$ agrees with behaviour of

$$I^\omega = \begin{array}{c} I \\ | \\ I \\ | \\ I \\ | \\ \vdots \end{array} \to I^\omega.$$

but confluence then breaks down:

$A(x) \to x$

$B(x) \to x$



II. nothing, i.e. $I^{\circlearrowleft}$ is a normal form.

III 

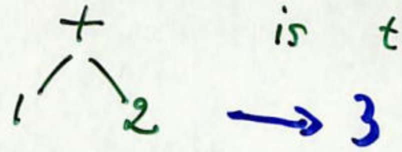
IV. O

A new object

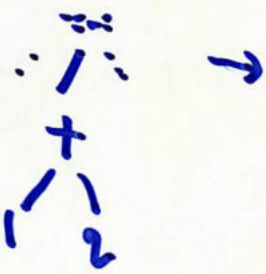
Example.

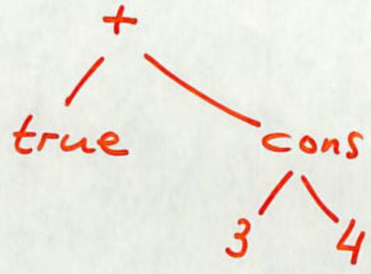
assume the usual rules for arithmetic, booleans, etc.:

Integers, Booleans, Nil, anything of the form $\mathrm{Cons}(t_1, t_2)$ are transparent values.



is transparent





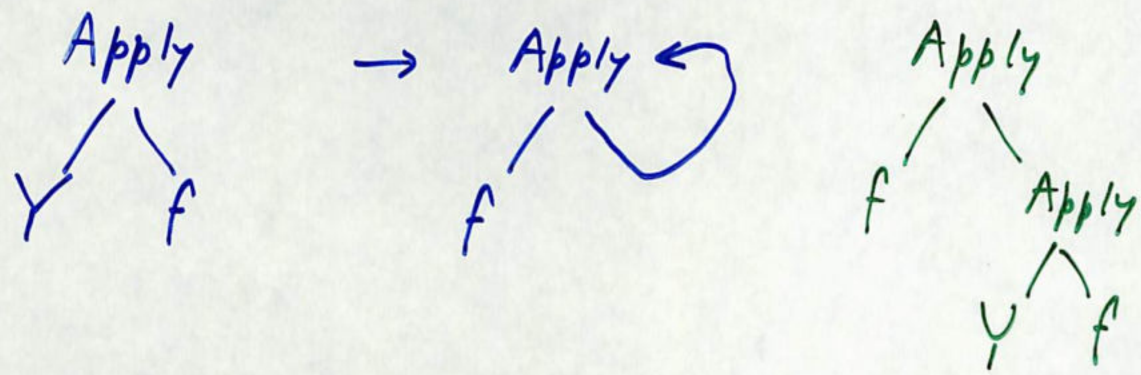
is opaque

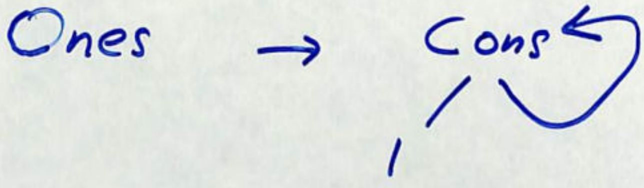
(type-checking without types?)

THEOREM.
Let M be opaque, and let $C[\ ]$ be a non-trivial context.
Then:
if $C[M]$ is transparent, then so is $C[N]$ for any N.

For acyclic graphs (dags), nothing new happens w.r.t. notions of undefinedness.

Cycles can be introduced by rules like





Circular redexes:

a persistent technical problem.

Given the rule $I(x) \to x$

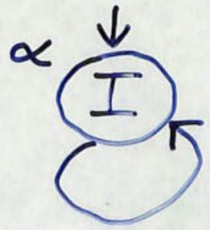
what does  reduce to?

The same question arises for all "circular redexes", e.g.

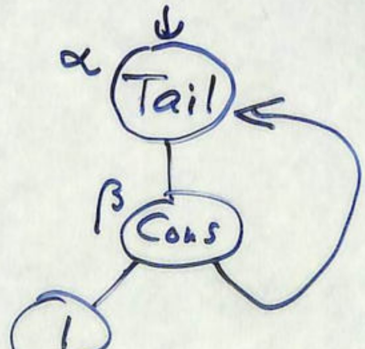
Solution: define all circular redexes to be meaningless. Equate them all to some special object

$$\bigcirc.$$

(This is consistent / suggested also by presenting graphs as recursion equations:)



$$\boxed{\alpha = I(\alpha)}$$
$$\downarrow$$
$$\boxed{\alpha = \alpha}$$

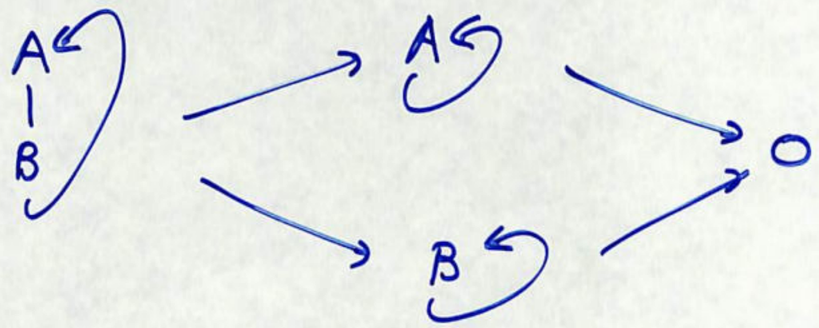


$$\boxed{\begin{cases} \alpha = \mathrm{Tail}(\beta) \\ \beta = \mathrm{Cons}(1, \alpha) \end{cases}}$$

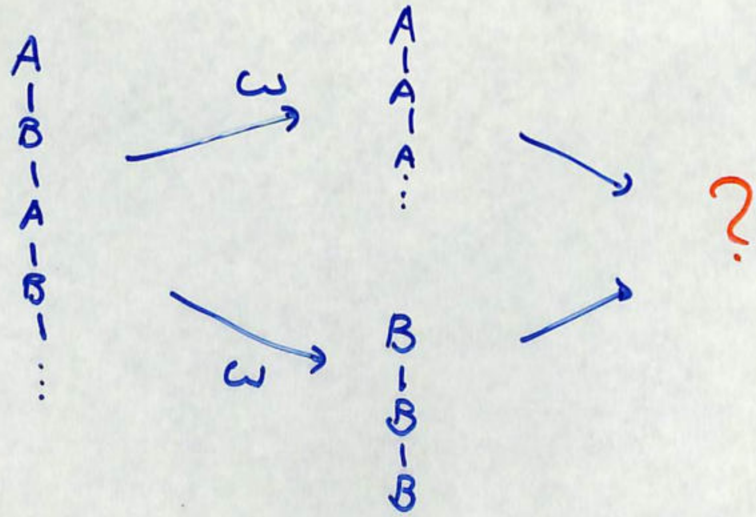
$$\boxed{\alpha = \mathrm{Tail}(\mathrm{Cons}(1, \alpha))}$$
$$\downarrow$$
$$\boxed{\alpha = \alpha}$$

Equating all circular redexes restores the Church-Rosser property for <u>finitary</u> graph rewriting.

Rules: $A(x) \to x$, $B(x) \to x$



But CR still fails for <u>infinitary</u> term rewriting:



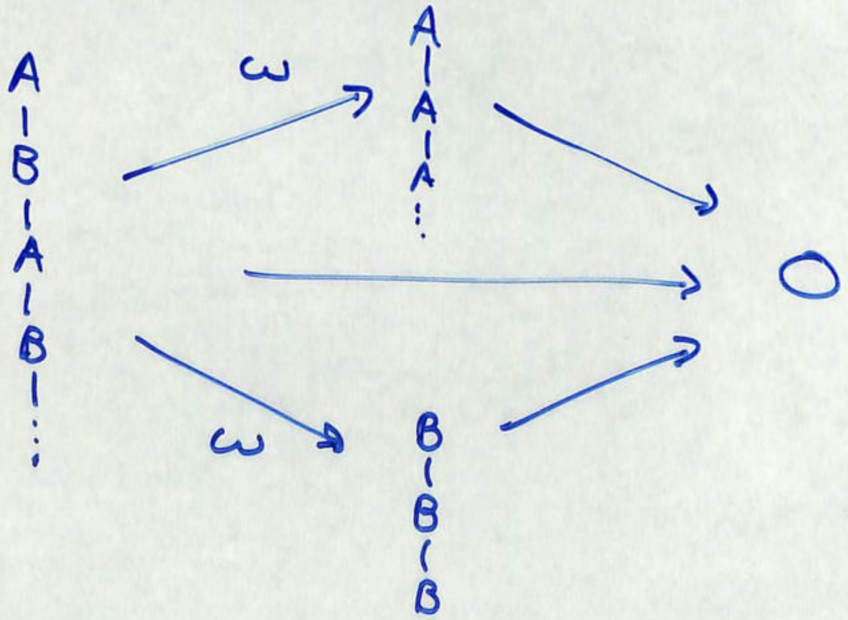
Call a term t *infinitely collapsing*

if there is a reduction sequence from t in which infinitely many steps are collapsing steps

(i.e. reduction at the root by a rule whose RHS is a variable)

If the infinitely collapsing terms are considered meaningless, and therefore are identified, then the CR property is restored for infinitary orthogonal term rewriting.

$$A(x) \to x \qquad B(x) \to x$$



circular redexes $\qquad\qquad$ I

$\cap\neq$ $\qquad\qquad$ I^{ω}

infinitely collapsing

$\cap\neq$ $\qquad\qquad$ $A \to A$

A

no hnf

$F(A) \to A \qquad B \to B$

$\cap\neq$ $\qquad\qquad$ $F(B)$

Ω-undefined

$F(A) \to B$

$\cap\neq$ $\qquad\qquad$ F^{ω}

projectively Ω-undefined

$+$

$\cap\neq$

opaque $\qquad\qquad$ true $\quad$ Nil

Projective Ω-values.

Transfinite reduction leads to an alternative version of Ω-value.

Given the rule $F(A) \to A$ and the infinite term

$$F^{\omega} \;(= \; F(F(F(F(\cdots)))) \;).$$

F^{ω} is a normal form – but every finite approximation to it "melts" to Ω.

Approximations:

define
$$t/0 = \Omega$$
$$x/n = x \quad (x \text{ a variable, } n>0)$$

$$F(t_1,\dots,t_k)/n = F(t_1/n{-}1, \dots, t_k/n{-}1) \quad (n>0)$$

A **projective Ω-value** is a term t s.t. for some n, t/n is an Ω-value.

F^{ω} is an Ω-value, but not a projective Ω-value.