LICS 2012 DUBROVNIK

Tutorial

Term Rewriting & Lambda Calculus

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VU University Amsterdam

0. A few words on history

1. rewriting dictionary

2. two theorems in abstract rewriting

3. word rewriting: monoids and braids

TEA, COFFEE

4. term rewriting: divide et impera; termination by stars

5. Lambda calculus and combinatory logic

6. Infinitary rewriting

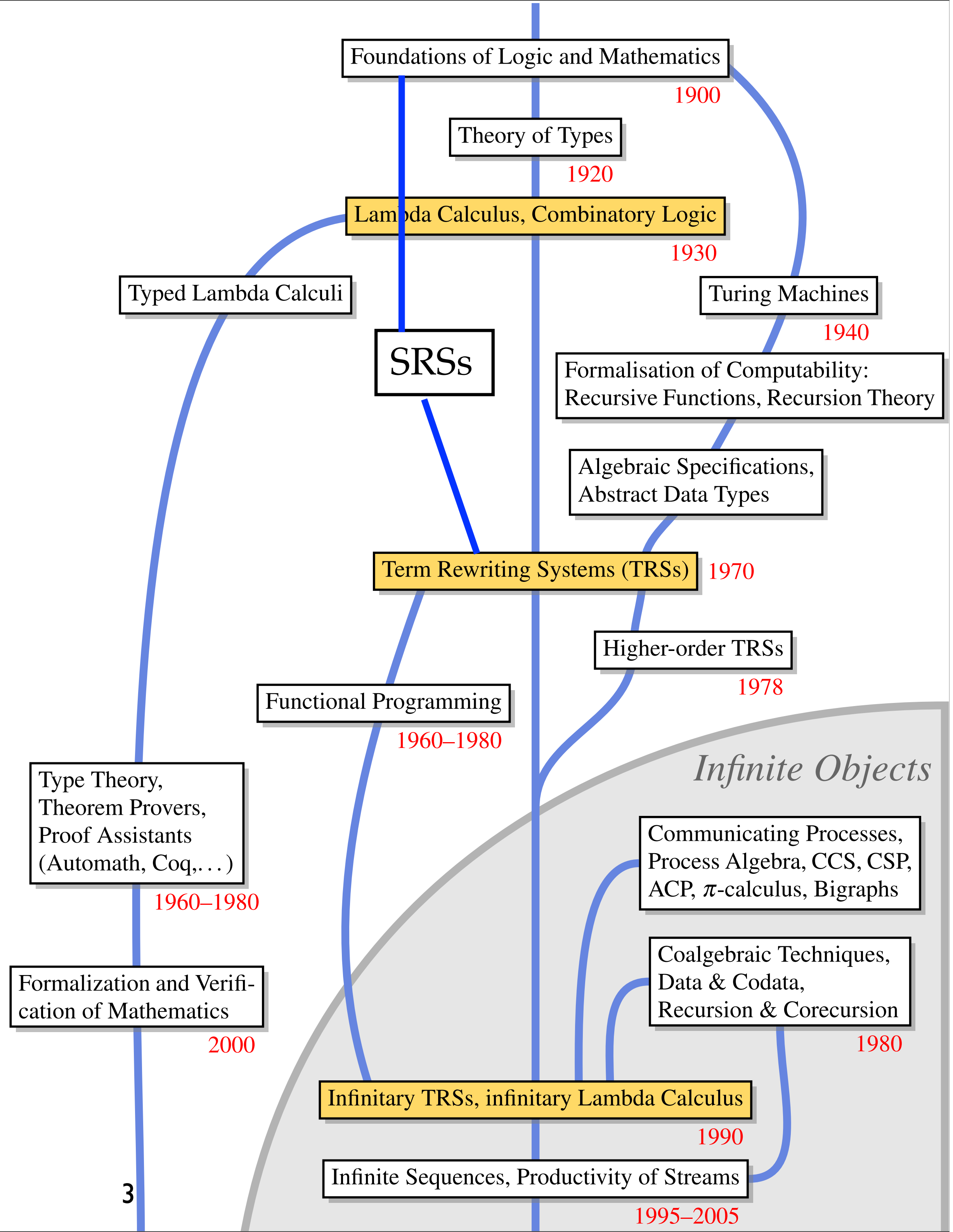
TEA, COFFEE

7.infinitary lambda calculus and the threefold path

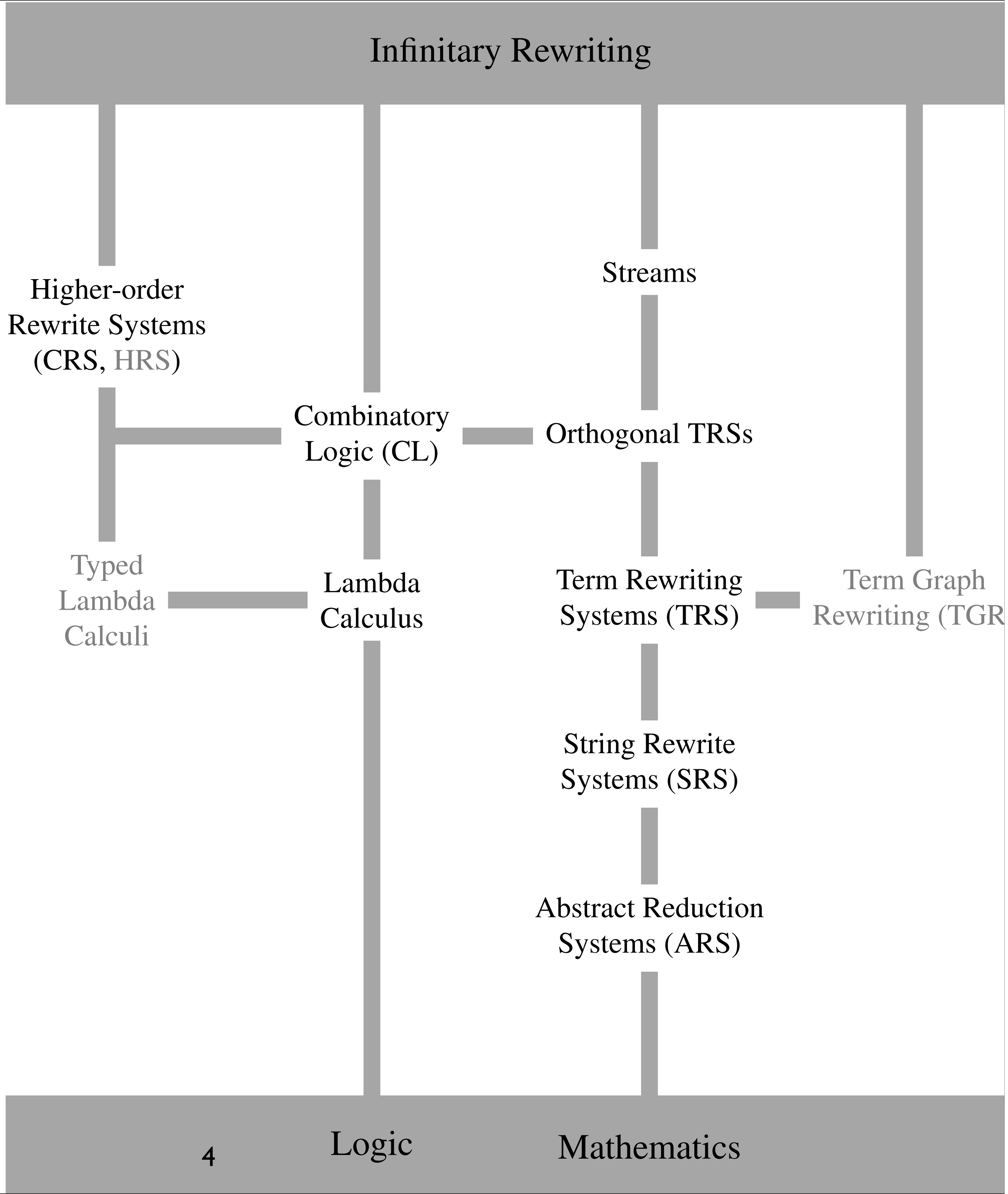
8.clocked semantics of lambda calculus

9.streams running forever

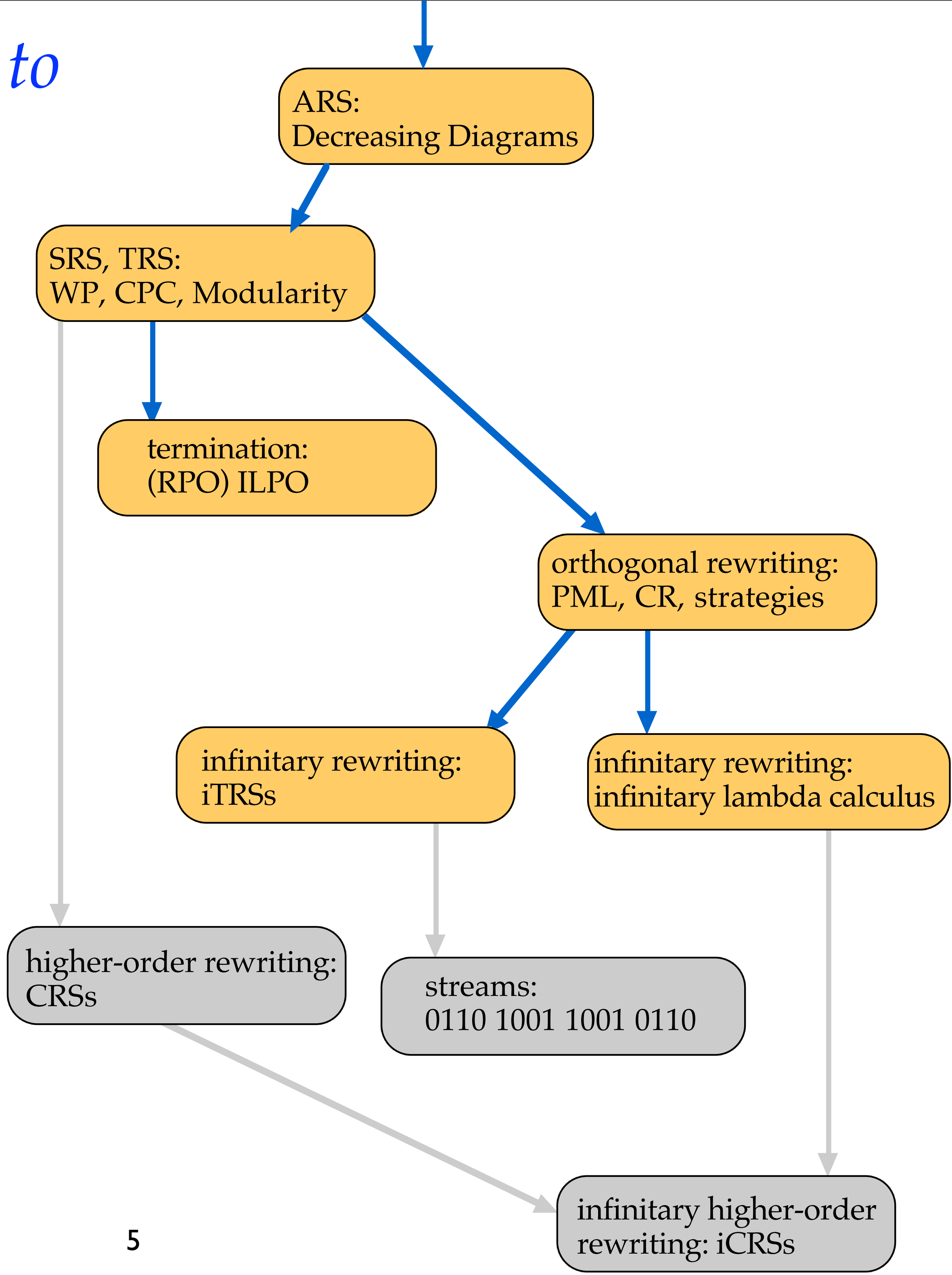
Some historical lines...



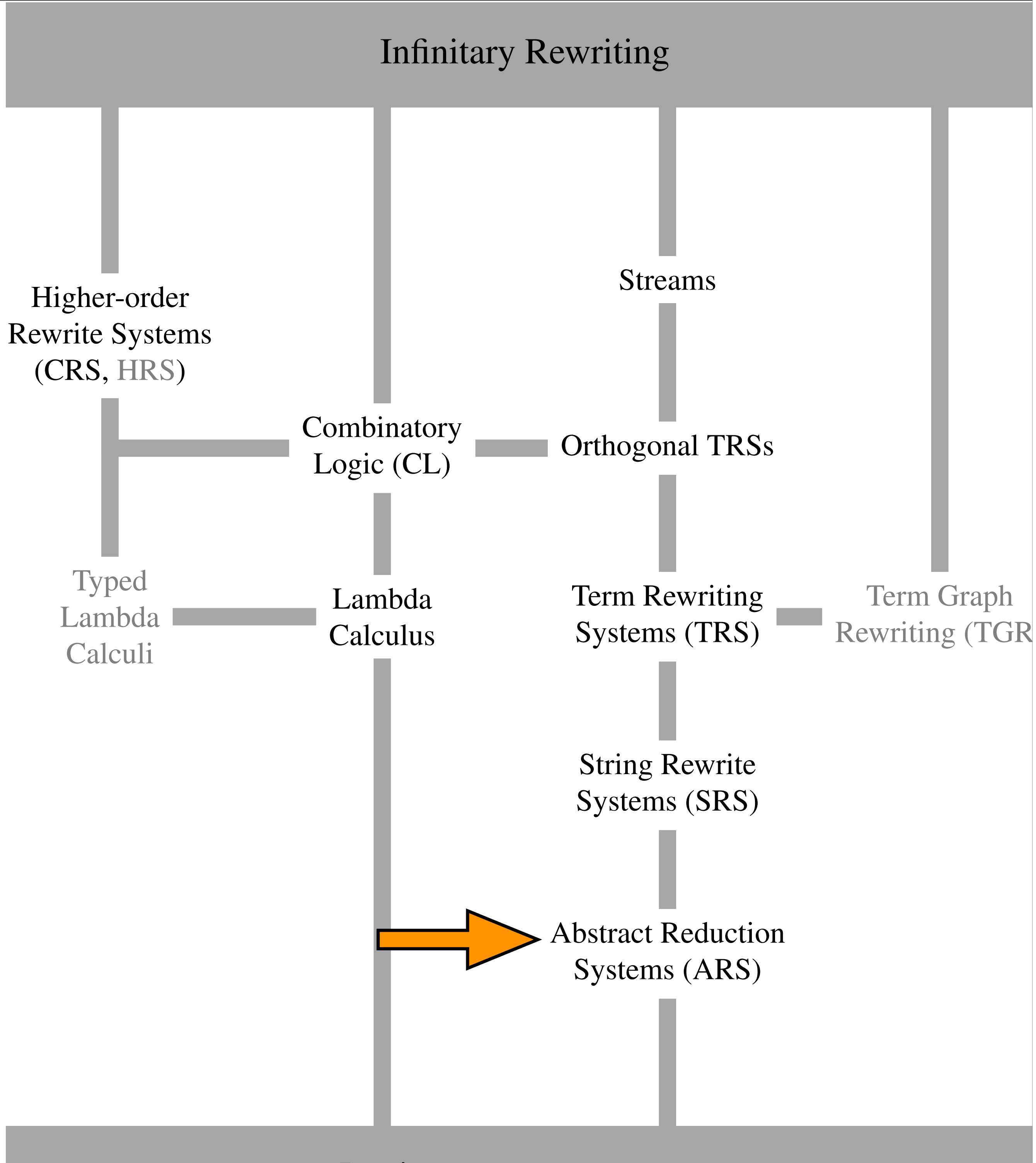
Some streets we want to walk



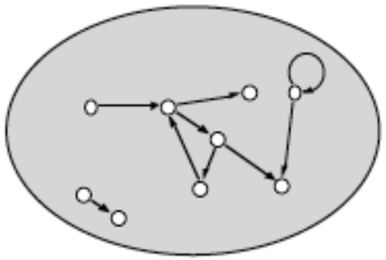
capita that we would like to discuss



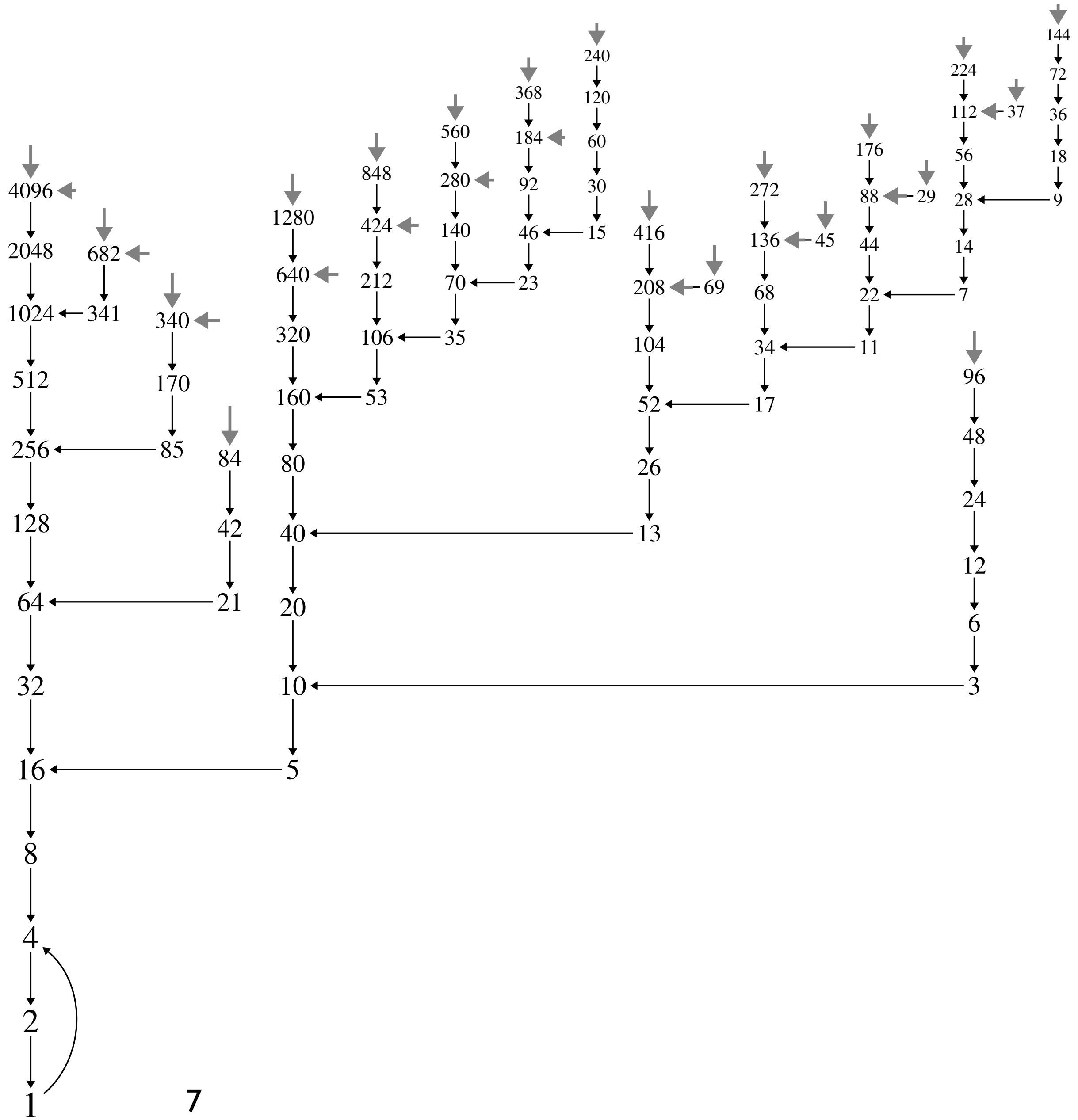
Some streets we want to walk



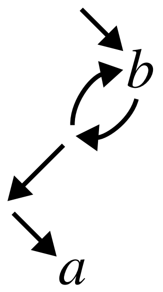
The famous Collatz ARS: 3n+1-problem



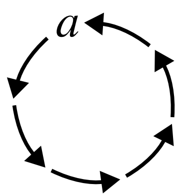
An ARS



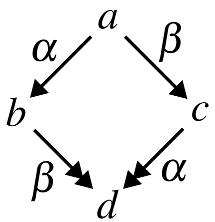
1. REWRITING DICTIONARY



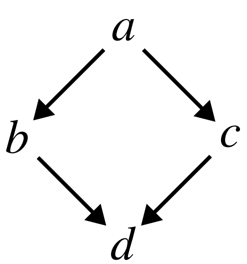
normal form



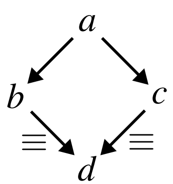
reduction cycle; loop if one step



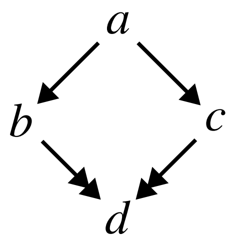
commuting



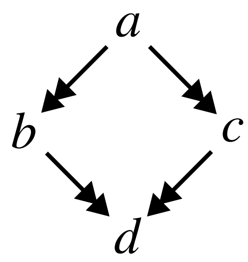
diamond property



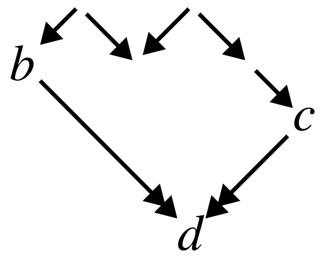
sub-commutative



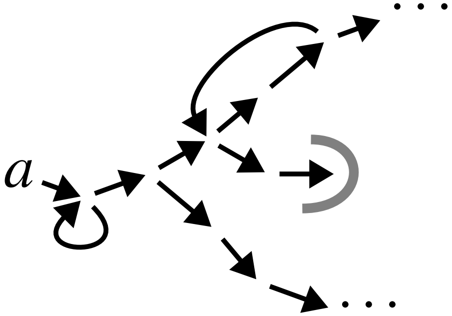
WCR, weakly Church-Rosser



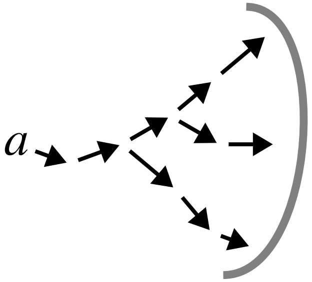
CR, Church-Rosser



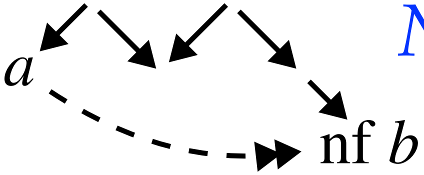
equivalent: CR, Church-Rosser



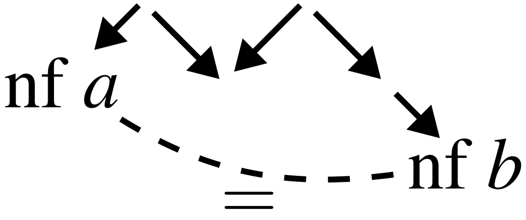
WN, weakly normalizing



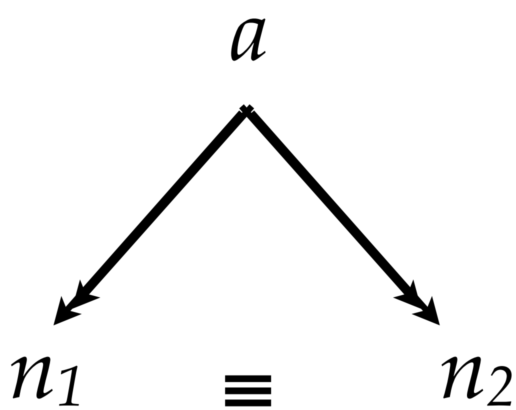
SN, strongly normalizing;terminating; noetherian



NF, normal form property

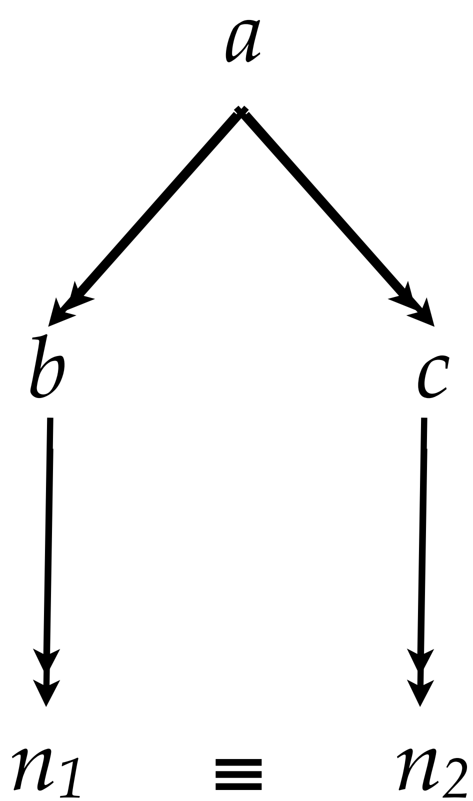


$UN^{=}$, unique normal form property wrt =

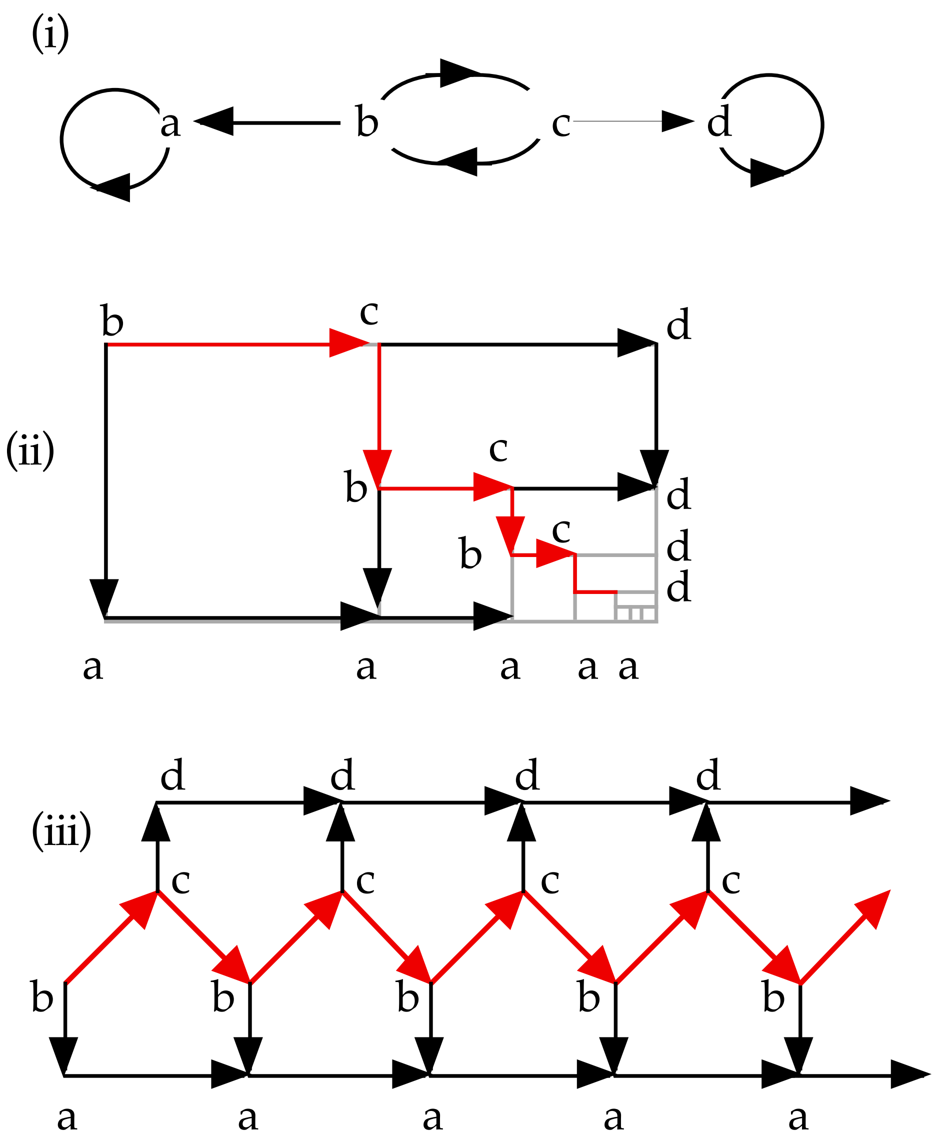


$UN^{\rightarrow}$, unique normal form property wrt $\rightarrow$

$UN^{\rightarrow} \,\&\, SN \Rightarrow CR$



CR ⇒ WCR, but not WCR ⇒ CR



shortest proof of Newman's Lemma:

$WCR \,\&\, SN \Rightarrow CR$

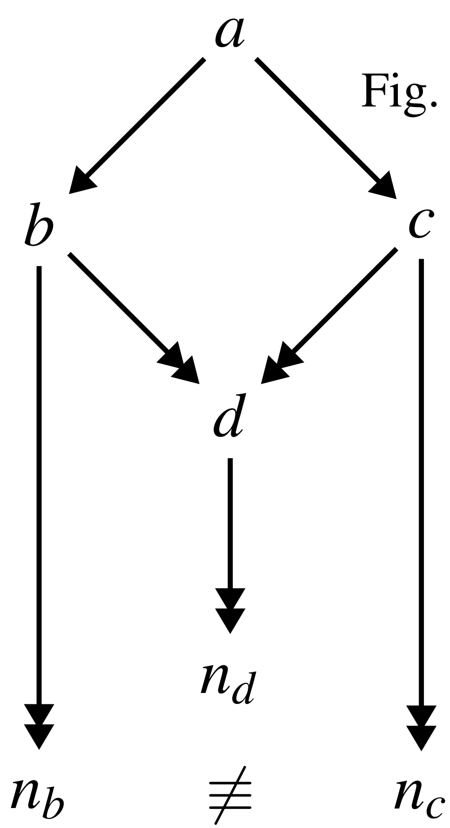
$WCR \,\&\, SN \Rightarrow UN^{\rightarrow} \,\&\, SN \Rightarrow CR$

Call a point bad if it reduces to two different nf's.

A bad point a has a bad one step reduct, b or c.

Hence by SN there are no bad points, i.e. $UN^{\rightarrow}$ holds.

Fig. 3.7: *Max Newman, 1897-1984.*



normal forms

Church (1903-1995)
Studying mathematics at
Princeton 1922 or 1924

Supervisor	Oswald Veblen
Suggested topic	find an algorithm for the genus of a manifold $\{\vec{x} \in K^n \mid p(\vec{x}) = 0\}$ (e.g. $K = \mathbb{R}$, $n = 3$)

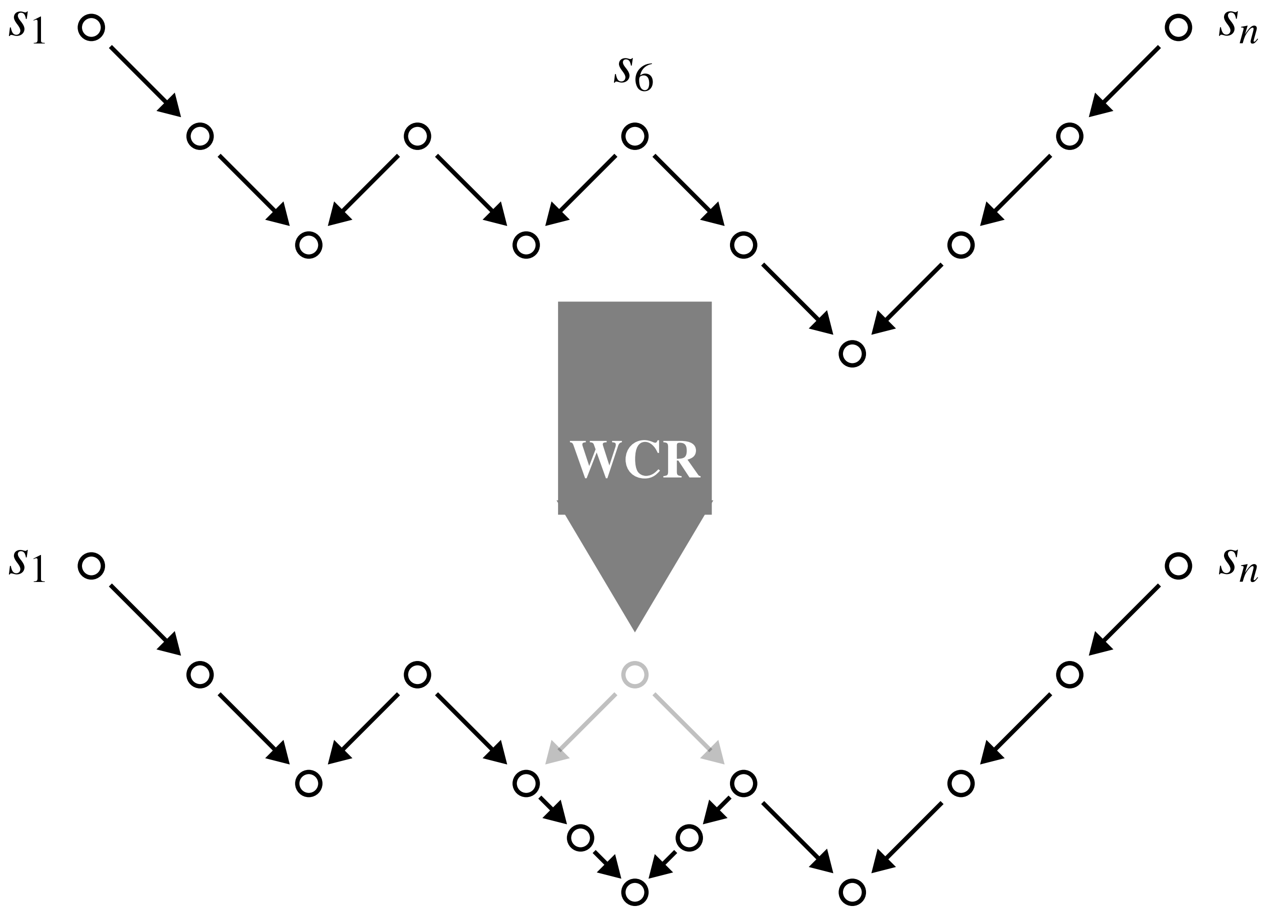
0 1 2 3 ?

Church could not do it
Started to wonder what computability is after all
Invented lambda calculus
Formulated Church's Thesis:

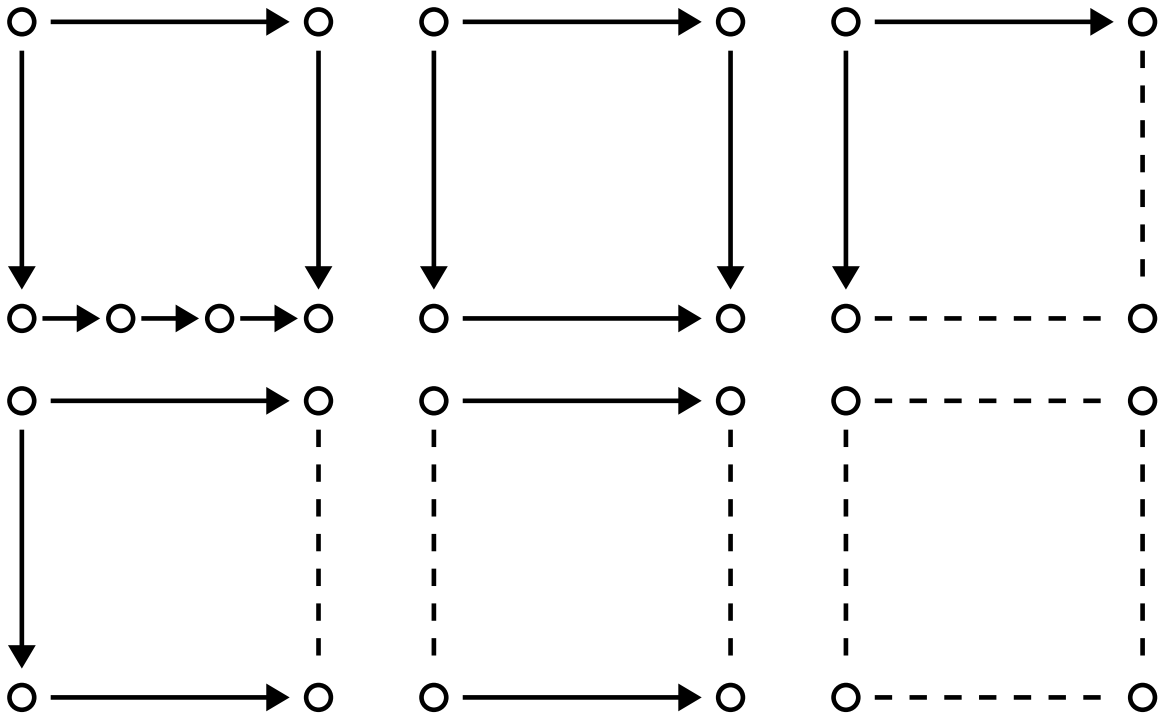
Given a function $f : \mathbb{N}^k \to \mathbb{N}$

Then f *is computable iff* f *is lambda definable*

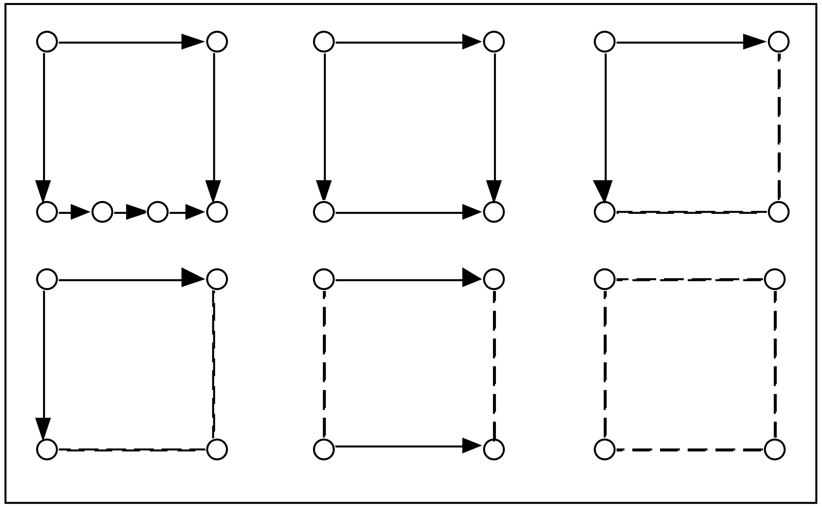
sophisticated multiset proof of Newman's Lemma:

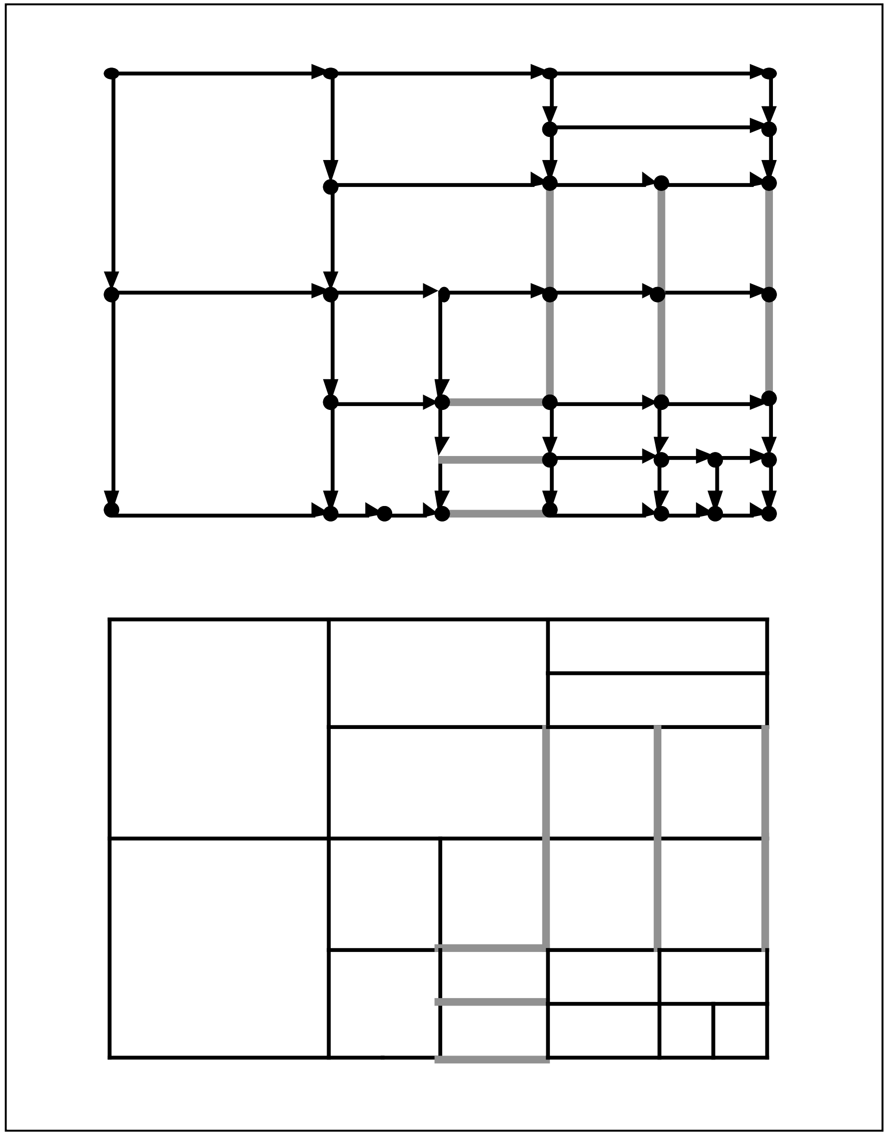


elementary diagrams to build reduction diagrams, given WCR

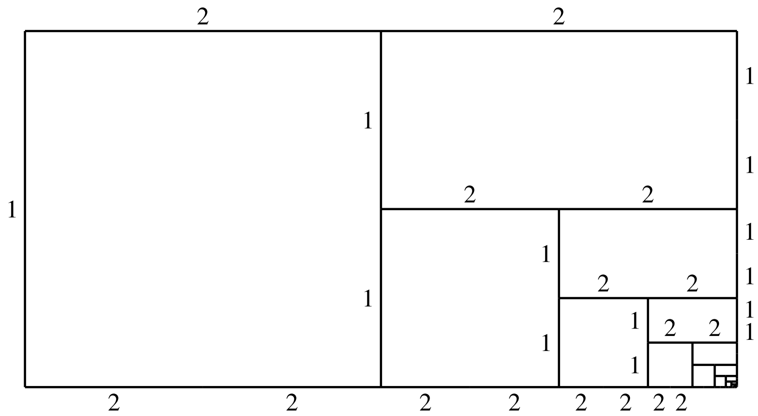


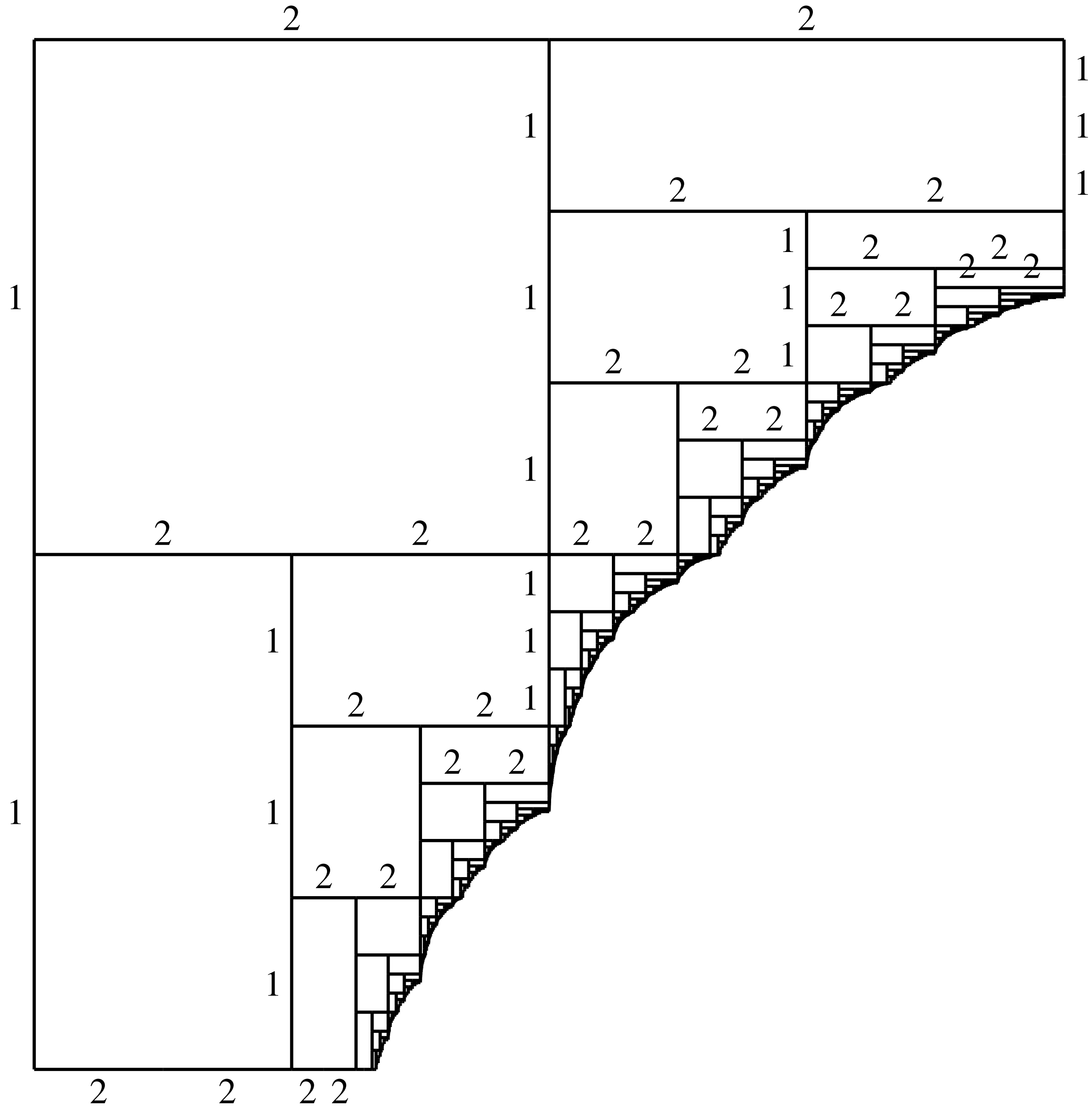
completed reduction diagrams



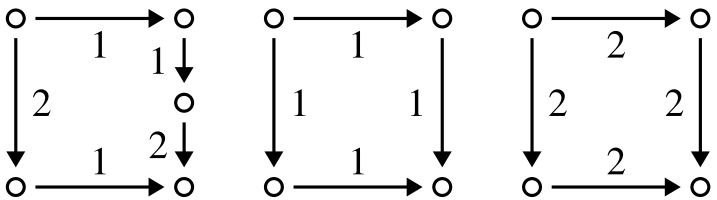


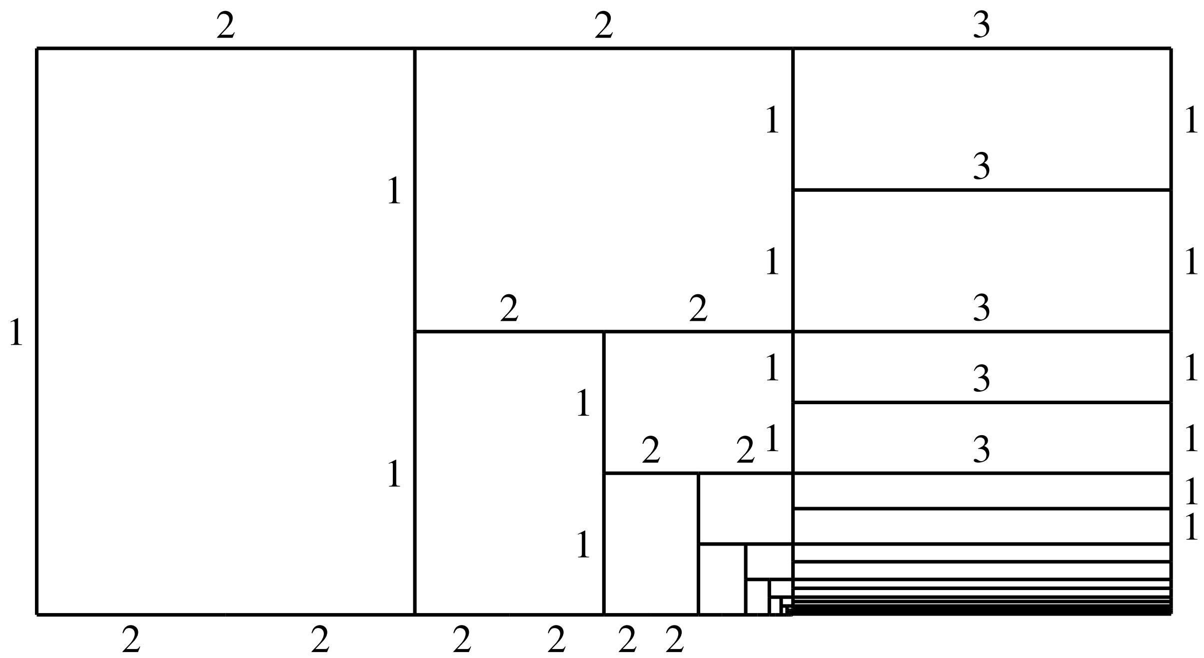
failed reduction diagrams



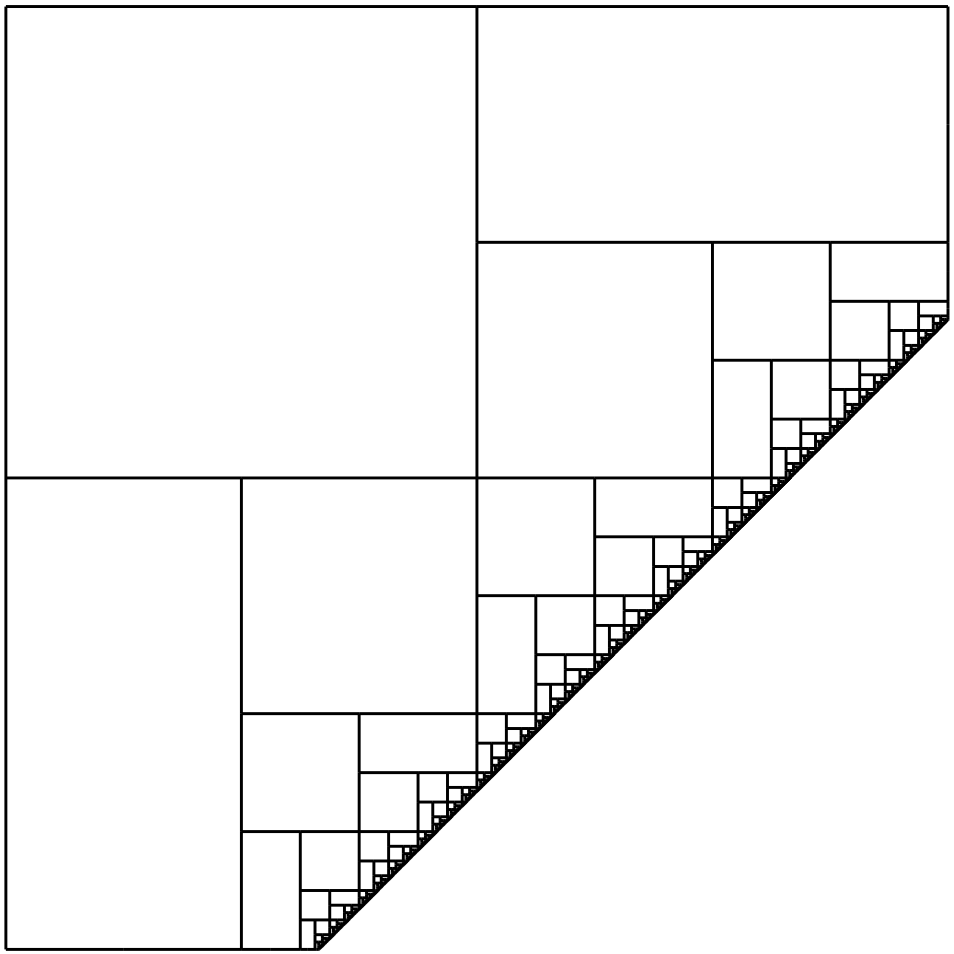


another failure

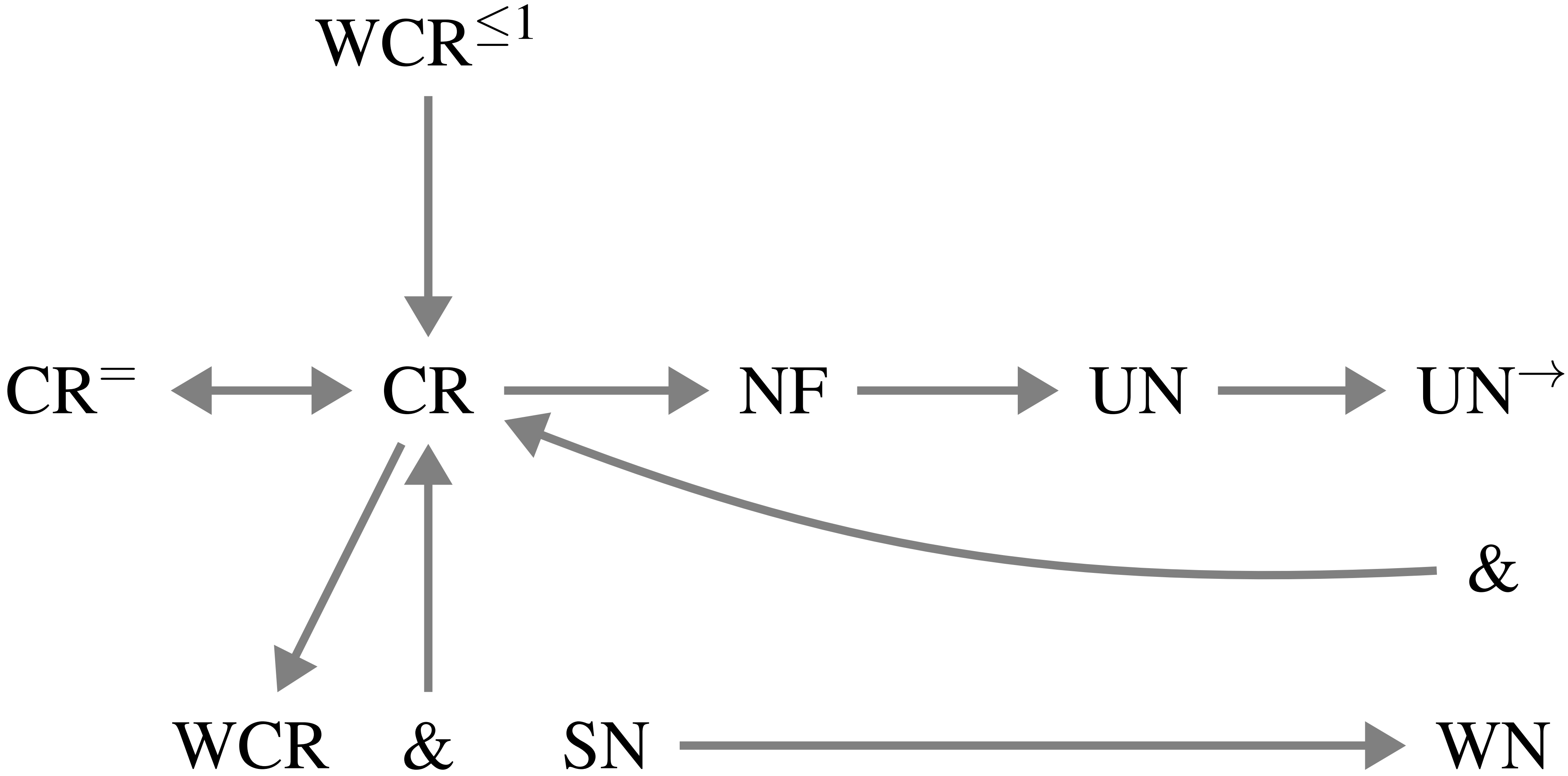




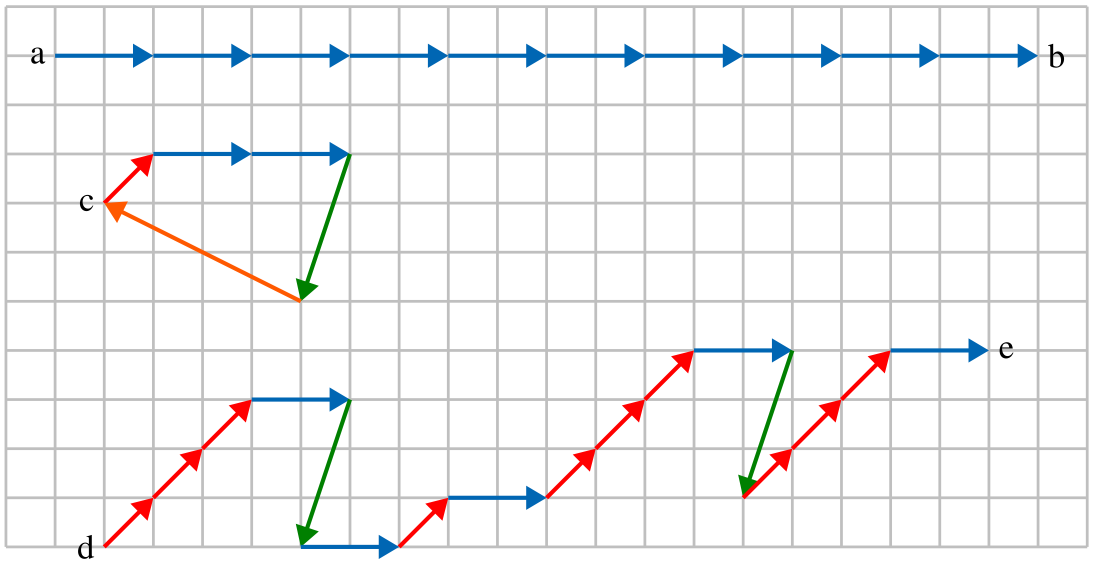
and one more

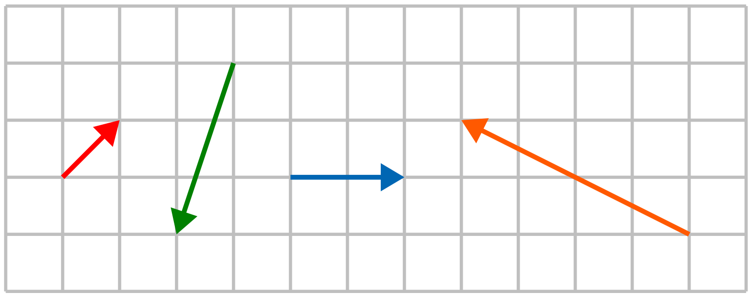


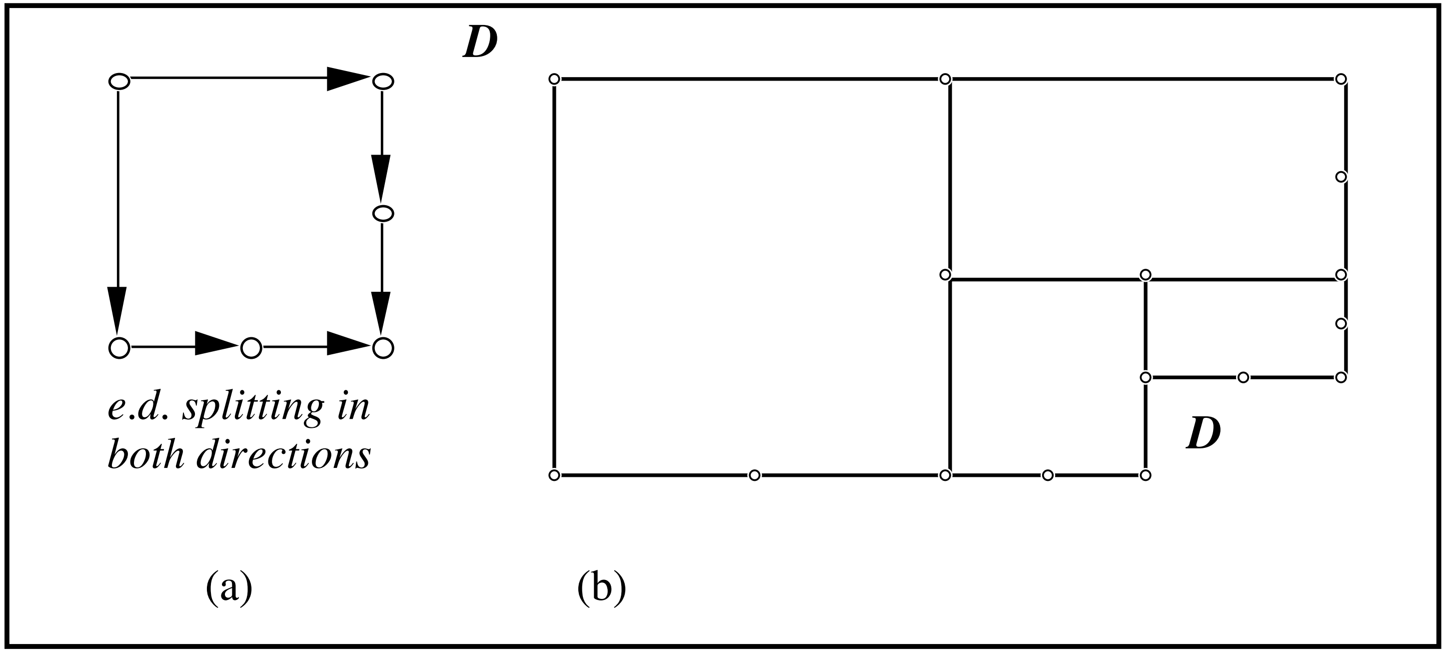
speaking for itself



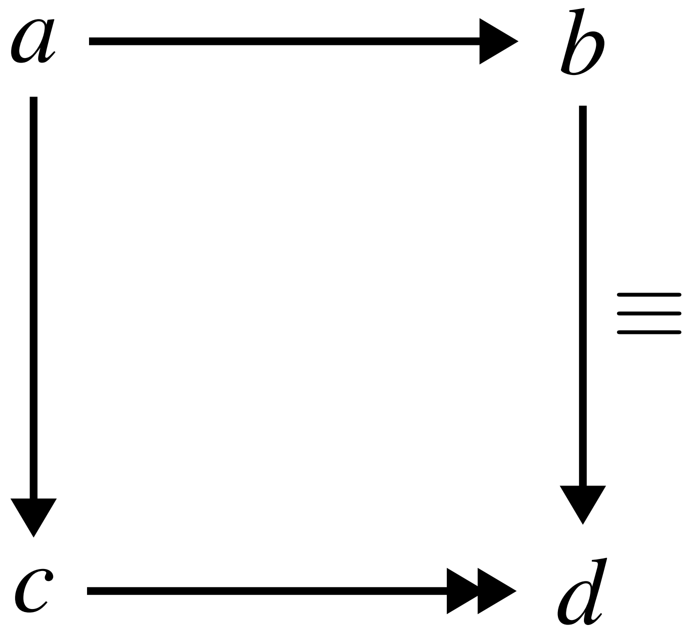
a vector addition system: indexed ARS







$$\forall a, b, c \in A \exists d, e, f \in A (c \leftarrow a \rightarrow b \Rightarrow c \rightarrow d \rightarrow e \leftarrow f \leftarrow b)$$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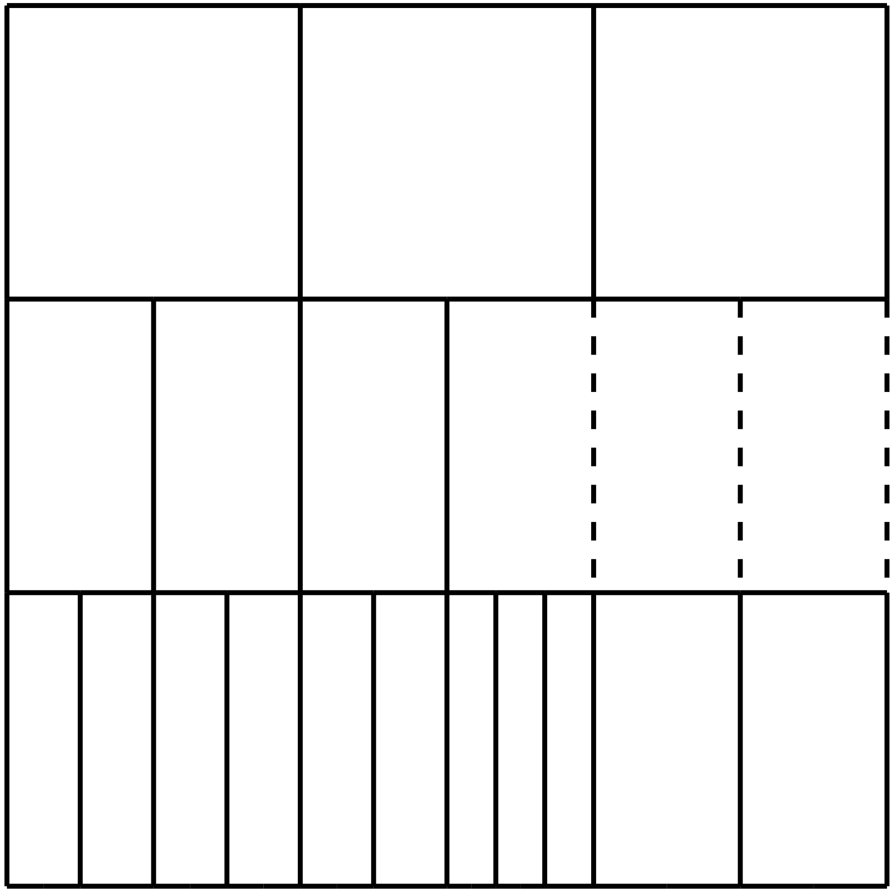
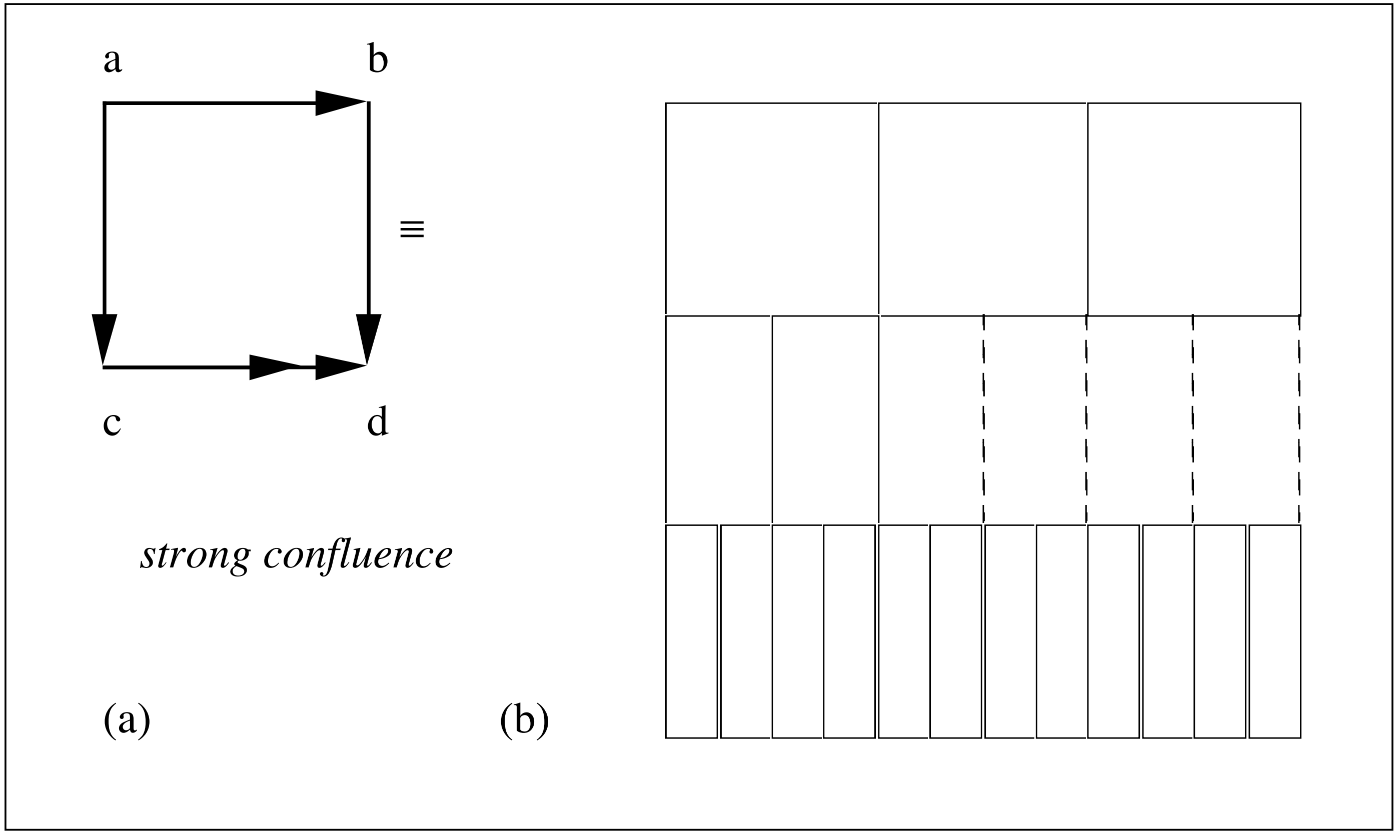


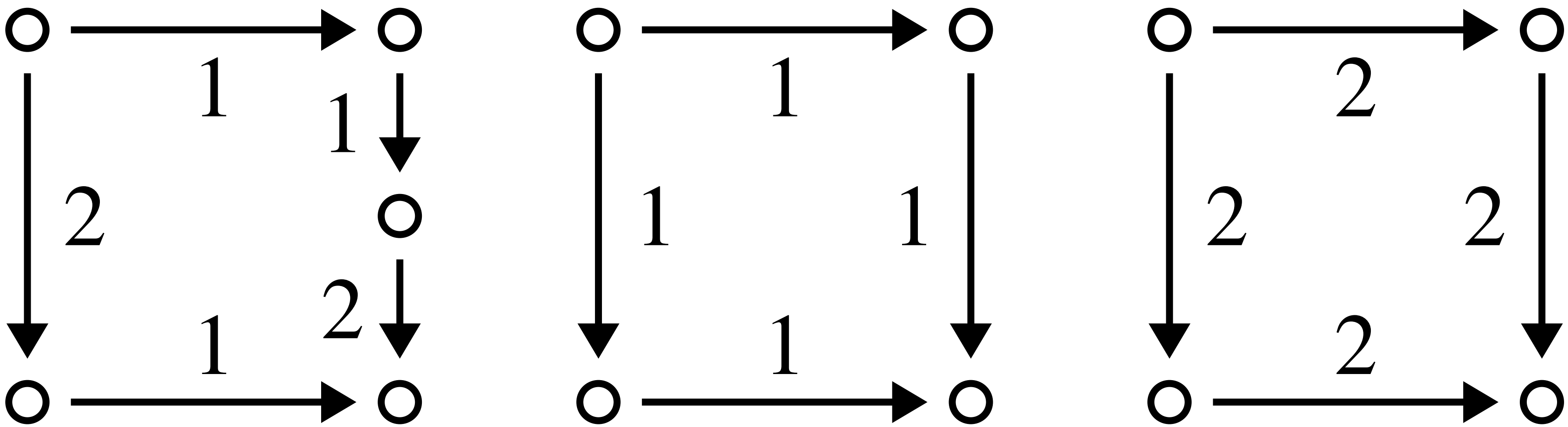


1.2.1. EXAMPLE. 1.2.2. DEFINITION. For an ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ we define: $\rightarrow$ is *strongly confluent* if

$$\forall a, b, c \in A\, \exists d \in A (b \leftarrow a \rightarrow c \Rightarrow c \twoheadrightarrow d \twoheadleftarrow^{\equiv} b)$$

(See Figure 1.9(a)) (Here $\twoheadleftarrow^{\equiv}$ is the reflexive closure of $\twoheadleftarrow$, so $b \twoheadrightarrow^{\equiv} d$ is zero or one step.)

1.2.3. LEMMA. *(Huet [80]). Let A be strongly confluent. Then A is CR.*



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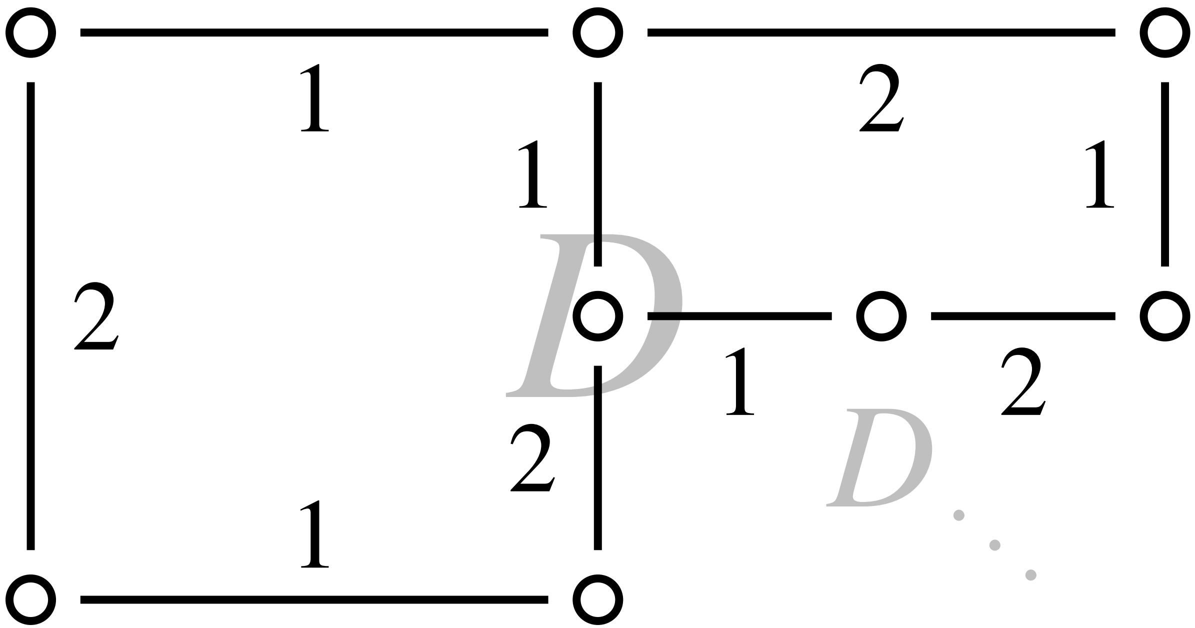
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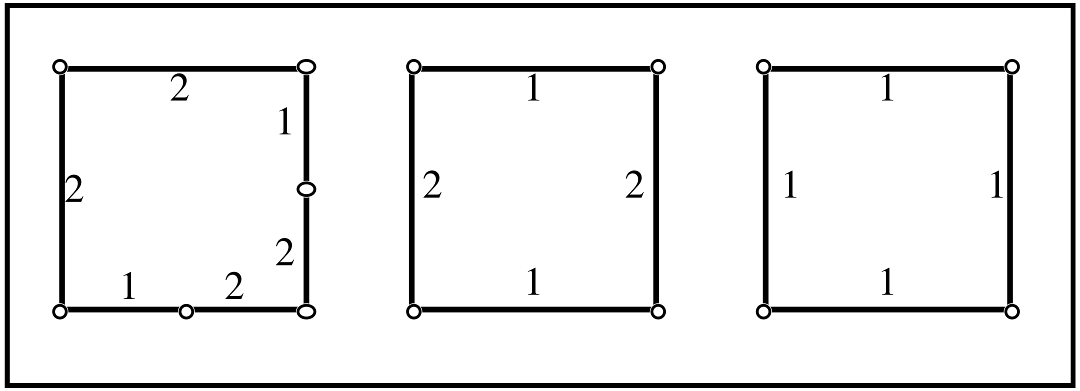


Is tiling succesful? YES!

Dick de Bruijn

1918 - 2012

AUTOMATH ARCHIVE

Institute in Nijmegen and the Formal Methods section of Eindhoven University of Technology. Started by prof. H. Barendregt, in cooperation with Rob Nederpelt, this archive project was launched to digitize valuable historical articles and other documentation concerning the Automath project.

Initiated by prof. N.G. de Bruijn, the project Automath (1967 until the early 80's) aimed at designing a language for expressing complete mathematical theories in such a way that a computer can verify the correctness. This project can be seen as the predecessor of type theoretical proof assistants such as the well known Nuprl and Coq.

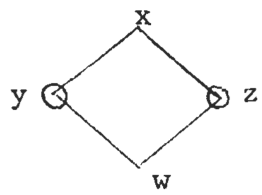
A note on weak diamond properties.

1. Introduction. Let S be a set with a binary relation >. We assume it to satisfy x > x for all $x \in S$. We are interested in establishing a property CR (named after its relevance for the Church-Rosser theorem of lambda calculus, cf. [1]). We say that $x \sim y$ if x > y or y > x. We say that $x >^* y$ if there is a finite sequence $x_1,\ldots,x_n$ with $x=x_1 > x_2 > \ldots > x_n=y$, and also if x=y. We say that (S,>) satisfies CR if for any sequence $x_1,\ldots,x_n$ with

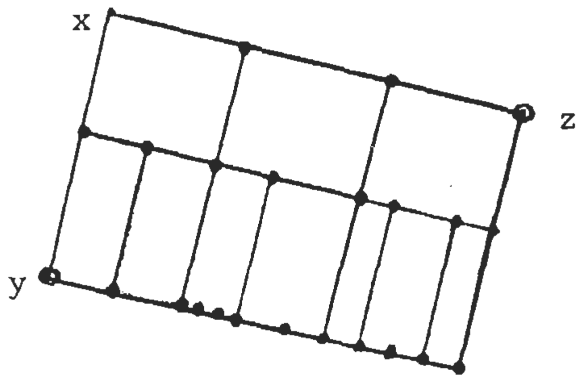
$$x_1 \sim x_2 \sim \ldots \sim x_n$$

there exist an element $x \in S$ with both $x_1 >^* z$ and $x_n >^* z$.

It is usual to say that (S,>) has the diamond property (DP) if for all x,y,z with x > y, x > z there exists a w with y > w, z > w. This is depicted in the following diagram:



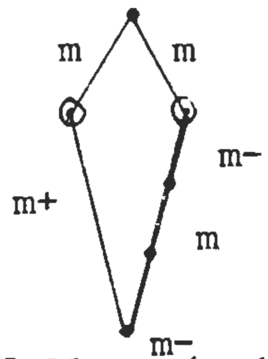
This example also shows that CR neither follows from WDP_2 where WDP_2 is slightly stronger than WDP_1 and says:"if $x > y$ and $x > z$ then w exists such that $y >^* w$ and $z >^* w$ and at least one of $y > w$ and $z > w$". Stronger again is WDP_3, expressing:"if $x > y$ and $x > z$ then w exists such that $y >^* w$ and $z > w$." This WDP_3 does imply CR. Actually WDP_3 implies WDP_4, which says: "if $x >^* y$ and $x >^* z$ then w exists such that both $y >^* w$ and $z >^* w$." This WDP_4 is the DP for $(S, >^*)$, and therefore implies CR for $(S, >^*)$, and that is the same thing as CR for $(S, >)$. The derivation of WDP_4 from WDP_3 is illustrated by the following picture (cf. [2] p. 59) which speaks for itself:



In this note we go considerably further. Instead of having just one relation $>$ we consider a set of relations $>_m$ where m is taken from an index set M. The idea behind this is that in the Church-Rosser theorem the relations represent lambda calculus reductions; there may be reductions of various types, and diamond properties may depend on these types. It is our purpose to establish weak diamond properties which guarantee CR (where CR has to be interpreted as in section 4.

5. <u>The basic diamond properties</u>. If $m \in M$, the diamond property $D_1(m)$ is defined by the following diagram.

$D_1(m)$:

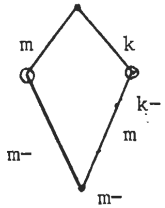


This has to be read as follows (and further diagrams have to be interpreted analogously: If x,y,z are such that $x >_m y$, $x >_m z$, then u,v,w exist such that

$$y >_{m+} w, \quad z >_{m-} u >_m v >_{m-} w.$$

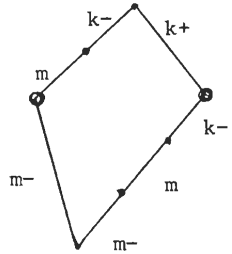
(so on the left we have a chain from y to w with all links $\leq m$; on the right we have a chain from z to w with all links $\leq m$ but with at most one $= m$).

$D_2(m,k)$:

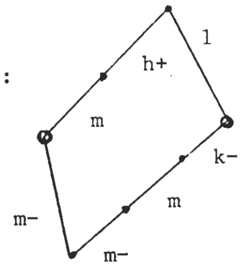


6. Some auxiliary diamond properties. We intend to show that $D_1(m)$ and $D_2(m,k)$ (for all m,k with $k < m$) lead to CR. In order to achieve this we formulate a number of diamond properties that will play a rôle in the proof.

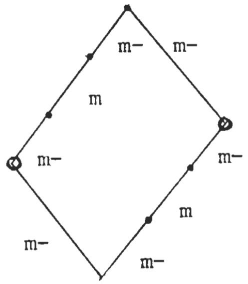
$D_3(m,k)$:



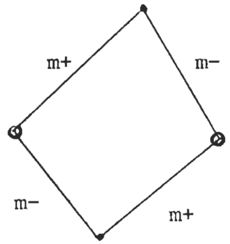
$D_4(m,k,l,h)$:



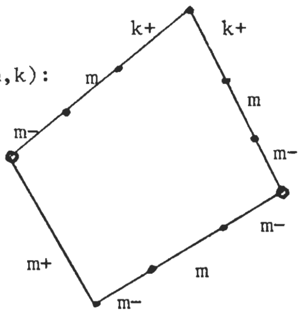
$D_5(m)$:



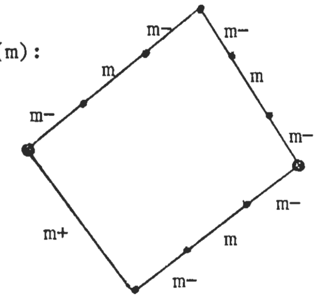
$D_6(m)$:



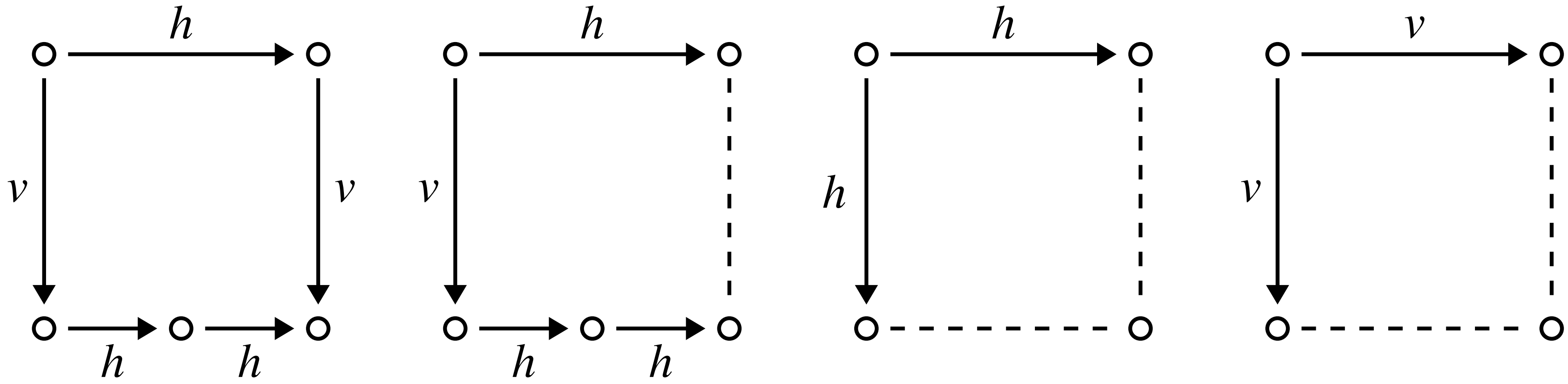
$D_7(m,k)$:



$D_8(m)$:



The diagrams D_3 and D_7 will play their rôle only if $k < m$, and D_4 only if $h < k' < m$, $l \leq m$.



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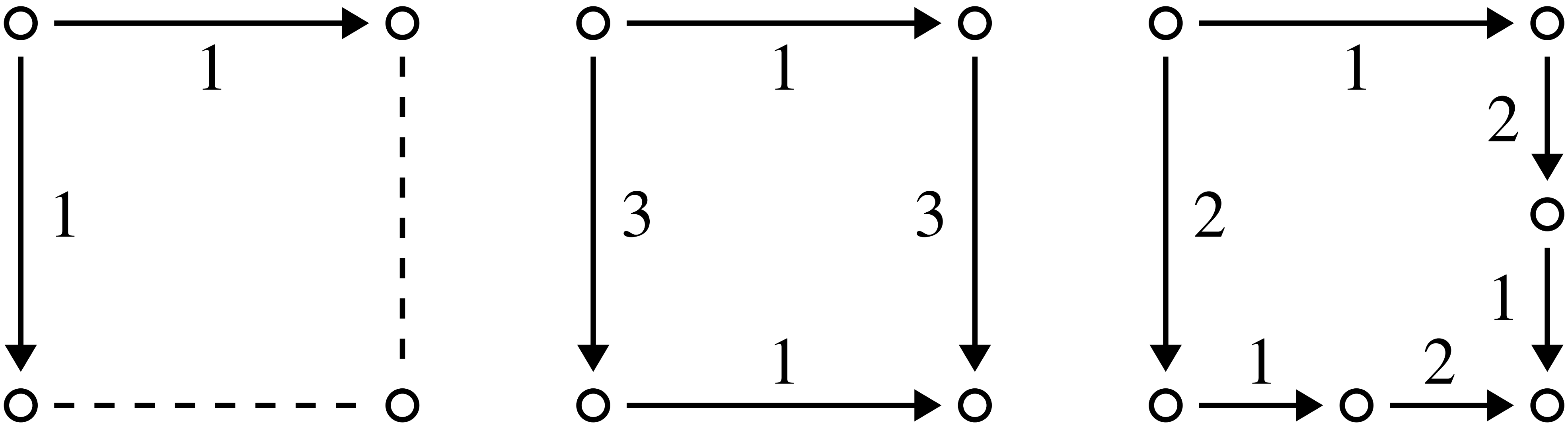
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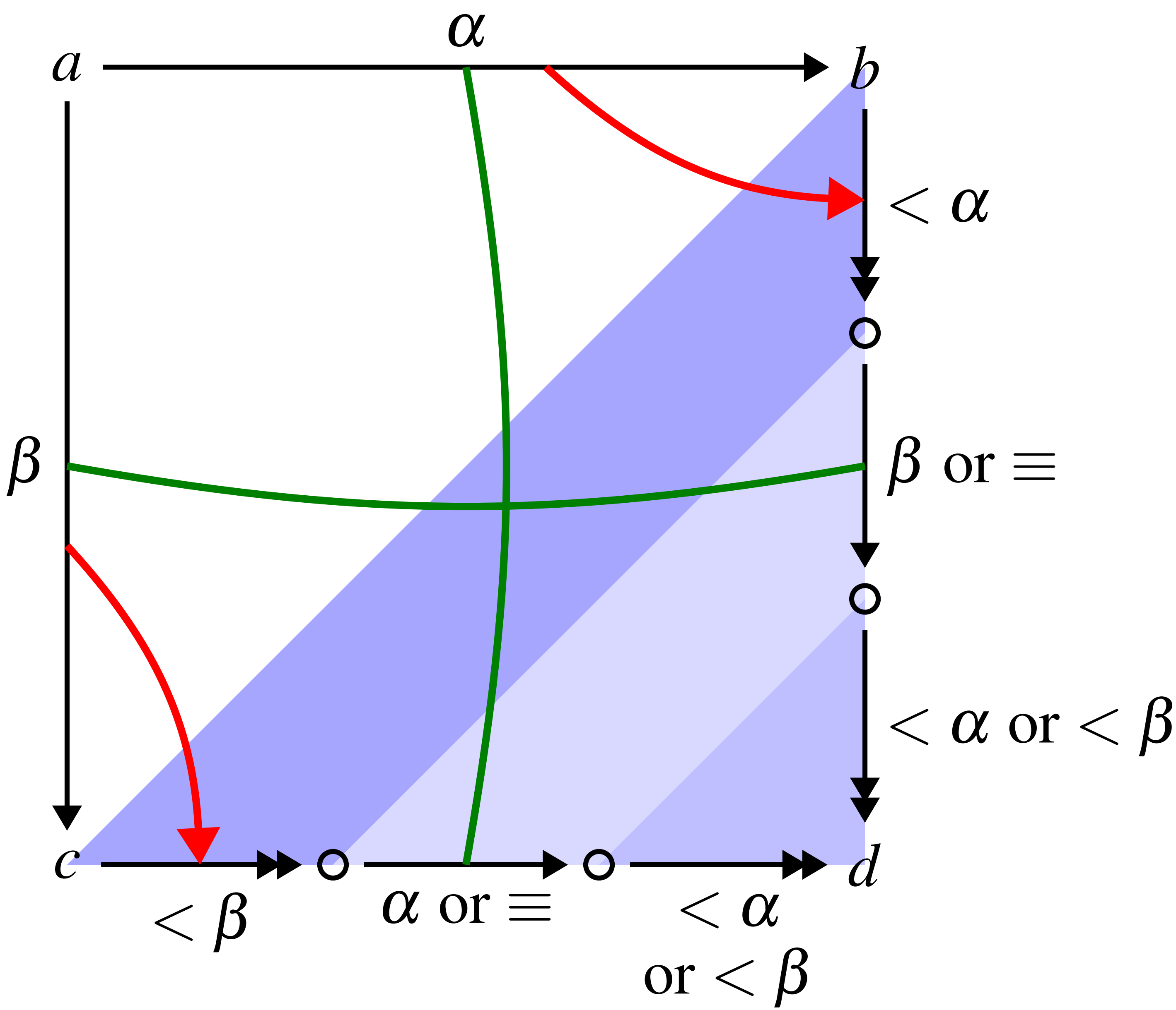

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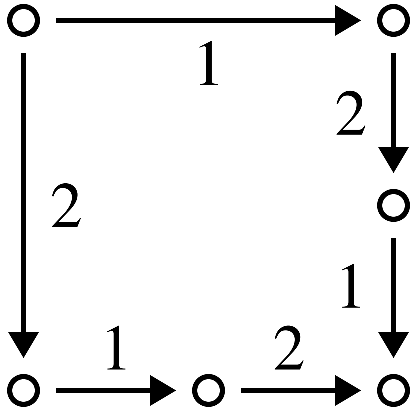
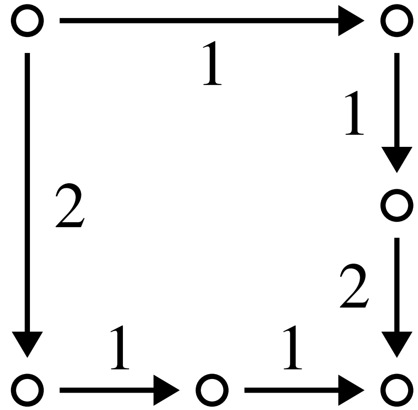
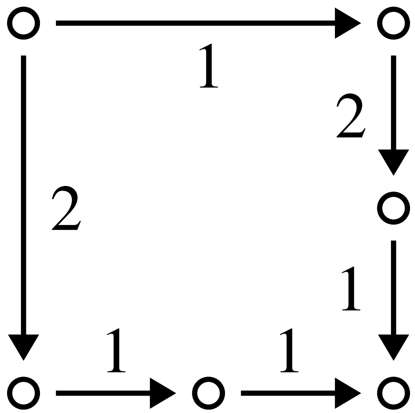
α or $\equiv$
$< \alpha$
or $< \beta$

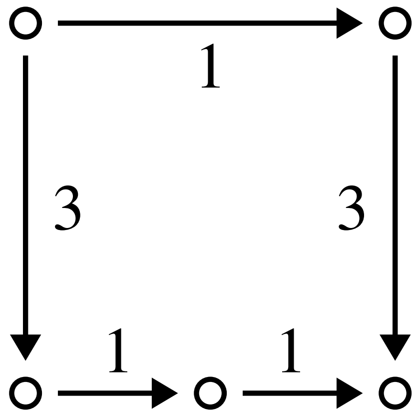
not decreasing

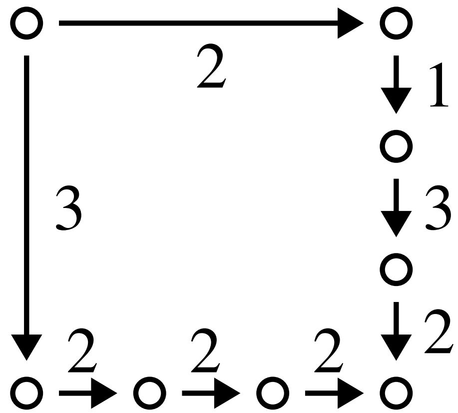




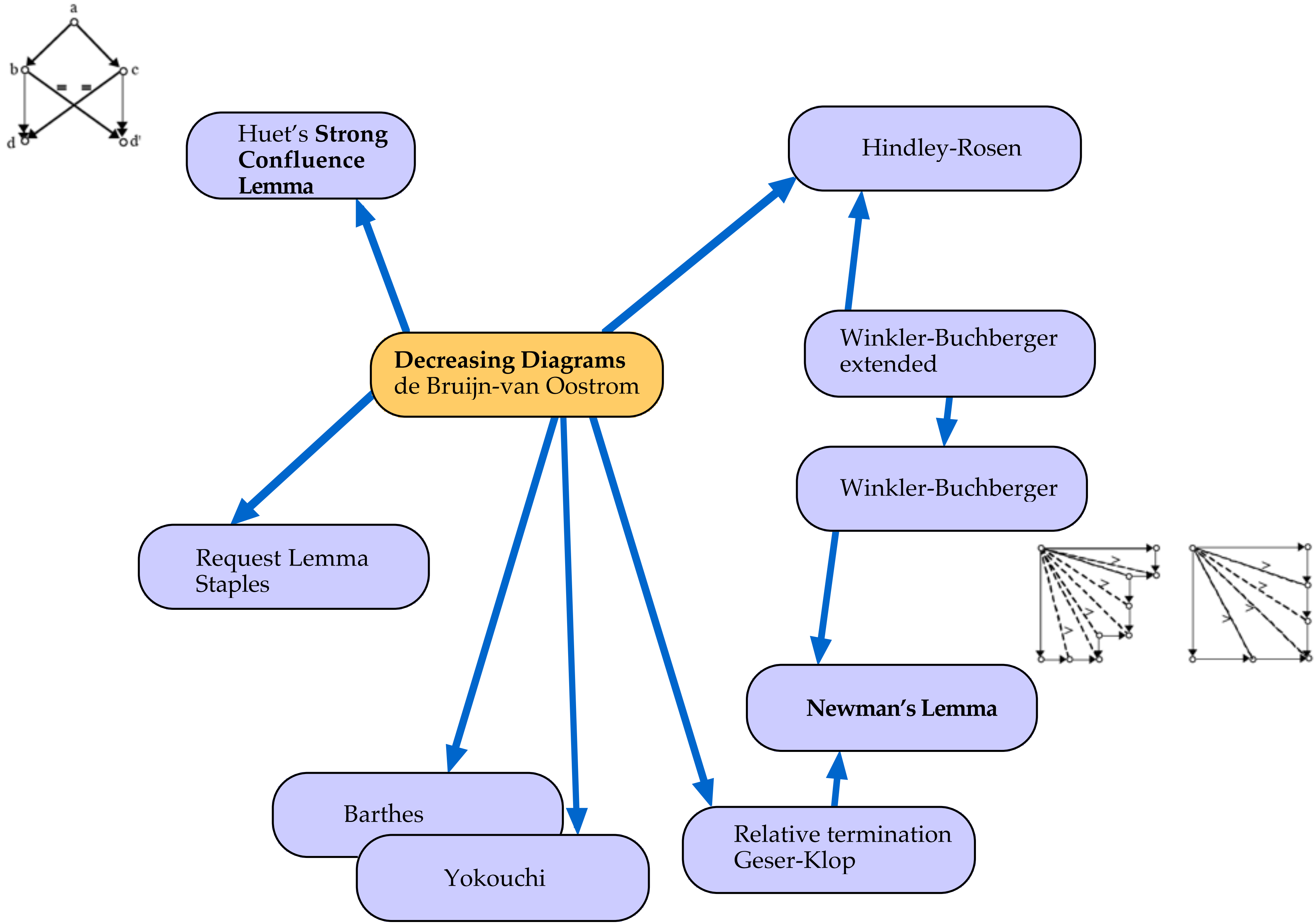
decreasing







1.2.14. THEOREM. *(De Bruijn - Van Oostrom) Every ARS with reduction relations indexed by a well-founded partial order I, and satisfying the decreasing criterion for its e.d.'s, is confluent.*


Huet's **Strong Confluence Lemma**
Hindley-Rosen
Decreasing Diagrams
de Bruijn-van Oostrom
Winkler-Buchberger extended
Winkler-Buchberger
Request Lemma
Staples
Newman's Lemma
Barthes
Yokouchi
Relative termination
Geser-Klop
a
b
c
d
d'

Theorem 3.3 (Decreasing Diagrams – De Bruijn). *Let $\mathscr{A} = (A, (\rightarrow_\alpha)_{\alpha \in I})$ be an ARS with reduction relations indexed by a well-founded* total *order $(I, >)$. If for every peak $c \leftarrow_\beta a \rightarrow_\alpha b$ there exists an elementary diagram joining this peak of one of the forms in Figure 3.13, then $\rightarrow$ is confluent.*

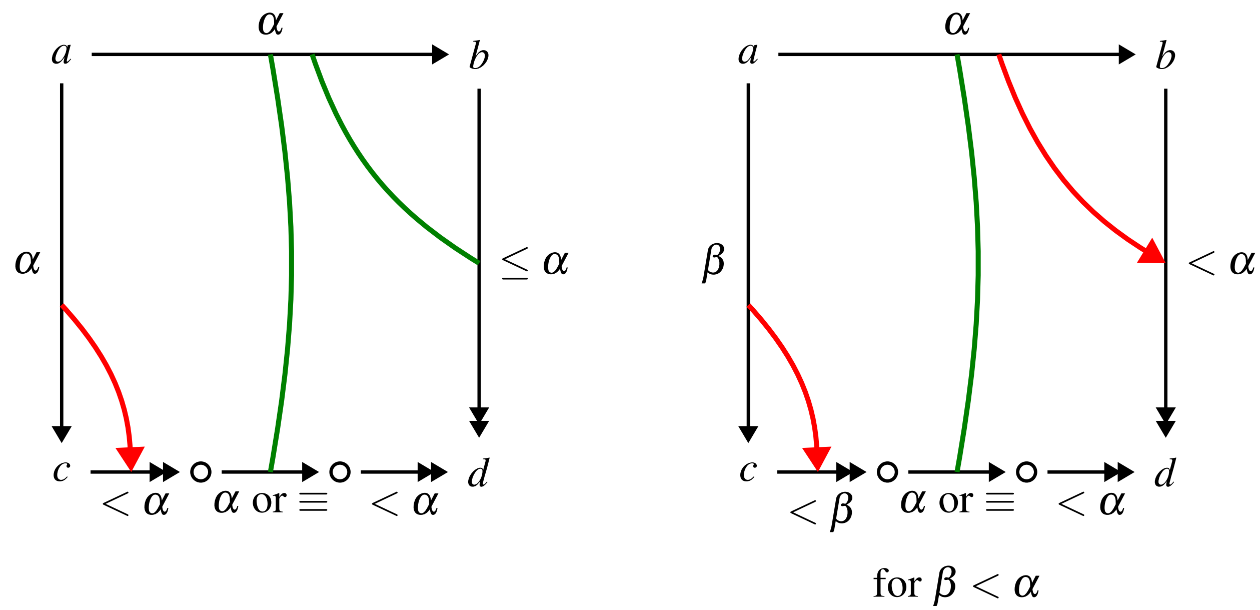


Fig. 3.13: *De Bruijn's asymmetrical decreasing elementary diagrams.*

Van Oostrom [vO94b, vO94a] presents a novel proof, and derives the following symmetrical version of decreasing elementary diagrams that allows for partial orders $>$, see Figure 3.14.

Theorem 3.4 (Decreasing Diagrams – Van Oostrom). *Let $\mathscr{A} = (A, (\rightarrow_\alpha)_{\alpha \in I})$ be an ARS with reduction relations indexed by a well-founded* partial *order $(I, >)$. An elementary diagram is called* decreasing *if it is of the form displayed in Figure 3.14. If for every peak $c \leftarrow_\beta a \rightarrow_\alpha b$ there exists a decreasing elementary diagram joining this peak, then $\rightarrow$ is confluent.*

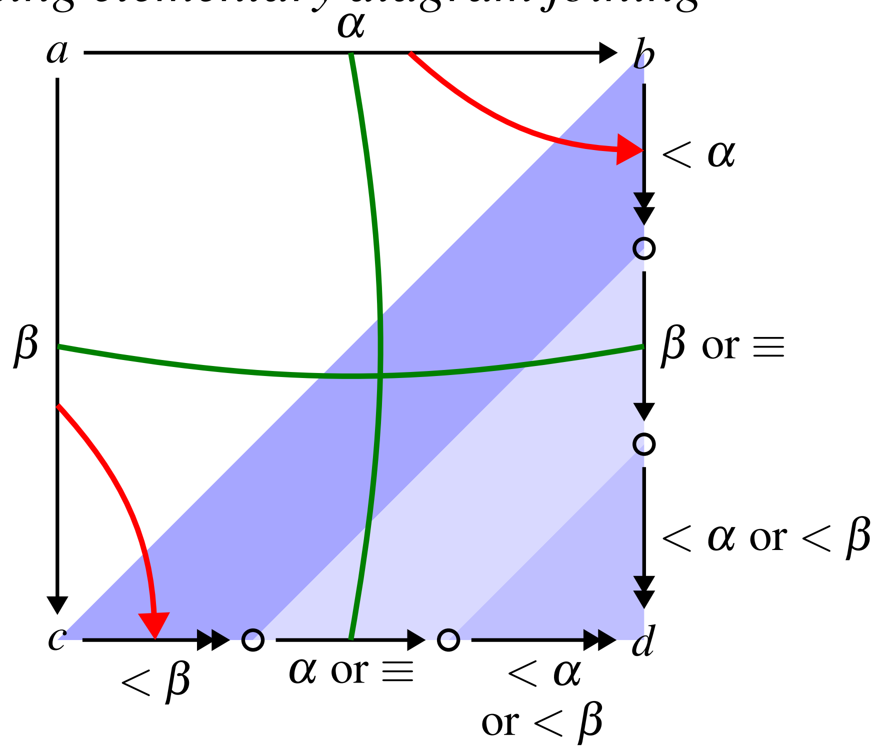


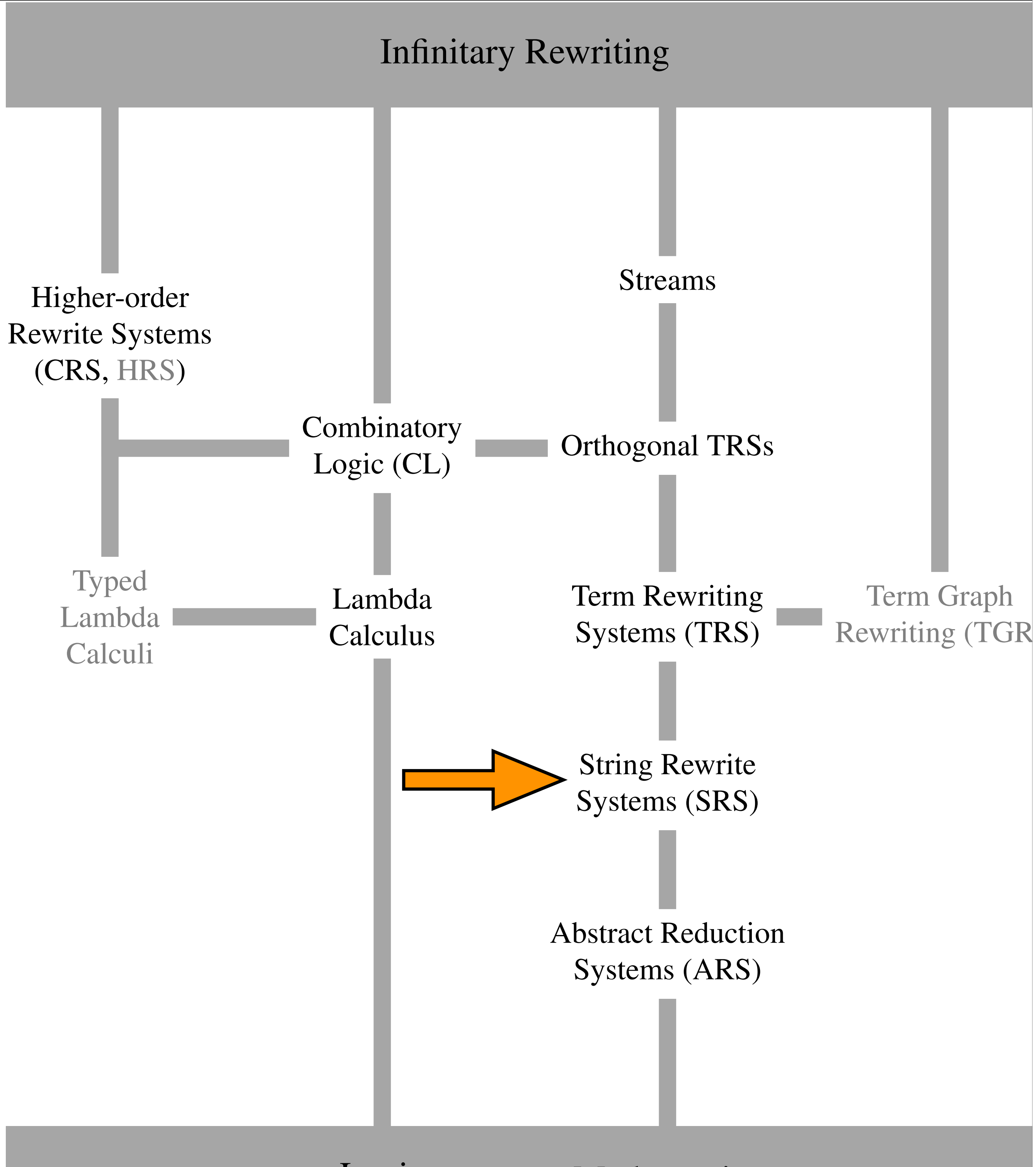
Fig. 3.14: *Decreasing elementary diagram.*

Definition 3.3. An ARS $\mathscr{A} = (A, \rightarrow)$ is said to be *decreasing Church-Rosser* (DCR), if there is an indexed ARS $\mathscr{B} = \langle A, (\rightarrow_\alpha)_{\alpha \in I} \rangle$ and a well-founded order $>$ on I such that $\mathscr{B}$ has decreasing elementary diagrams with respect to $>$, and $\rightarrow = \bigcup_{\alpha \in I} \rightarrow_\alpha$.

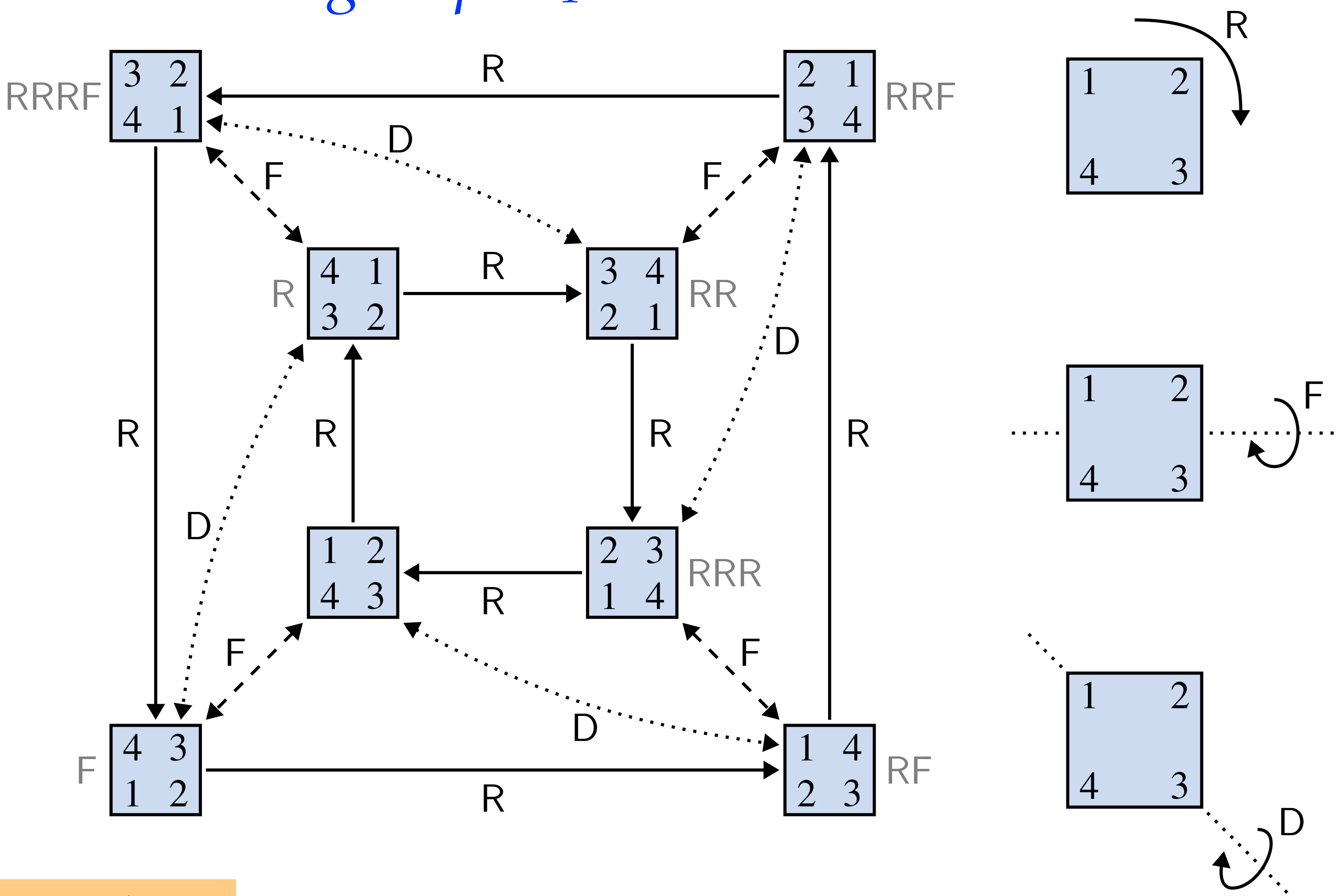
Theorem 3.5 (van Oostrom [vO94b]). *For countable ARSs: DCR $\Leftrightarrow$ CR.*

The proof, also present in Bezem, Klop & van Oostrom [BKvO98], employs the fact mentioned in chapter 1: CR $\Leftrightarrow$ CP for countable ARSs. It seems to be a difficult exercise to establish the (conjectured) result that the condition 'countable' is necessary.

Some streets we want to walk



dihedral group D_4



$FF \to \lambda$

$RRRR \to \lambda$

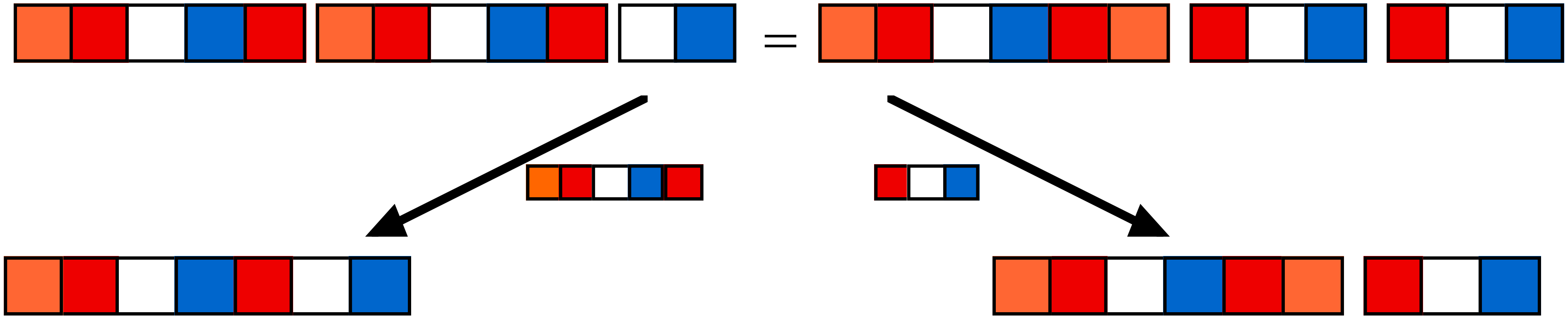
$FR \to RRRF$

is a complete TRS for this equality, thus solving its word problem

Other presentations of D_4

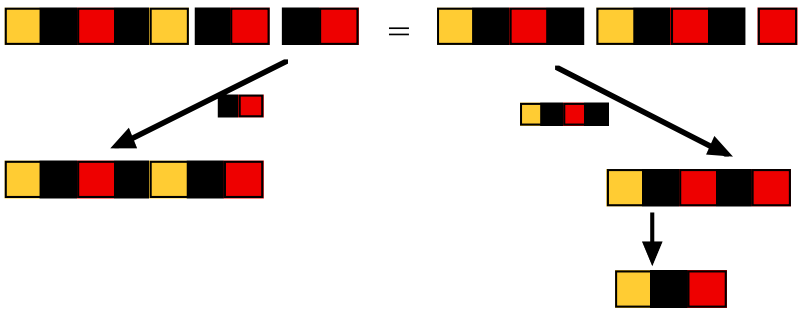
$$A \simeq B \;\Leftrightarrow\; A \Longleftrightarrow_{Tietze} B$$

free idempotent monoid: xx → x



$$dabcabc \leftarrow (dabca)(dabca)bc = dabcad(abc)(abc) \rightarrow dabcadabc$$

by Vincent van Oostrom



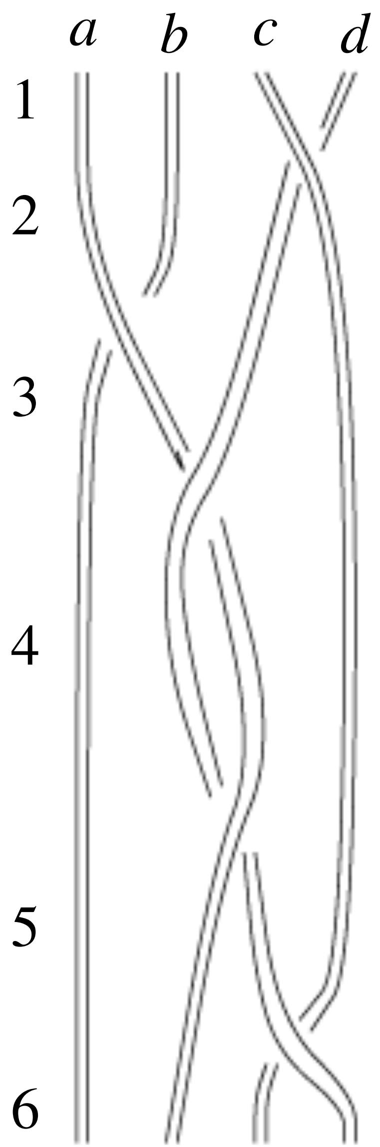
Zantema-Geser: does the rule 0011 → 111000 terminate?

the one-rule SRS $0^p 1^q \to 1^r 0^s$ terminates if and only if

(a) $p \geq s$ or $q \geq r$ or
(b) $p < s < 2p$ and $q < r$ and q is not a divisor of r or
$q < r < 2q$ and $p < s$ and p is not a divisor of s.

(so, does it terminate?)

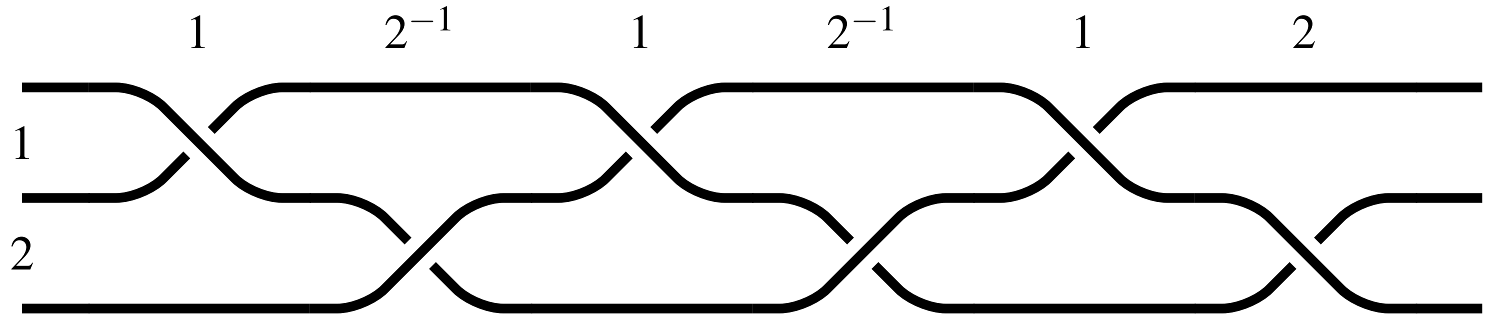
from the Notebook of Gauss

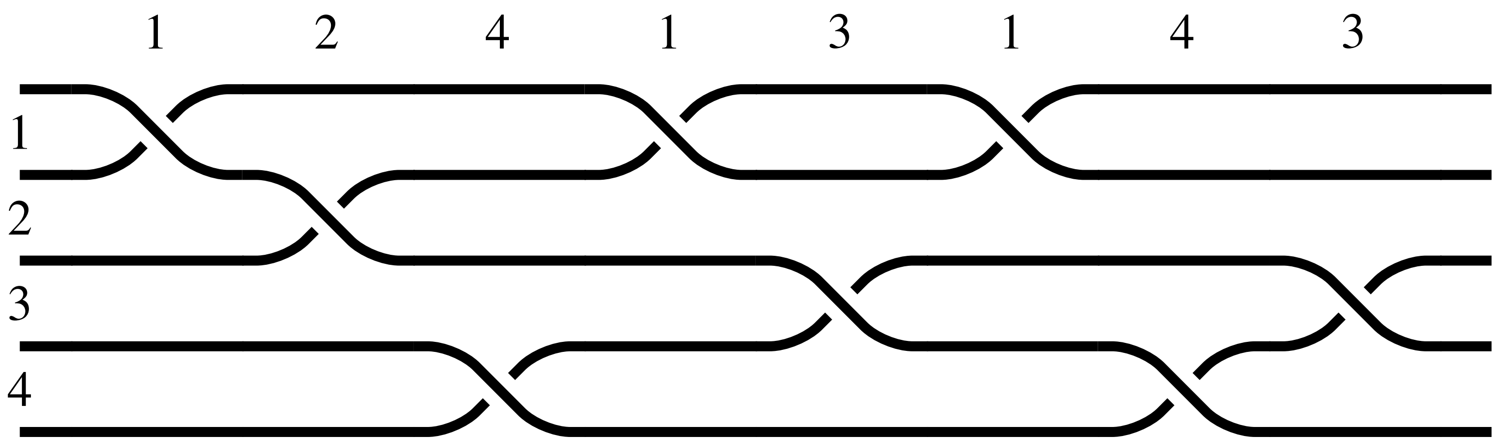


Veraindrung der Coordiniz

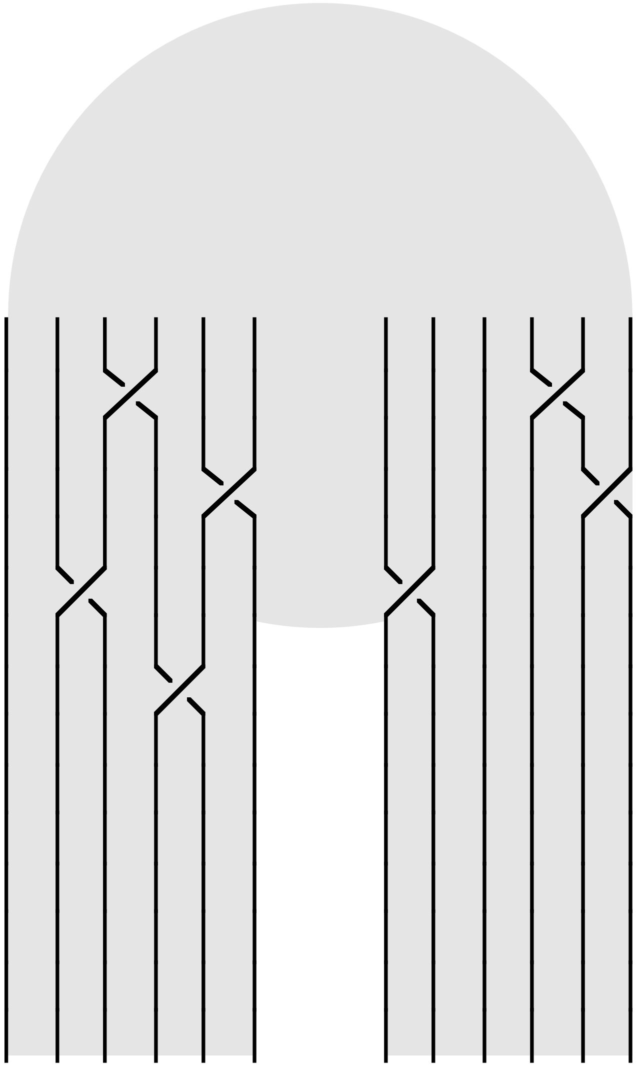
a	1	1	$2+i$	$3+i$	$2+2i$	$2+2i$
b	2	2	1	1	1	1
c	3	4	4	4	4	3
d	4	$3+i$	$3+i$	$2+2i$	$3+2i$	$4+3i$

notation of Braids



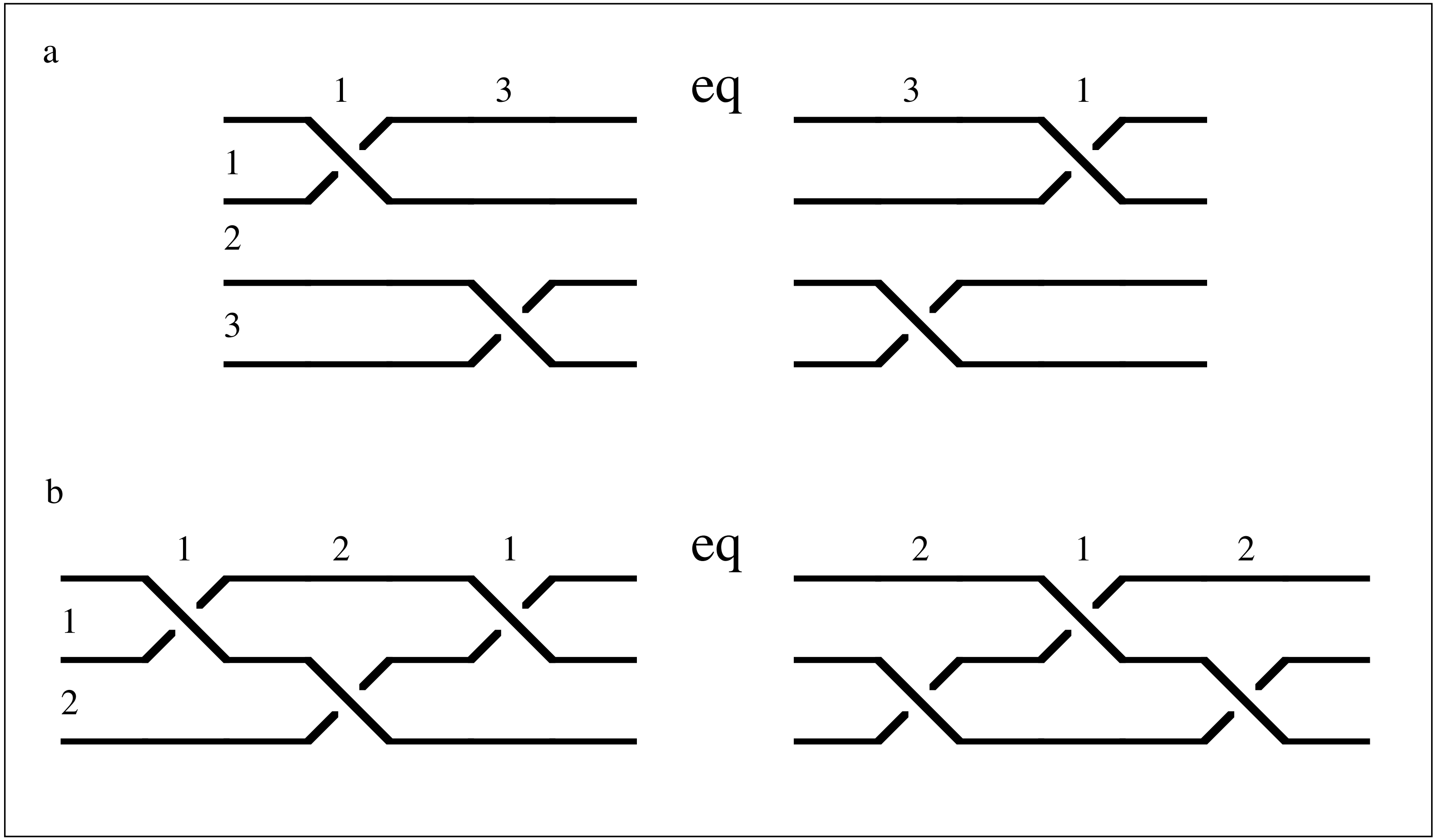


braiding problem



Girl with two braids

Artin's braid equations



braid equations as e.d.'s

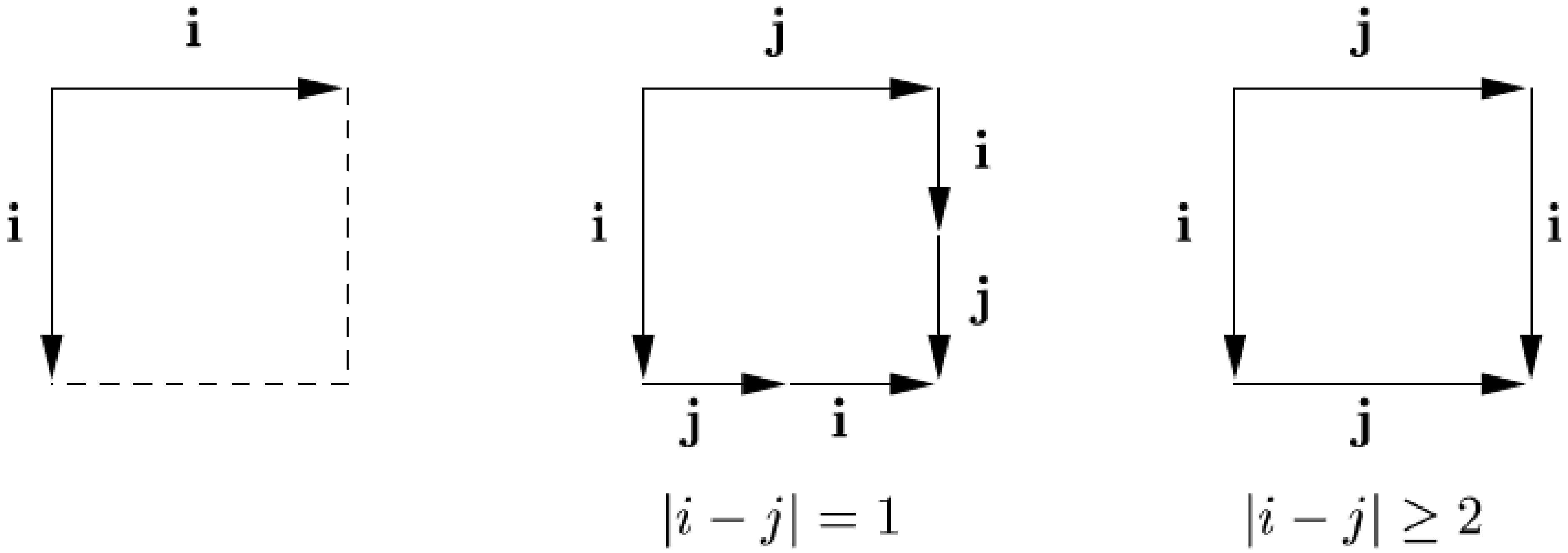
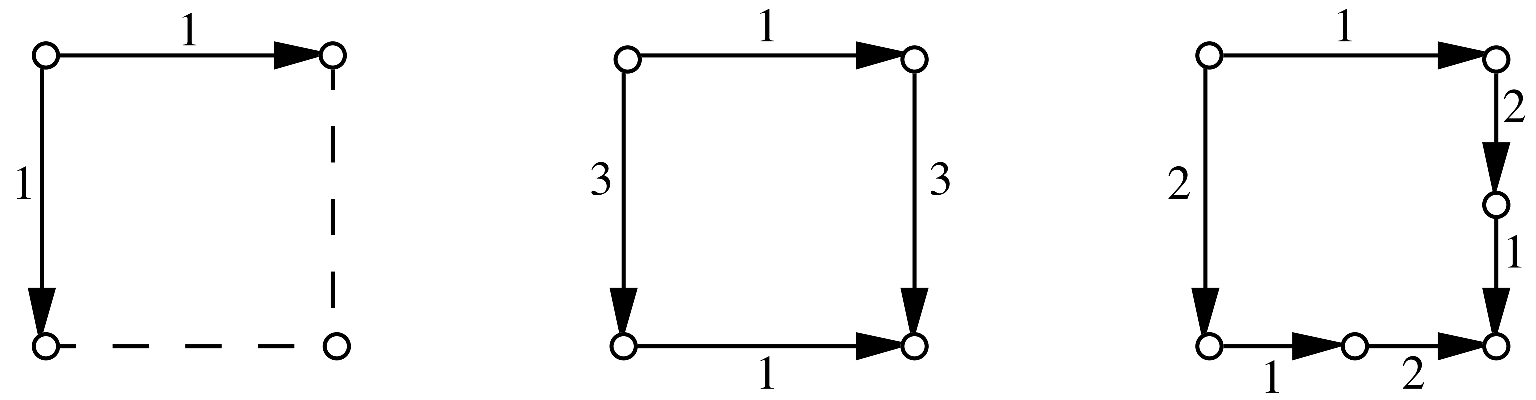
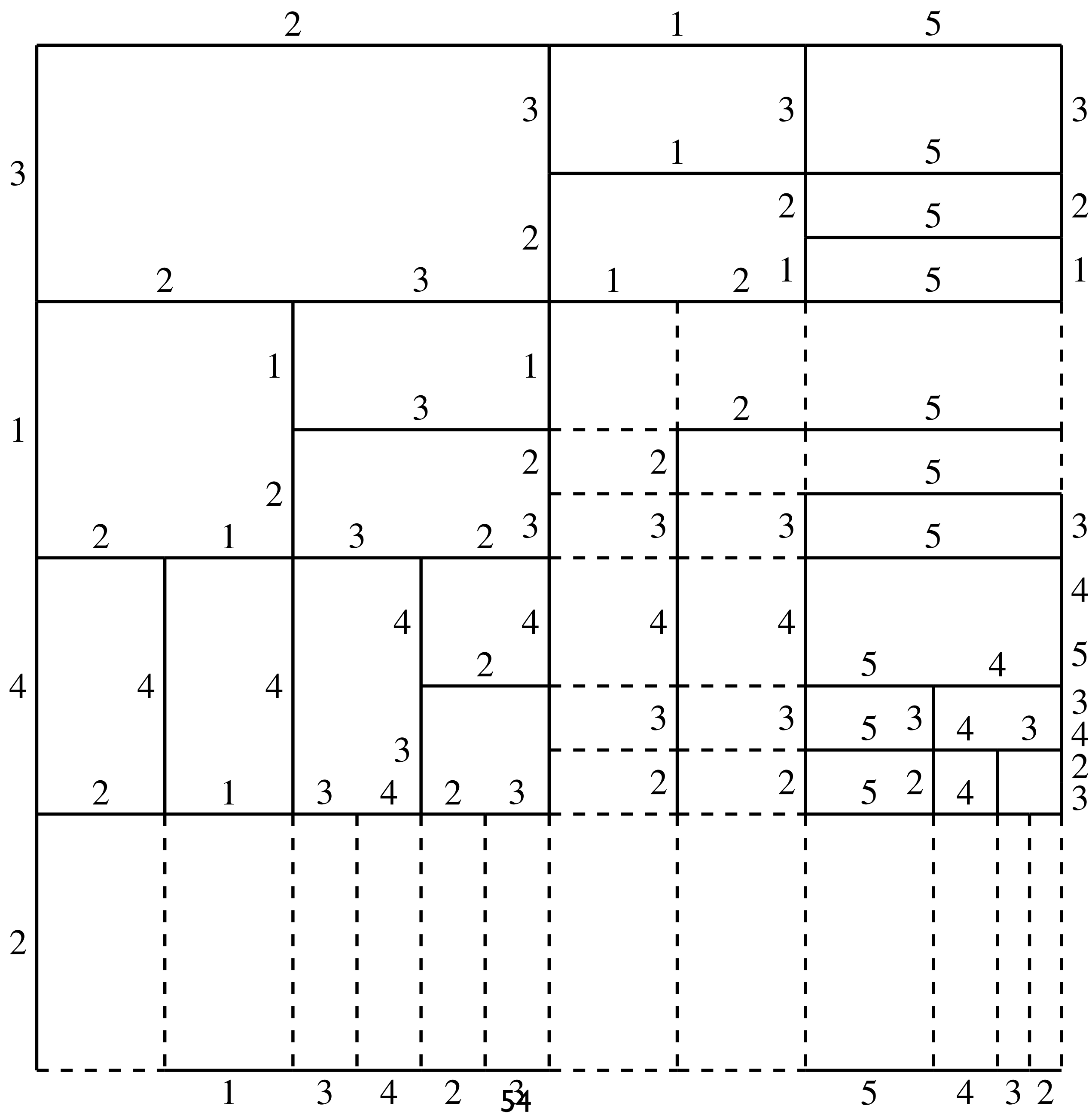


Figure 4: Elementary diagrams $(1 \leq i, j < n)$



elementary diagrams for confluence problem in braid semi-group

completed braid reduction diagram



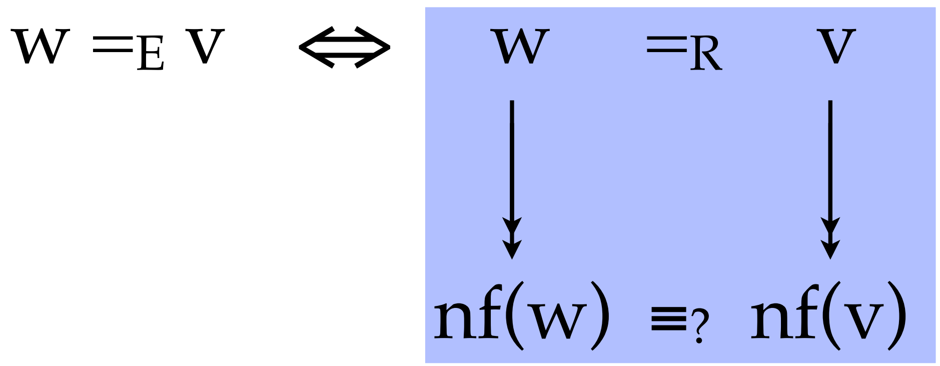
aba = bab and the need for signature extension

Kapur-Narendran 1985:
the monoid aba=bab has decidable equality (word problem), but there is no complete SRS generating this equality, like for D_4.

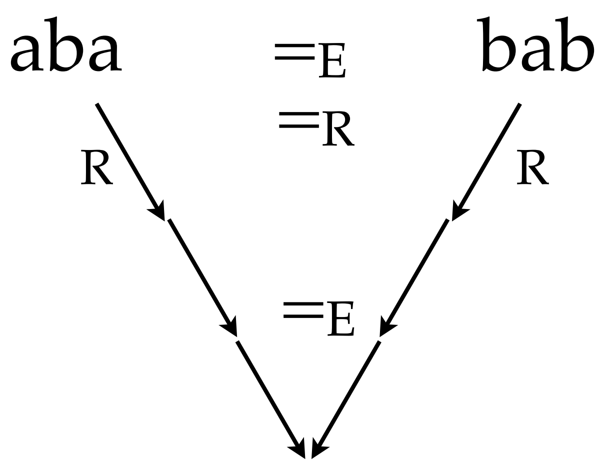
However, with extra symbols (signature extension) there is.
ab = c, ca = bc.
After completion:
ab=c, ca=bc, bcb=cc, ccb=acc.

Equality given by E = {aba =bab} on a,b-words is decidable, as each E-equivalence class is finite, because applying E preserves length.

Can we implement the decidability by a complete TRS R such that



NO!



u, in E-equivalence class of aba and bab, must be either aba or bab. In both case R is cyclic, hence not SN.

Another SRS with this phenomenon is $abba = e$*, defining even a group.*

Question: what signature extension plus equations would admit a complete TRS?

Same question for: $E = \{f(x,y) = f(y,x)\}$*, generator* o*.*
Closed terms are finite commutative trees; decidable equality, but no complete TRS in same signature.

In algebraic data type theory / universal algebra similar: if the equality is decidable, a signature extension yields a complete orthogonal TRS for it. (Hidden sorts and functions.)

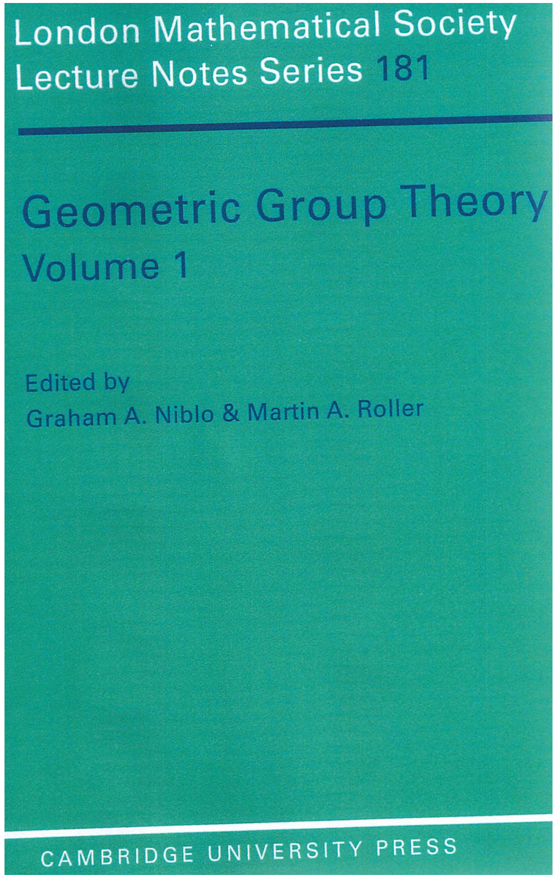
Theorem 2.14 ((Bergstra & Tucker (80)). *Let $\mathscr{A}$ be a minimal Σ-algebra, Σ a finite signature. Then the following are equivalent:*

(i) $\mathscr{A}$ is a computable algebra;

(ii) there is an extension of Σ to a finite Γ, obtained by adding some function and constant symbols, and there is a complete TRS (Γ, R) such that

$$\mathscr{A} \equiv I(\Gamma, R^{=}) \mid_{\Sigma} .$$

Another solution by Burckel-Riviere 2001:
1 → *1,*
212 → 12*1*
2122 → 1212
1211 → 2121

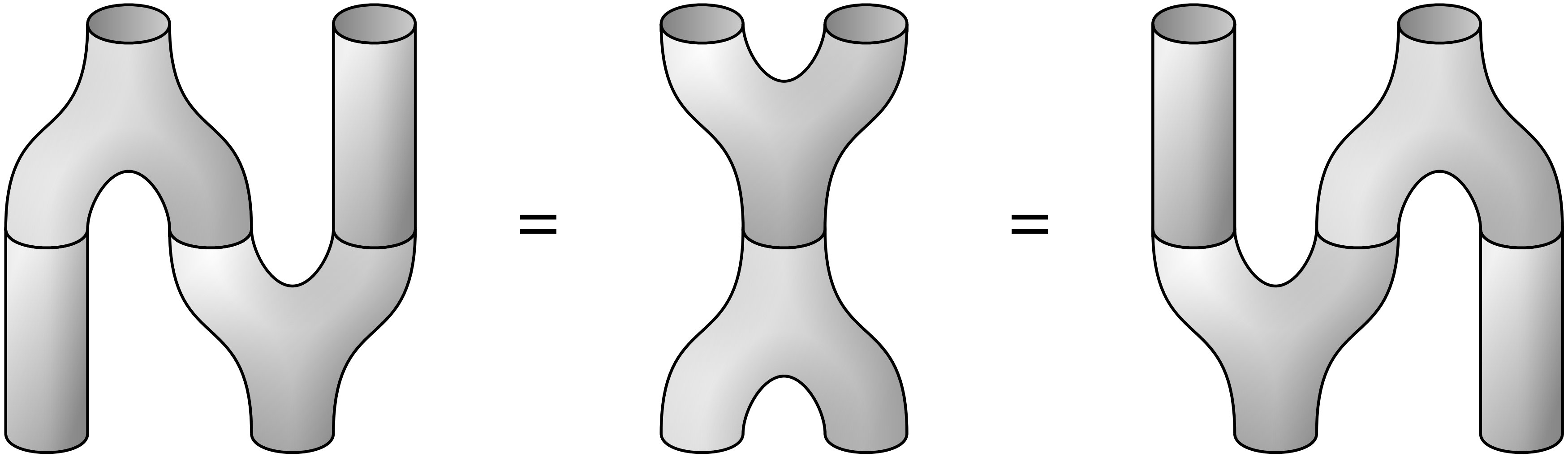


Remarkably, the word problem for monoids is not dependent on the actual presentation.

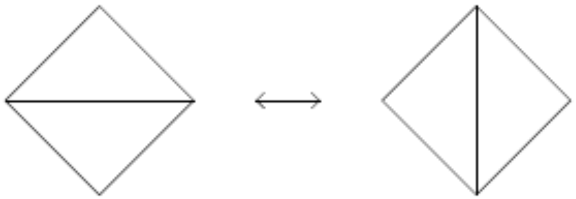
Shown by Tietze transformation rules.

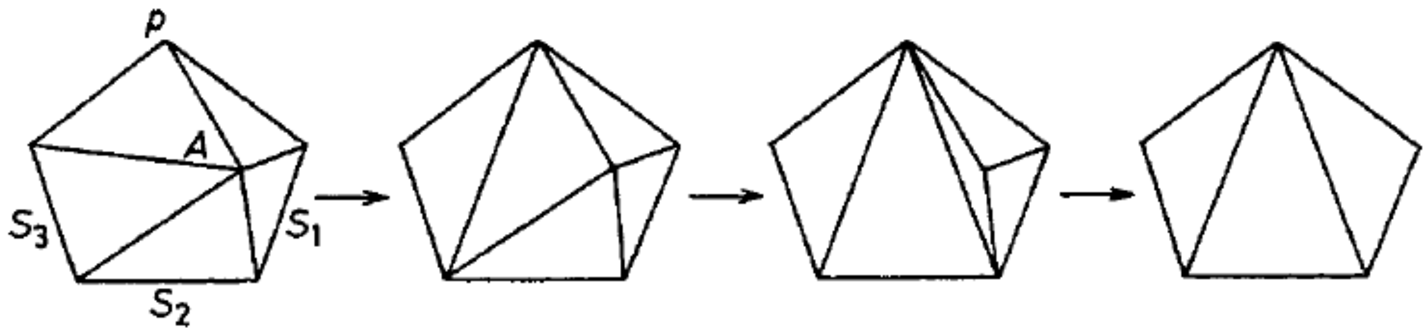
The same holds for a large class of Sigma-algebras.
(Pers. comm. by V. van Oostrom, June 2012.

axioms in Frobenius algebras



Pachner moves: for transforming different triangulations of topological surfaces into each other





Prijsvraag **Het Cola-gen**

Een team ... tische manipuleer-
ders ... riceren die
ge... n. Daartoe
moet... chte
DN... van het melkgen:
TAGCTAGCTAGCT
ombouwen tot het cola-
CTGACTGACT
Er zijn technieken ter beschikking om de volgende DNA-substituties – heen en weer – uit te voeren:

TCAT ↔ T
GAG ↔ AG
CTC ↔ TC
AGTA ↔ A
TAT ↔ CT

given DNA-string

transform it to

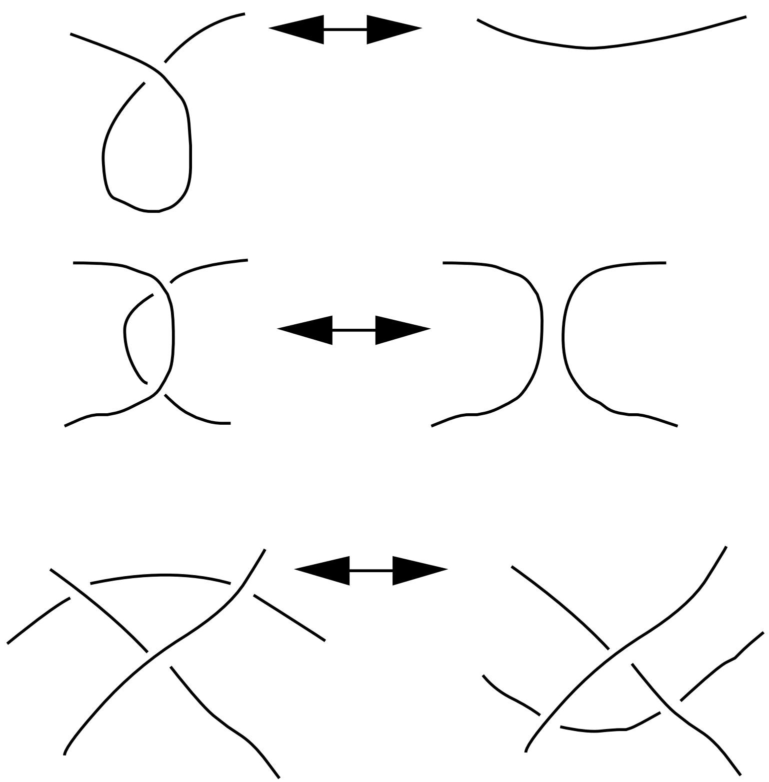
using

Kort daarvoor was echter o...
de gekke-koeienziekte wo...
zaakt door een retro-virus ...
DNA-volgorde:
CTGCTACTGACT
Wat nu, als onbedoeld koeien met dit virus ontstaan? Volgens de manipuleerders loopt dit zo'n vaart niet omdat het bij al hun experimenten nog nooit gebeurd is, maar diverse actiegroepen, zich beroepend op het voorzorgbeginsel, eisen keiharde garanties.
Hoe bewijs je dat dit virus nooit kan ontstaan? Het aantal mogelijke combinaties van substituties is vrijwel eindeloos, dus een slimme redenatie is hier nodig. Het maken van het cola-gen vergt wel behoorlijk wat gepuzzel.

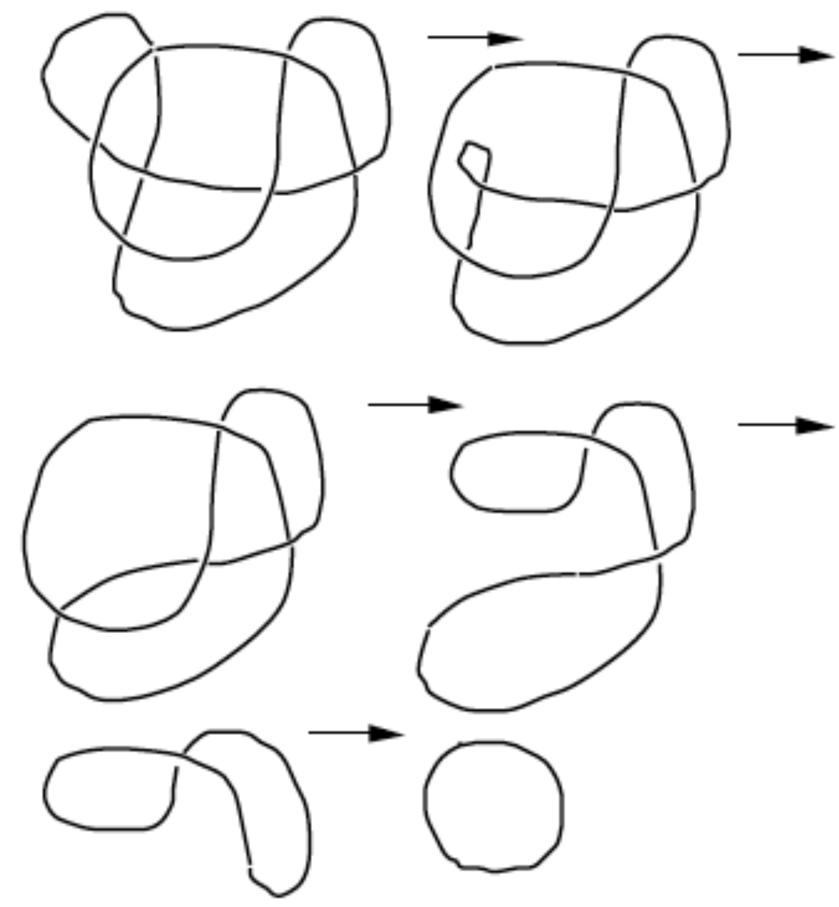
but avoid BSE virus

Zorg dat de oplossing uiterlijk 7 januari 2005 bij de Prijsvraagredactie is, NW&T, postbus 256, 1110 AG Diemen, of *prijsvraag@natutech.nl* o.v.v. Prijsvraag januari.
De winnaar ontvangt een cadeaubon voor Natuurwetenschap&Techniek-producten van€ 35,-.
De prijsvraag voor februari staat vanaf maandag 17 januari al op *www.natutech.nl*.

Reidemeister moves to transform knots into each other



Reidemeister moves



0. A few words on history

1. rewriting dictionary

2. two theorems in abstract rewriting

3. word rewriting: monoids and braids

tea, coffee

---

4. term rewriting: divide et impera; termination by stars

5. Lambda calculus and combinatory logic

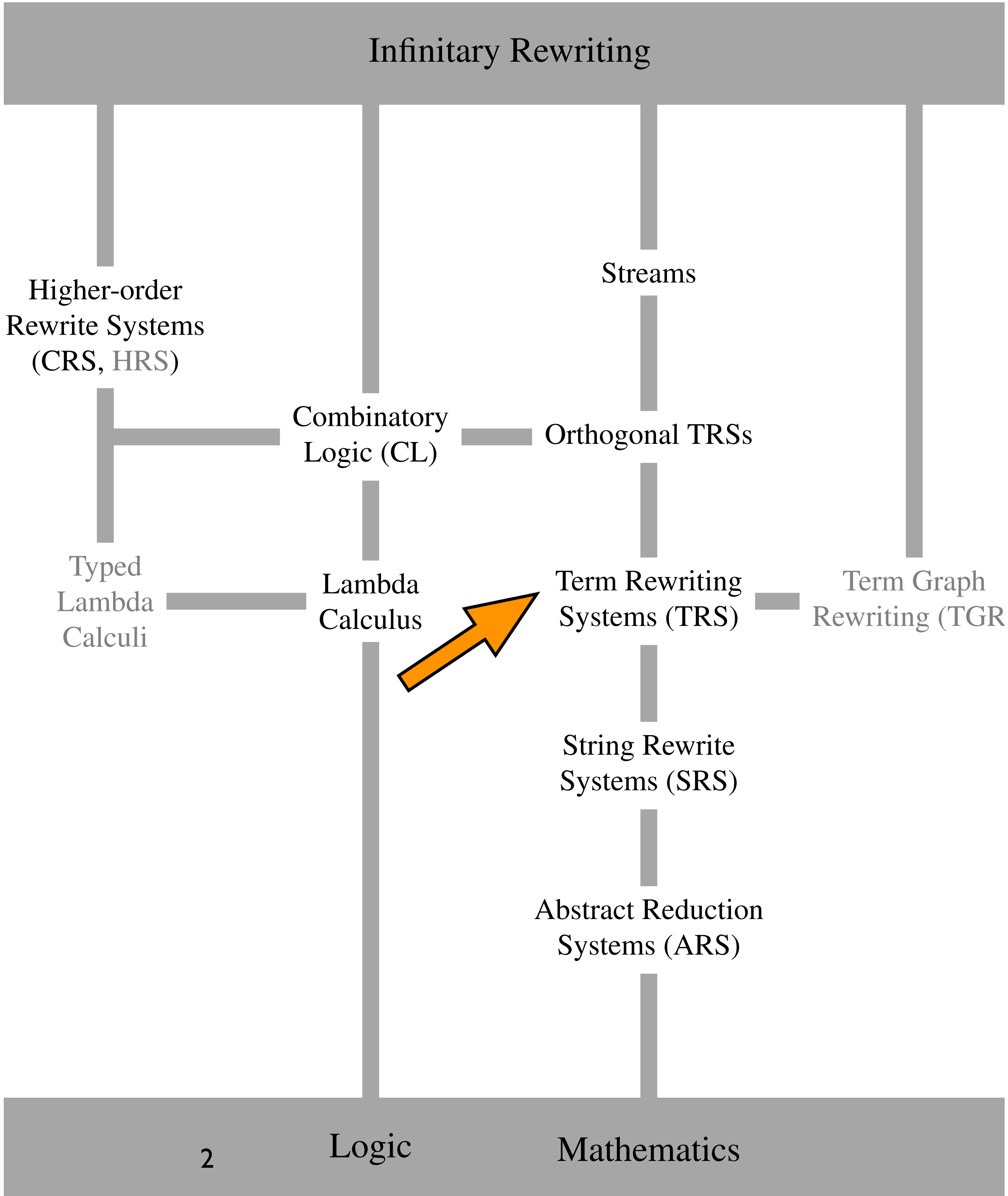
6. Infinitary rewriting

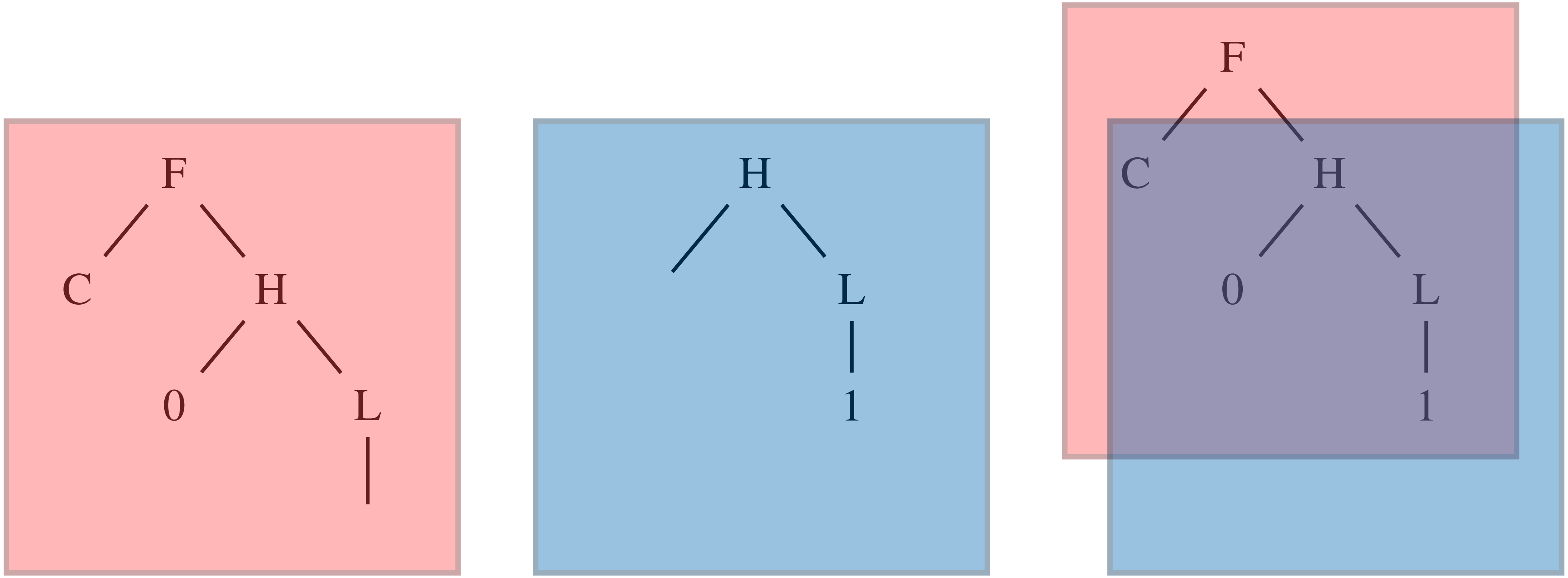
tea, coffee

7.infinitary lambda calculus and the threefold path

8.clocked semantics of lambda calculus

9.streams running forever

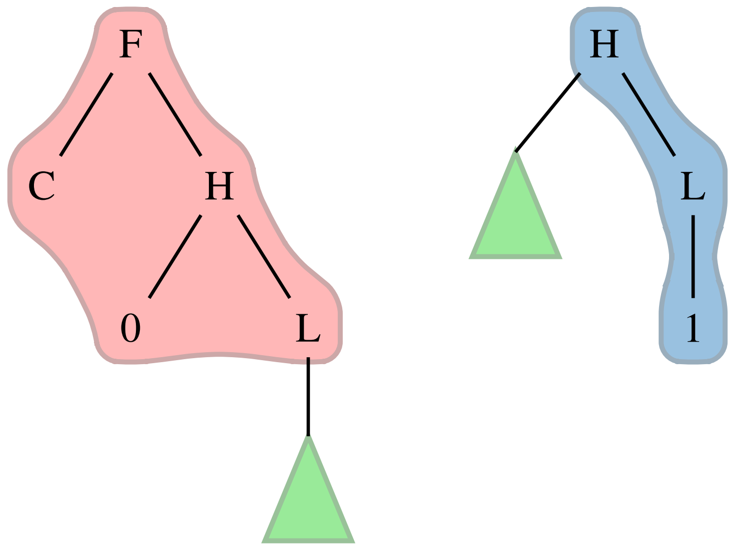

Infinitary Rewriting
Higher-order Rewrite Systems (CRS, HRS)
Streams
Combinatory Logic (CL)
Orthogonal TRSs
Typed Lambda Calculi
Lambda Calculus
Term Rewriting Systems (TRS)
Term Graph Rewriting (TGR
String Rewrite Systems (SRS)
Abstract Reduction Systems (ARS)
Logic
Mathematics

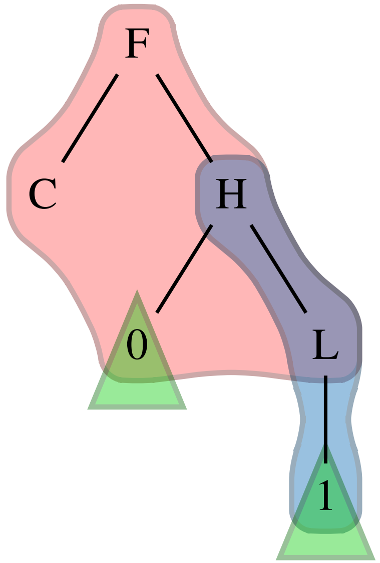


slide 1 slide 2 overlap

$$r_1 : F(C, H(0, L(x))) \rightarrow L(x)$$
$$r_2 : H(y, L(1)) \rightarrow H(y, y)$$

The term arising from this superposition, $F(C, H(0, L(1)))$, is now subject to two rewritings, as follows.

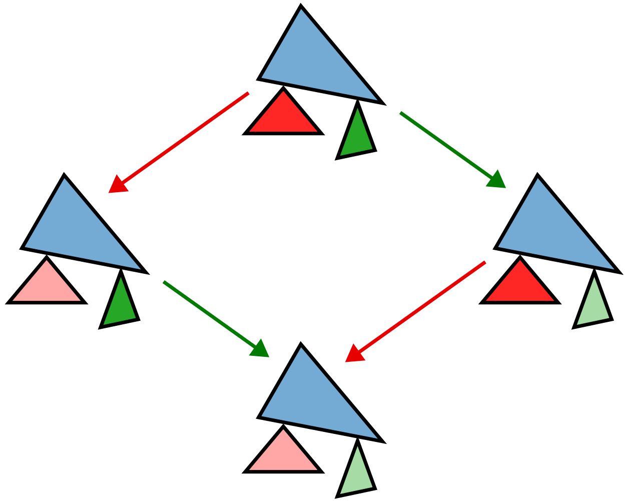




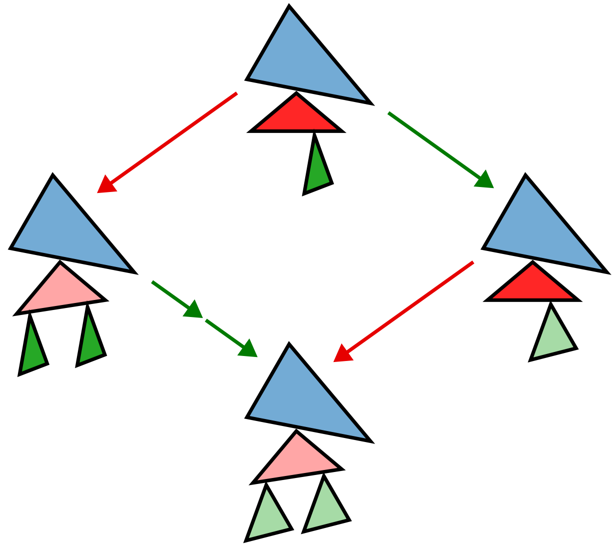
$$F(C,H(0,L(1))) \rightarrow_{r_1} L(1)$$
$$F(C,H(0,L(1))) \rightarrow_{r_2} F(C,H(0,0))$$

Now $\langle L(1), F(C,H(0,0))\rangle$ is the critical pair generated by this overlapping between r_1 and r_2.

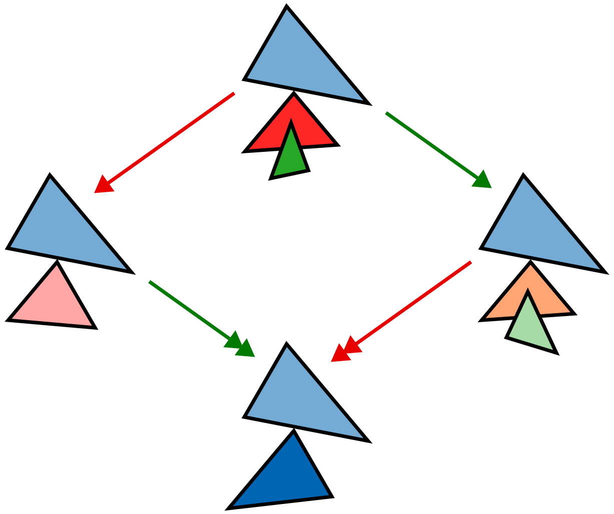
Theorem 5.3. *(Huet [Hue80]) A TRS is weakly confluent iff all its critical pairs $\langle s,t \rangle$ are convergent, i.e. $s \downarrow t$, in words: s and t have a common reduct.*



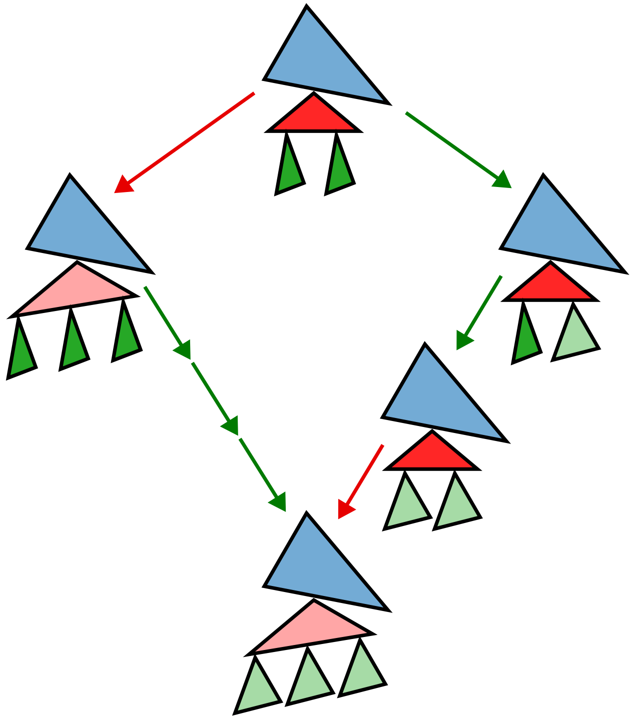
(a) Disjoint redexes



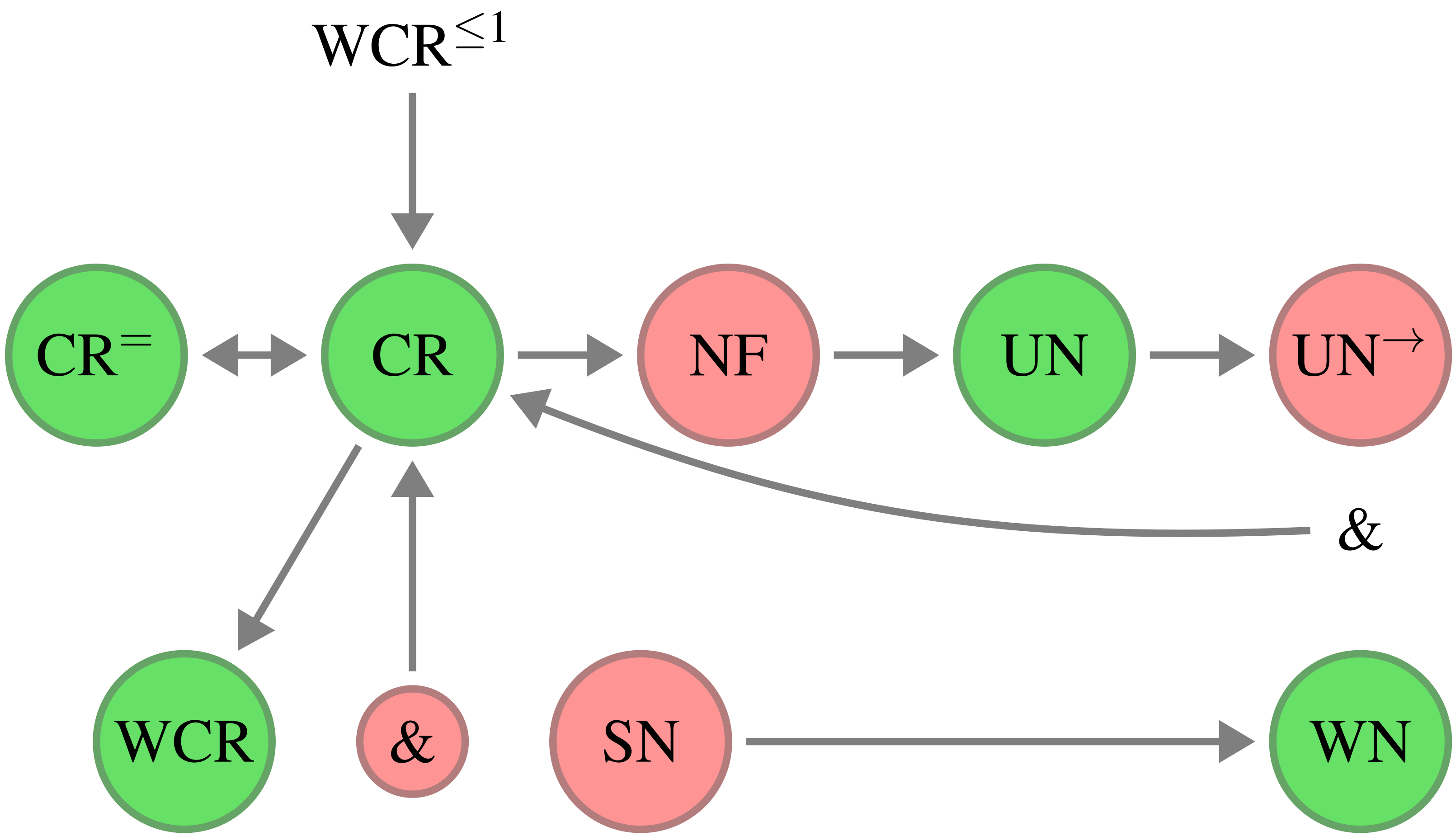
(b) Nested redexes

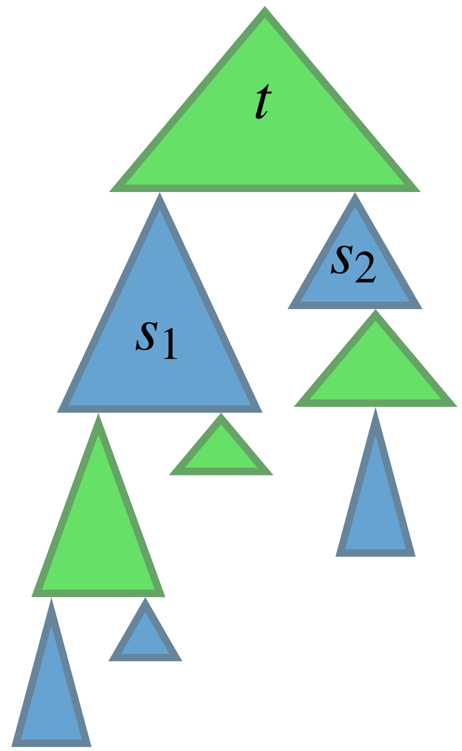


(c) Overlapping redexes



(d) Non-left-linear redexes


$WCR^{\leq 1}$
$CR^{=}$
CR
NF
UN
$UN^{\rightarrow}$
&
WCR
&
SN
WN


t
s1
s2

$A(x,0) \rightarrow x$

$A(x,S(y)) \rightarrow S(A(x,y))$

$F(0) \rightarrow 0$

$F(S(x)) \rightarrow A(F(x),S(x))$

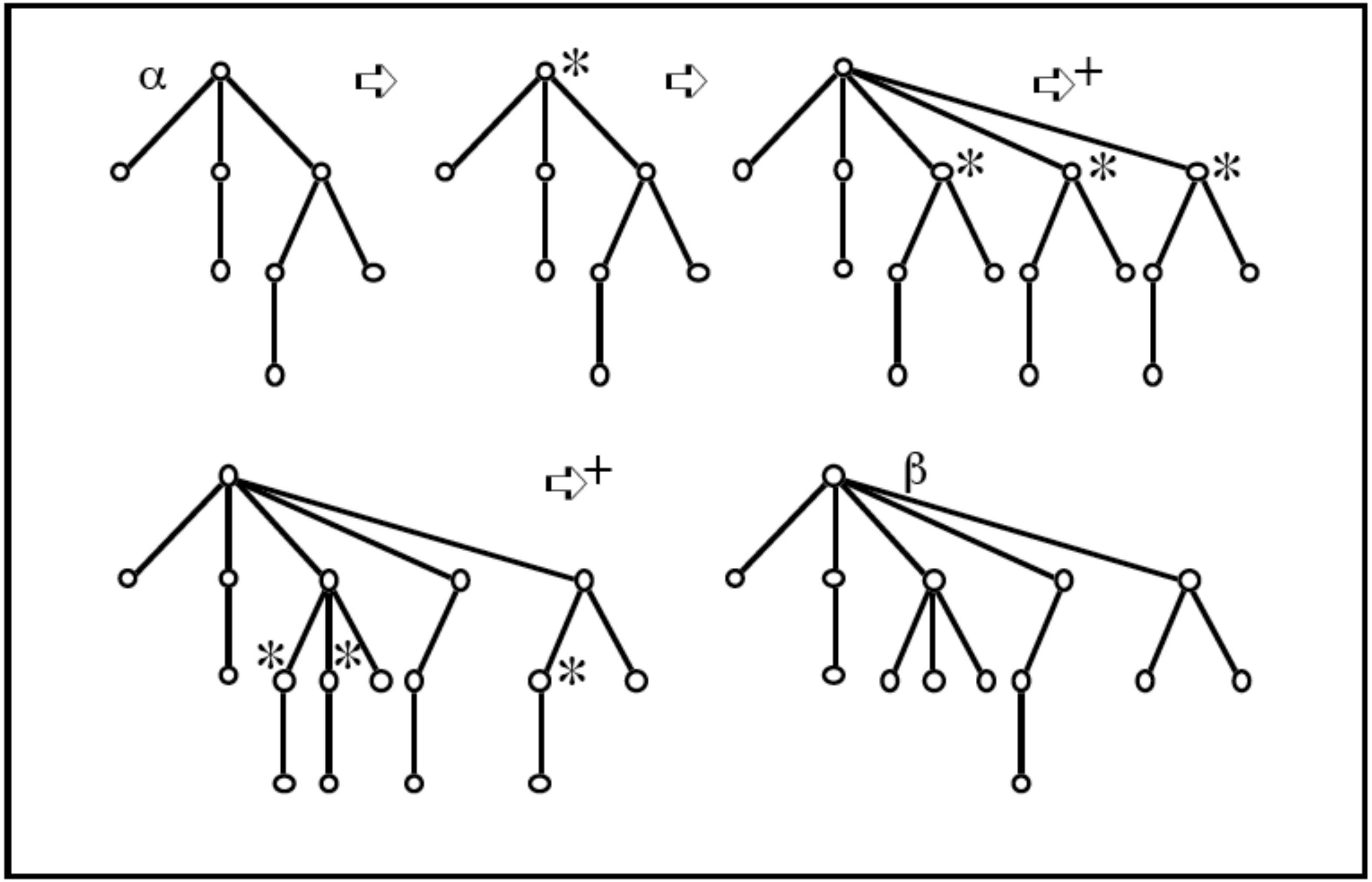
$\mathcal{R}_2$

$M(x,0) \rightarrow 0$

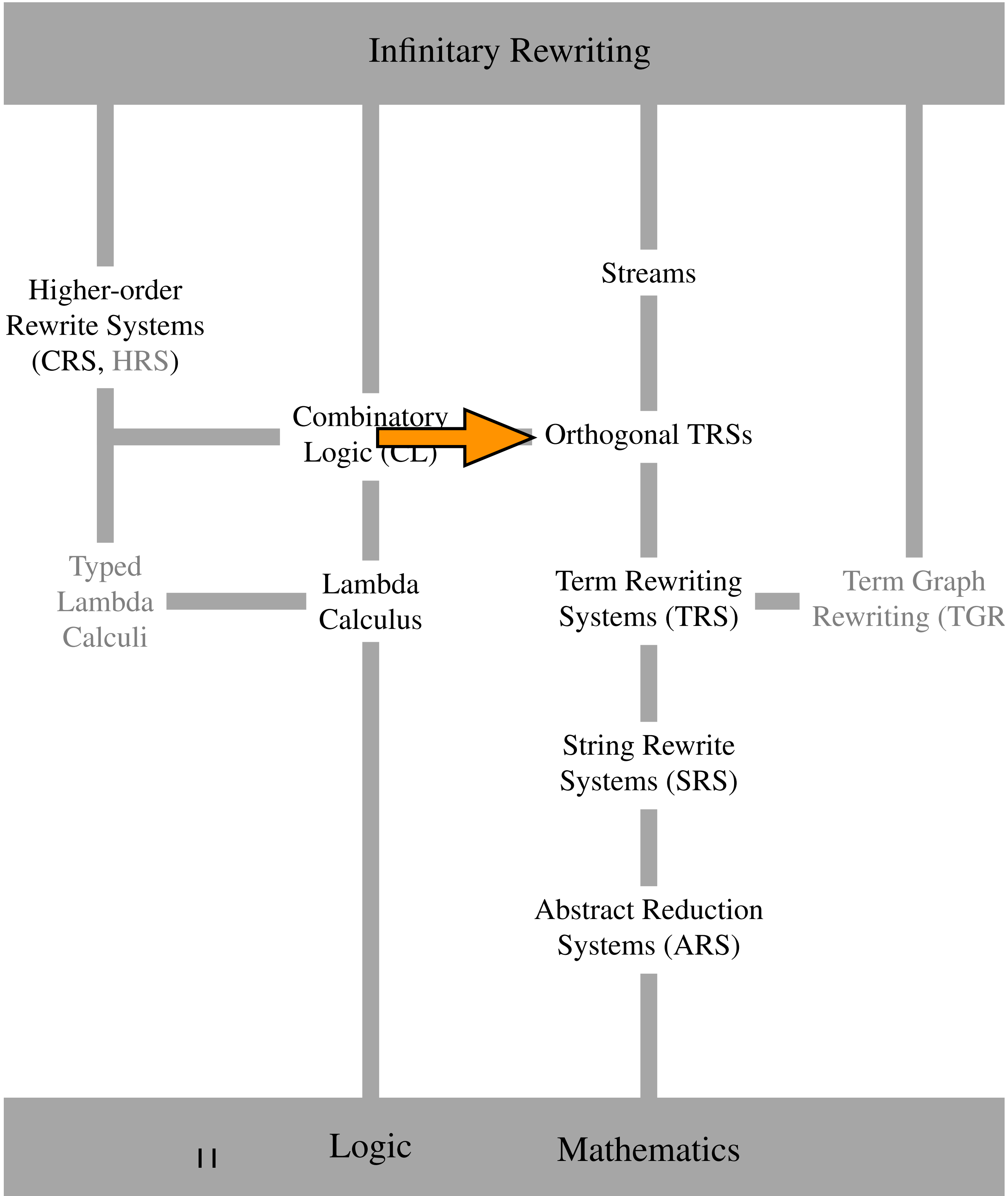
$M(x,S(y)) \rightarrow A(M(x,y),x)$

$\mathcal{D} = \mathcal{R}_1$

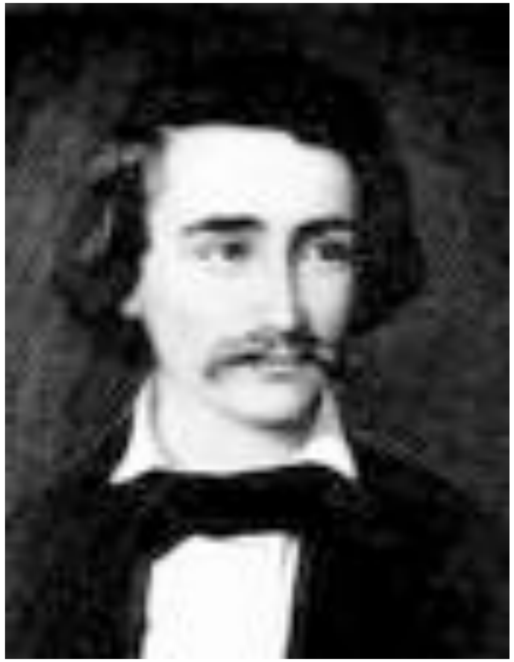
4. term rewriting: divide et impera; termination by stars



Some streets we want to walk



Grassmann 1861, Dedekind 1888



$A(x, 0) \to x$

$A(x, S(y)) \to S(A(x, y))$

$M(x, 0) \to 0$

$M(x, S(y)) \to A(M(x, y), x)$

$A(x, 0) \to x$

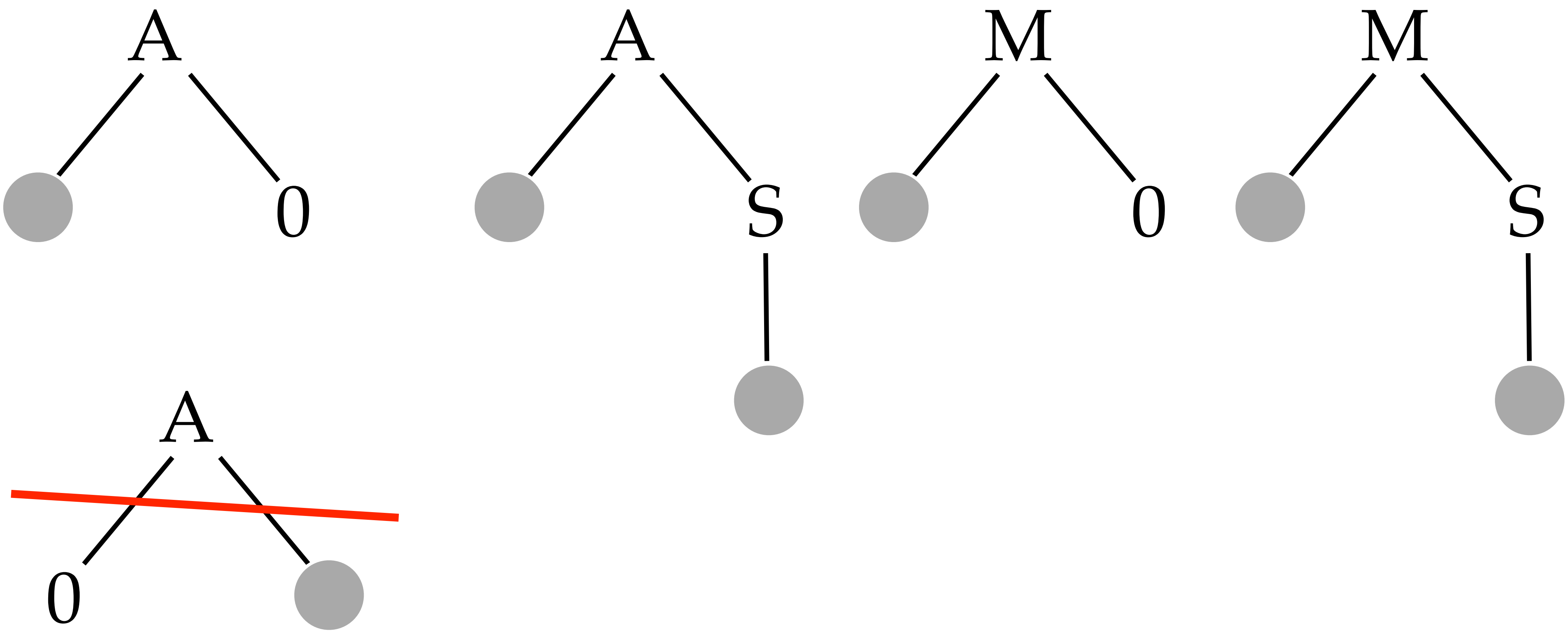
$A(x, S(y)) \to S(A(x, y))$

$M(x, 0) \to 0$

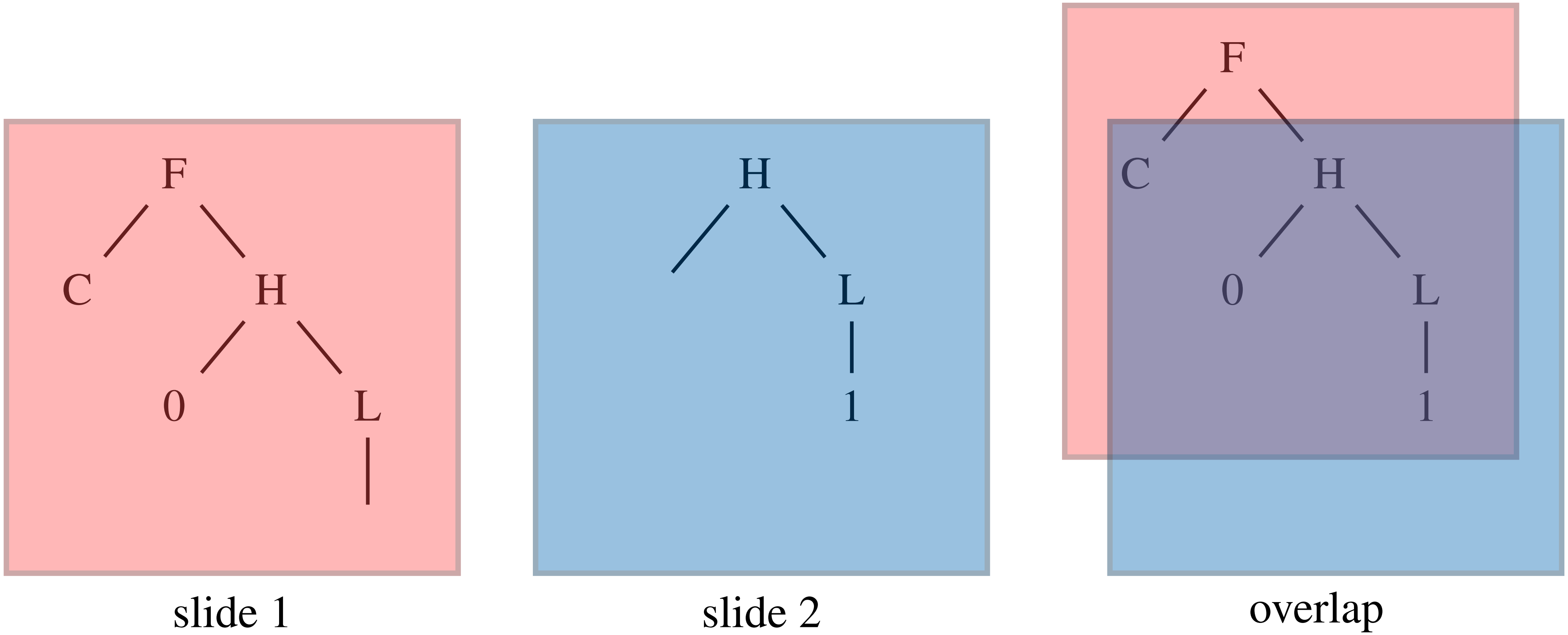
$M(x, S(y)) \to A(M(x, y), x)$

left linear

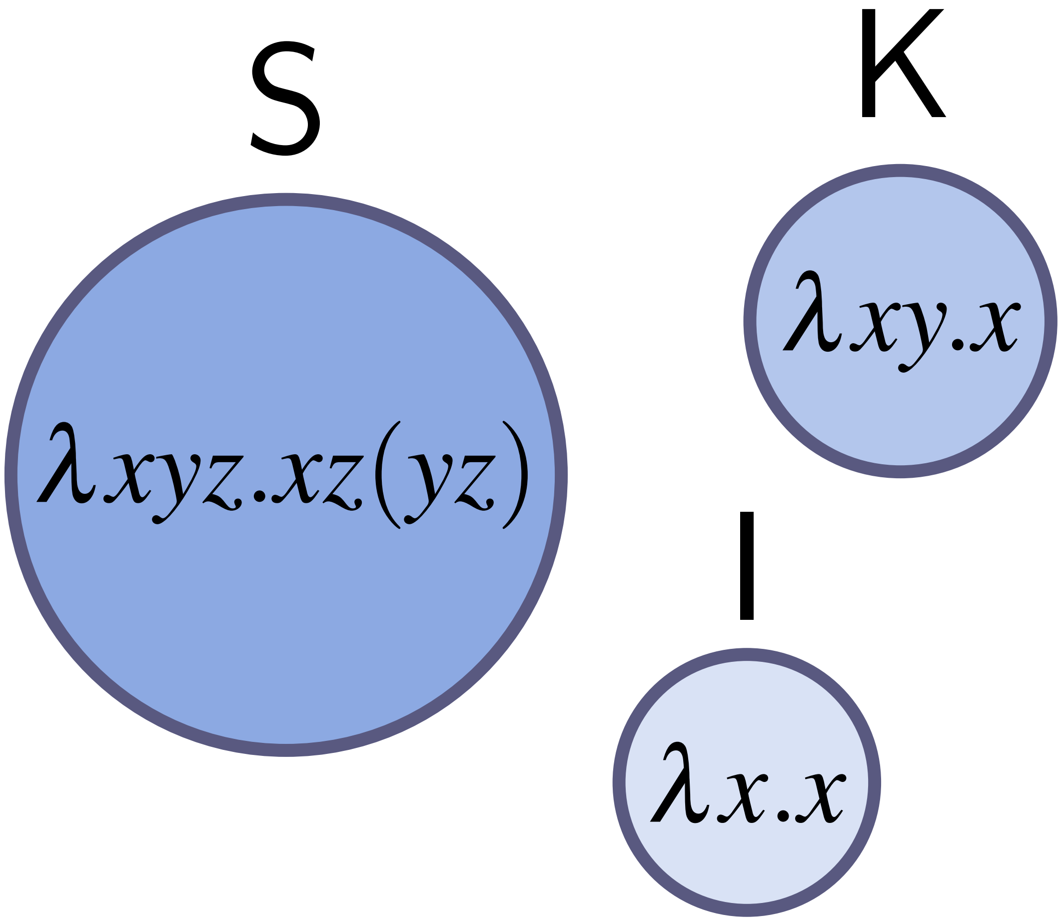
non-overlapping rules



orthogonal TRSs: no overlaps



and no repeated variables

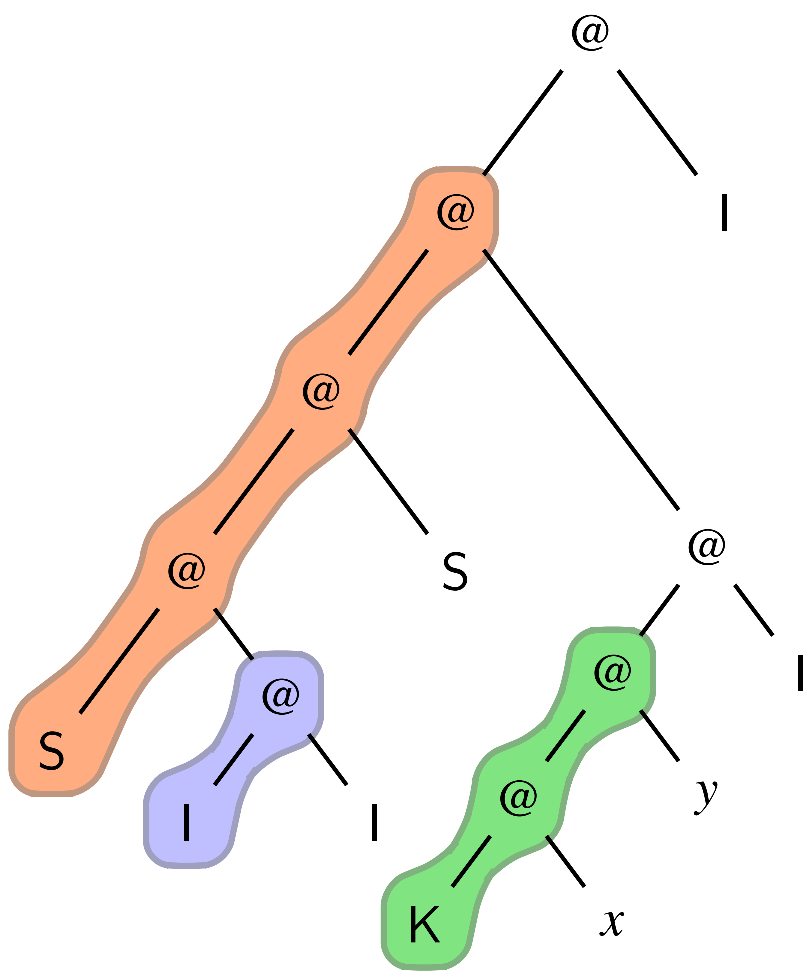


1924. "Über die Bausteine der mathematischen Logik"

Moses Schönfinkel

Combinatory Logic
Turing complete

$$Ix \longrightarrow x$$
$$Kxy \longrightarrow x$$
$$Sxyz \longrightarrow xz(yz)$$

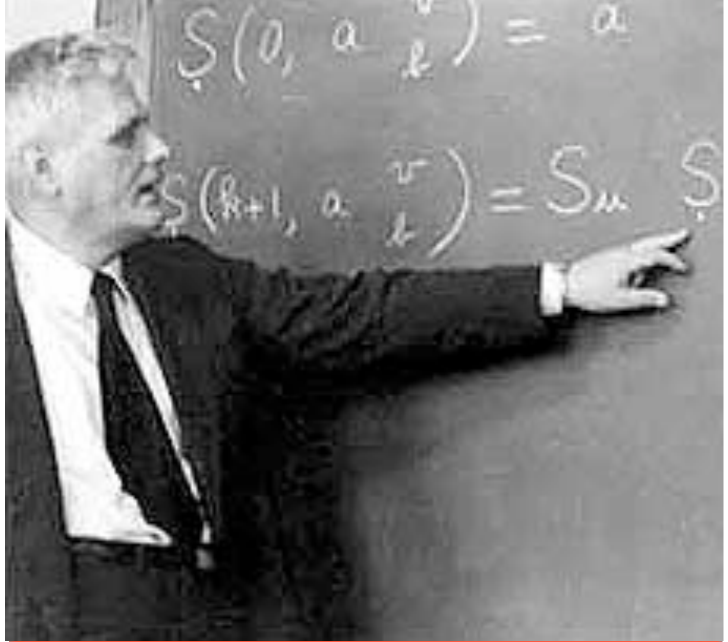


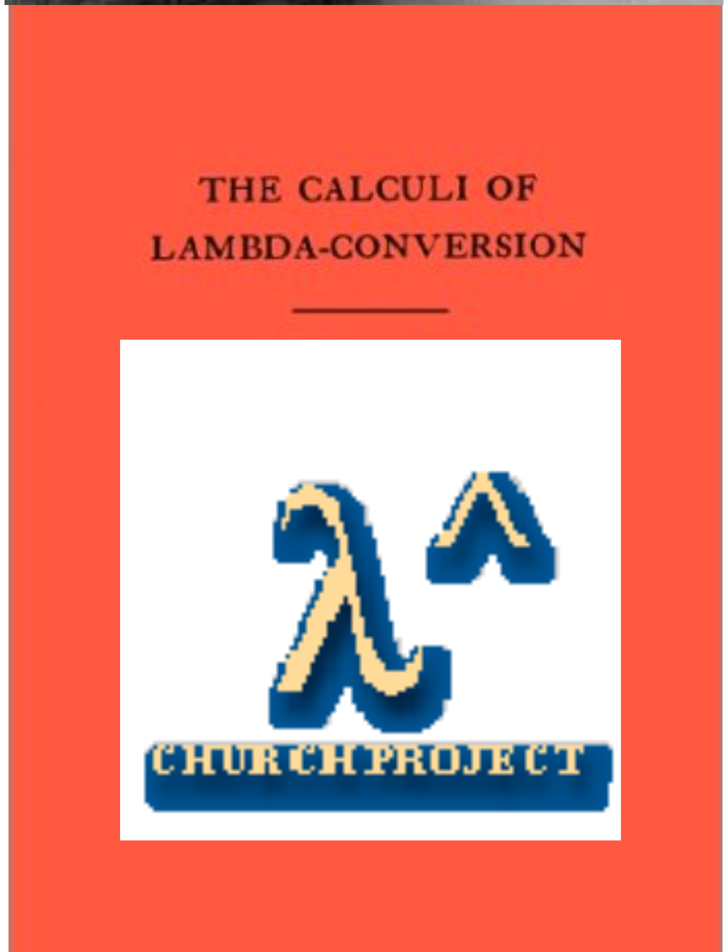
orthogonal, hence confluent

Alonzo Church

1903- 1995

At the time of his death, Church was widely regarded as the greatest living logician in the world

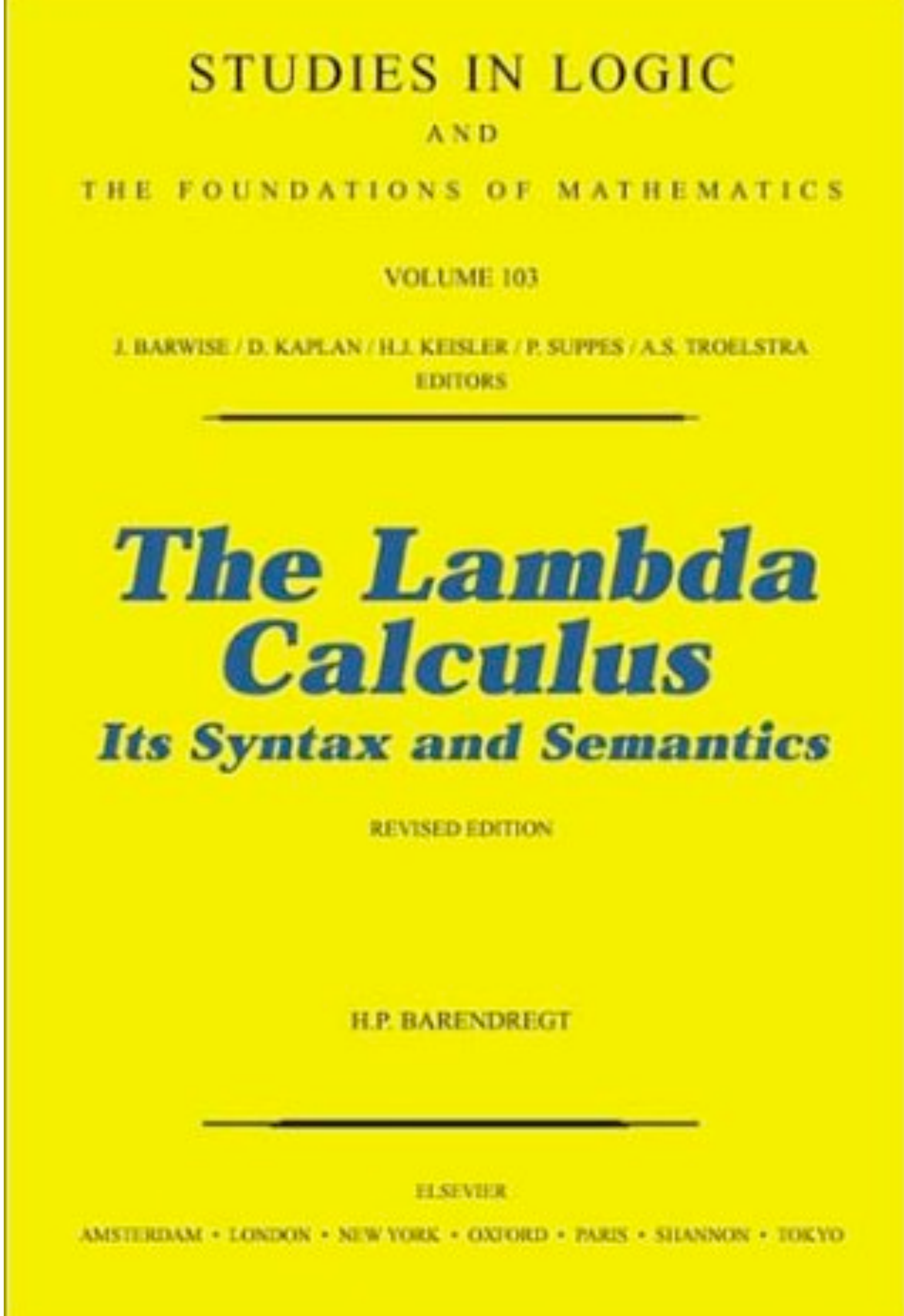


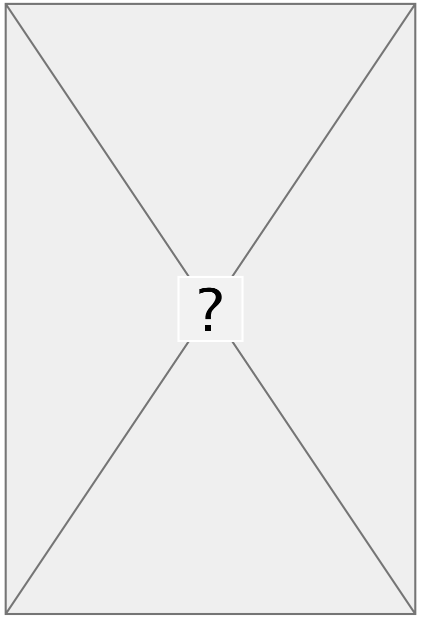


Lambda Calculus

$$(\lambda x.Z(x))Y \longrightarrow Z(Y)$$

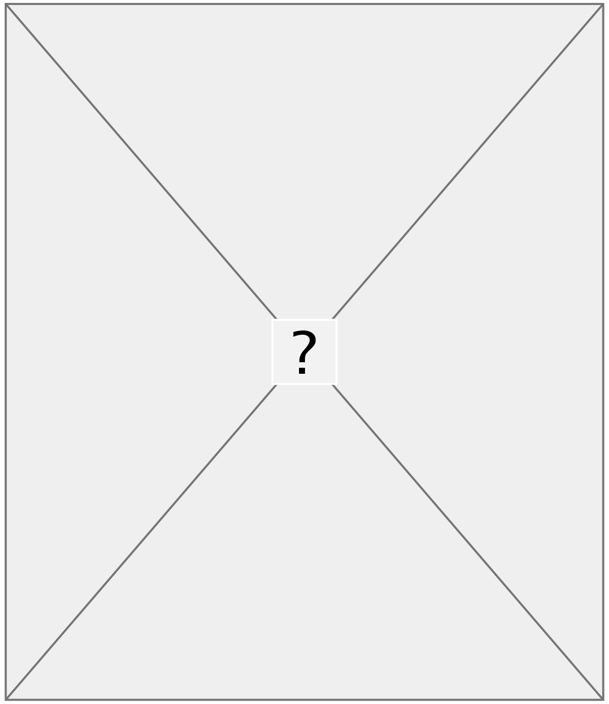
Turing complete





(λx.xx)(λx.xx)

ur-cycle



pure 3-cycle

Not in CL!

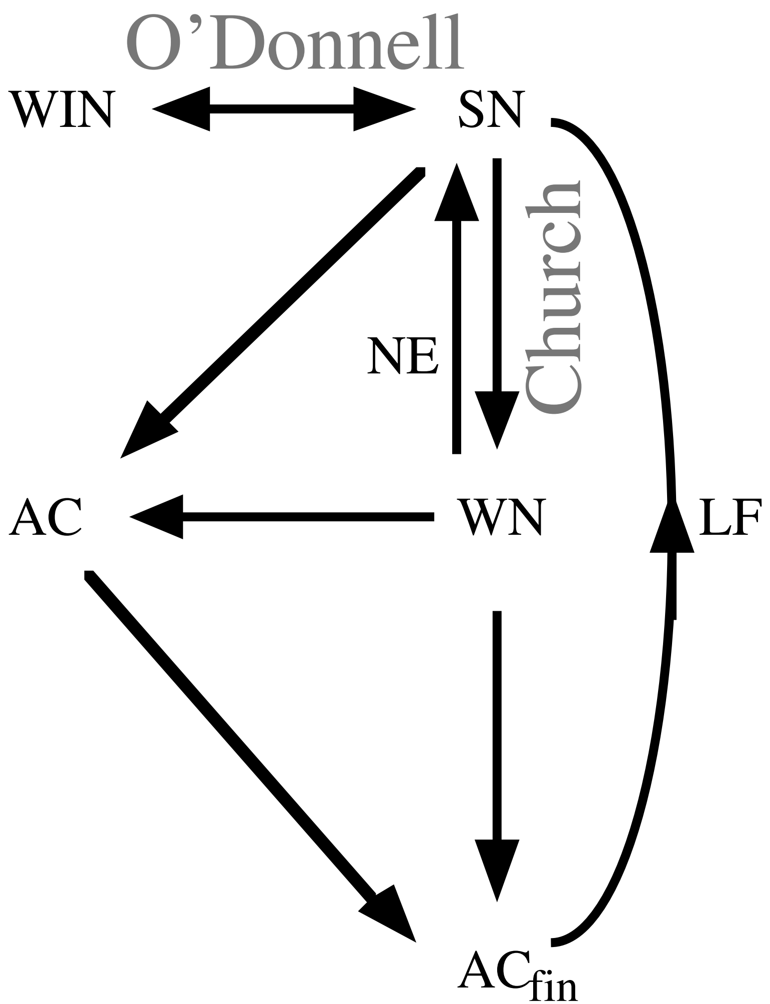
M.H. Sorensen:

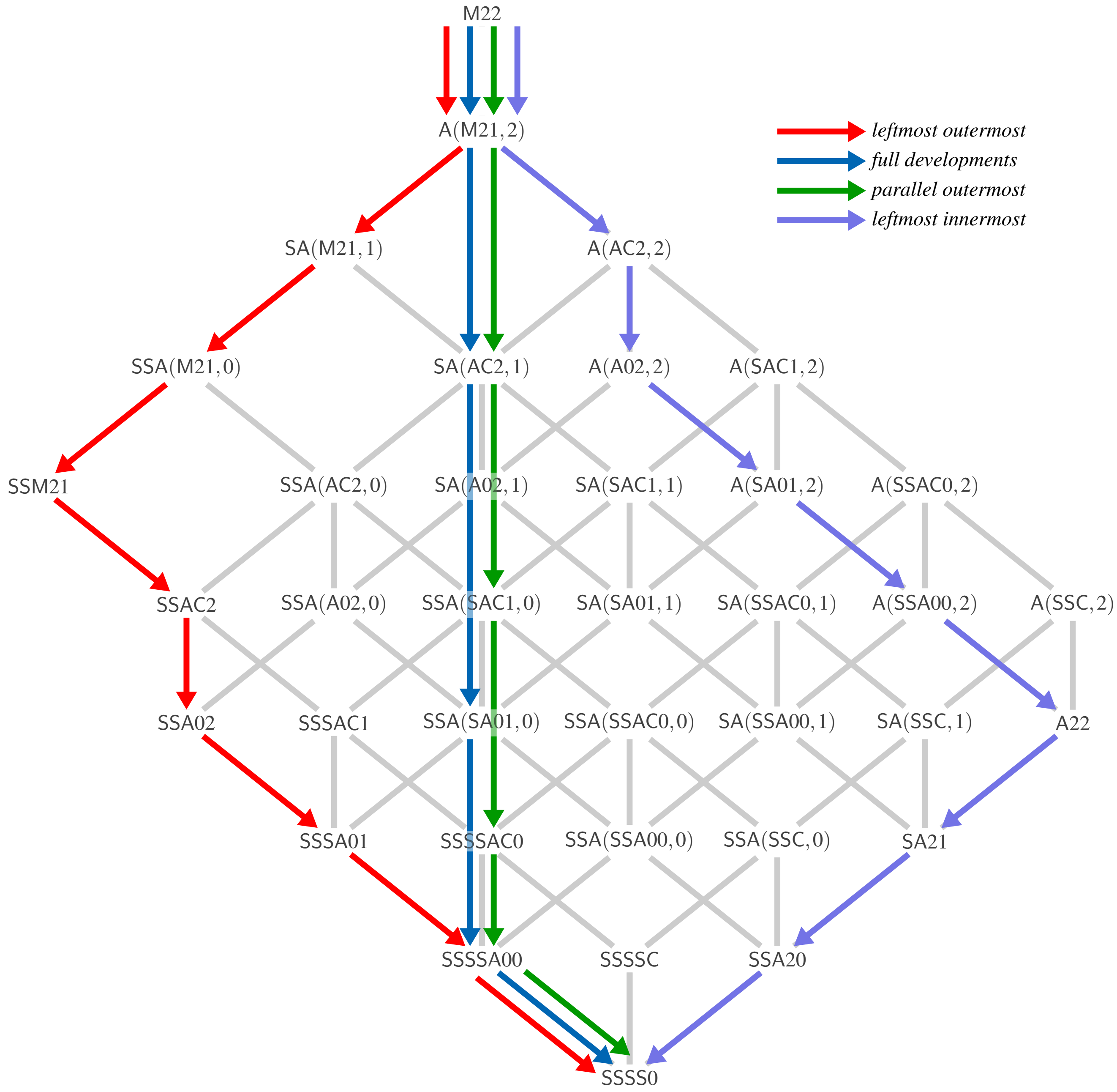
λ-*term has infinite reduction* $\Rightarrow$
$(\lambda x.xx)(\lambda x.xx)$ *is a subword*

$$(\lambda xy.y(xxy))(\lambda xy.y(xxy))$$

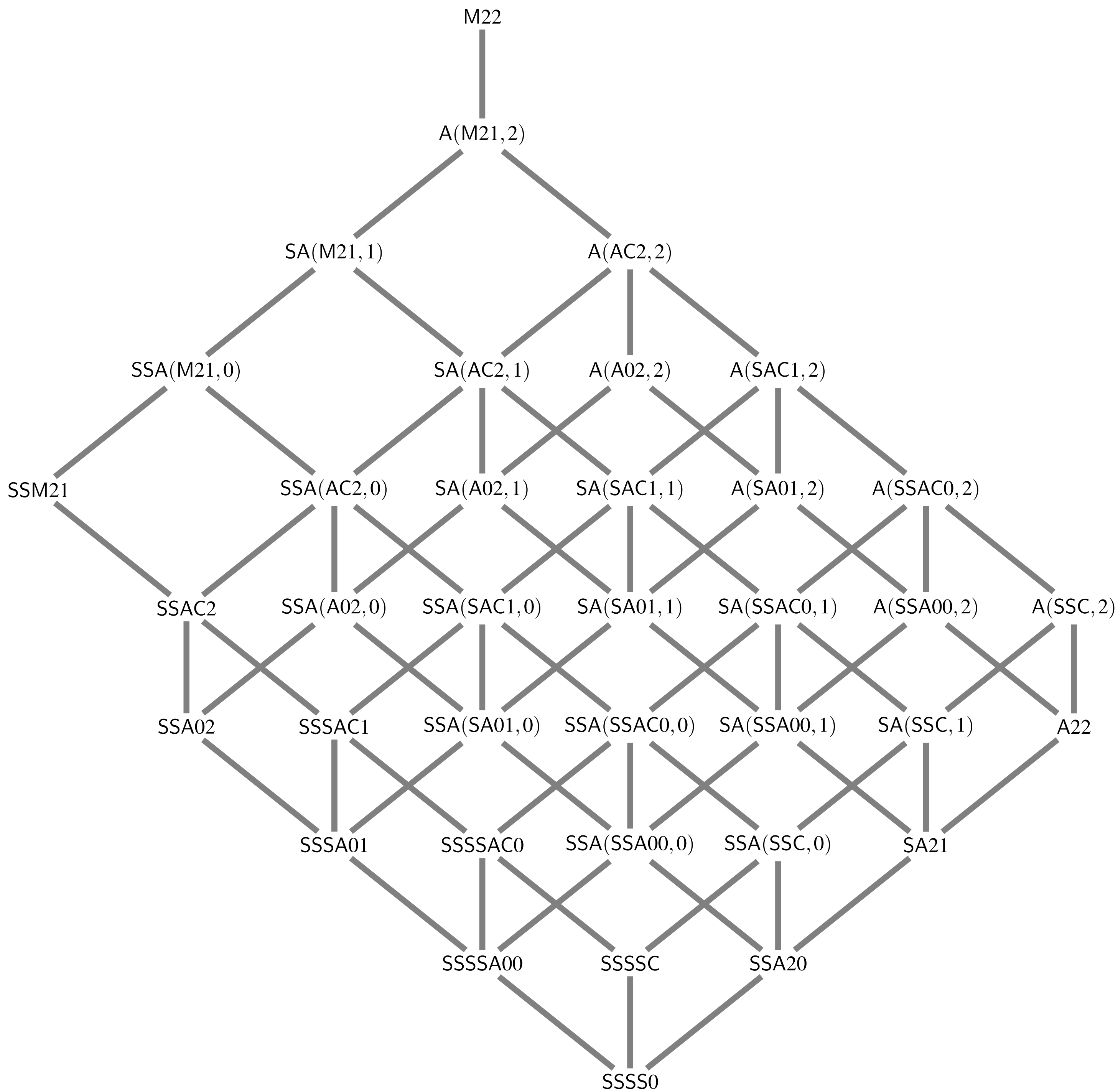
The TRS of S-terms, fragment of CL was another favourite passtime

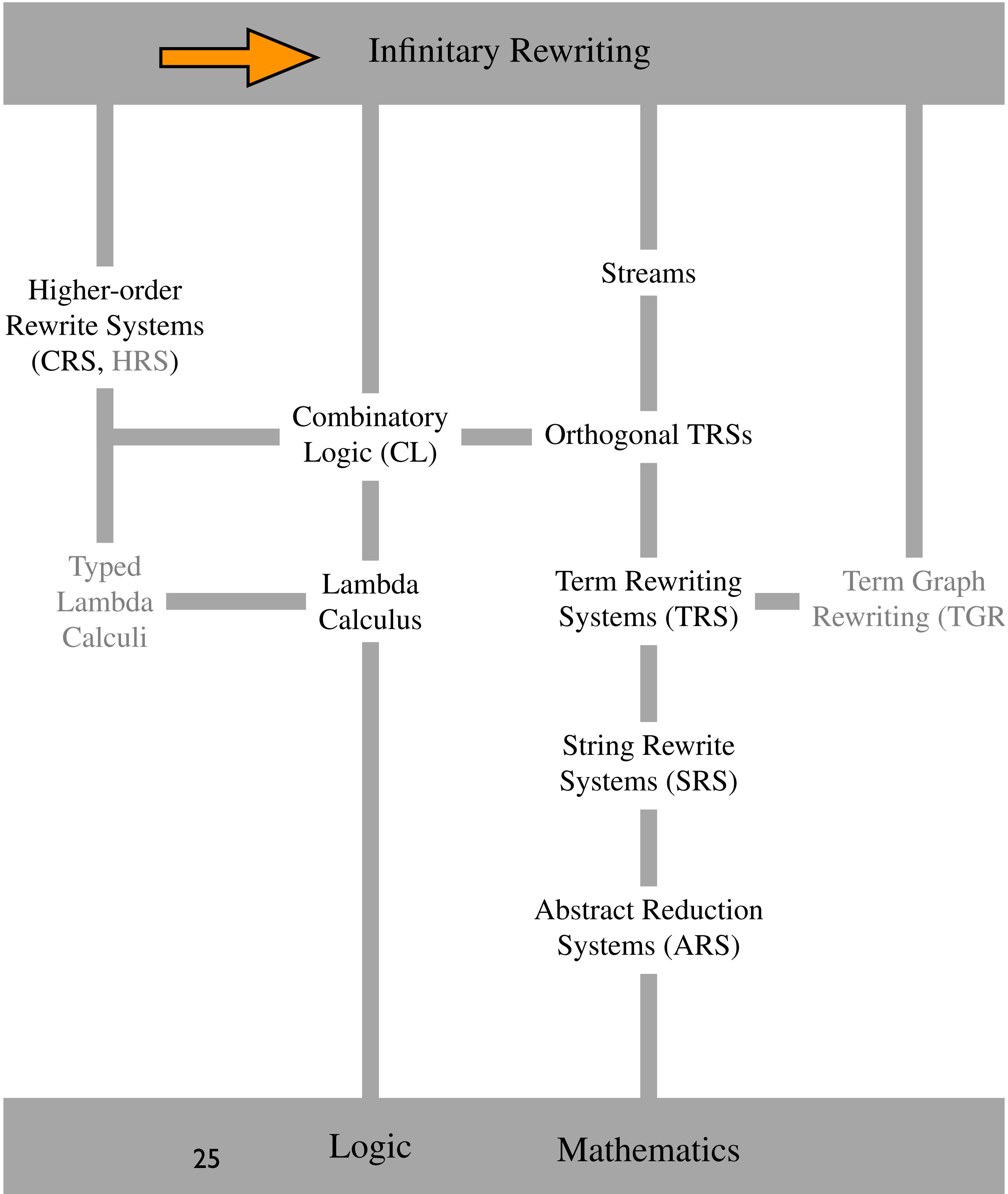
- *is not SN:* SSS(SSS)(SSS) has infinite reduction (Barendregt earns 25 guilders)
- *has no cycles* (Bergstra)
- *is top terminating* (Waldmann)


O'Donnell
WIN
SN
Church
NE
AC
WN
LF
AC$_{fin}$



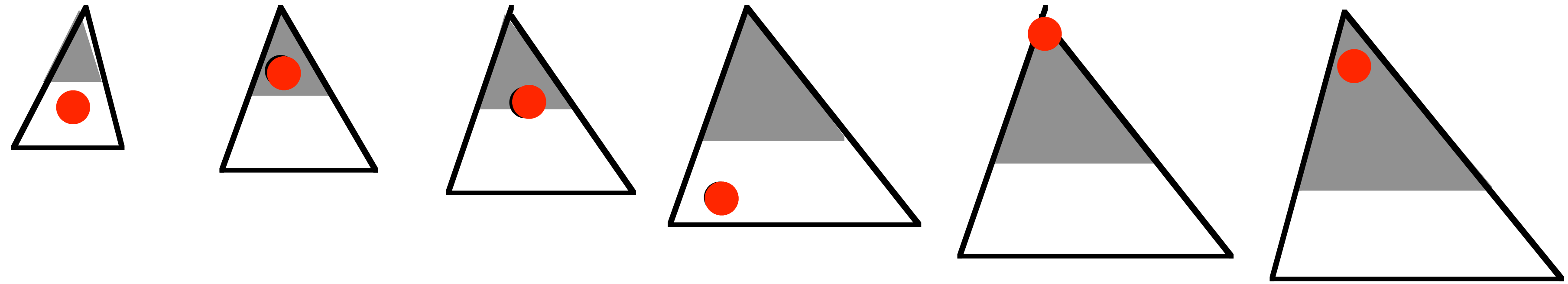
M22
A(M21, 2)
leftmost outermost
full developments
parallel outermost
leftmost innermost
SA(M21, 1)
A(AC2, 2)
SSA(M21, 0)
SA(AC2, 1)
A(A02, 2)
A(SAC1, 2)
SSM21
SSA(AC2, 0)
SA(A02, 1)
SA(SAC1, 1)
A(SA01, 2)
A(SSAC0, 2)
SSAC2
SSA(A02, 0)
SSA(SAC1, 0)
SA(SA01, 1)
SA(SSAC0, 1)
A(SSA00, 2)
A(SSC, 2)
SSA02
SSSAC1
SSA(SA01, 0)
SSA(SSAC0, 0)
SA(SSA00, 1)
SA(SSC, 1)
A22
SSSA01
SSSSAC0
SSA(SSA00, 0)
SSA(SSC, 0)
SA21
SSSSA00
SSSSC
SSA20
SSSS0


M22
A(M21,2)
SA(M21,1)
A(AC2,2)
SSA(M21,0)
SA(AC2,1)
A(A02,2)
A(SAC1,2)
SSM21
SSA(AC2,0)
SA(A02,1)
SA(SAC1,1)
A(SA01,2)
A(SSAC0,2)
SSAC2
SSA(A02,0)
SSA(SAC1,0)
SA(SA01,1)
SA(SSAC0,1)
A(SSA00,2)
A(SSC,2)
SSA02
SSSAC1
SSA(SA01,0)
SSA(SSAC0,0)
SA(SSA00,1)
SA(SSC,1)
A22
SSSA01
SSSSAC0
SSA(SSA00,0)
SSA(SSC,0)
SA21
SSSSA00
SSSSC
SSA20
SSSS0

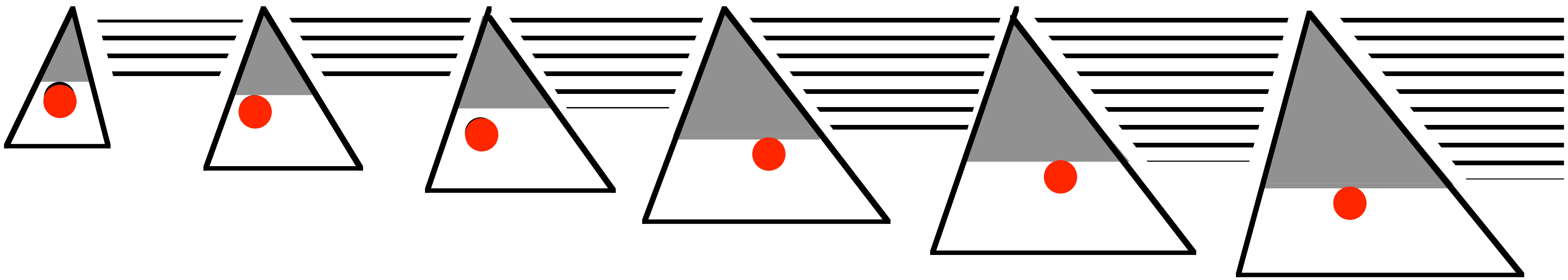

Infinitary Rewriting
Higher-order Rewrite Systems (CRS, HRS)
Streams
Combinatory Logic (CL)
Orthogonal TRSs
Typed Lambda Calculi
Lambda Calculus
Term Rewriting Systems (TRS)
Term Graph Rewriting (TGR
String Rewrite Systems (SRS)
Abstract Reduction Systems (ARS)
Logic
Mathematics

$$F(x) \rightarrow P(x, F(S(x)))$$

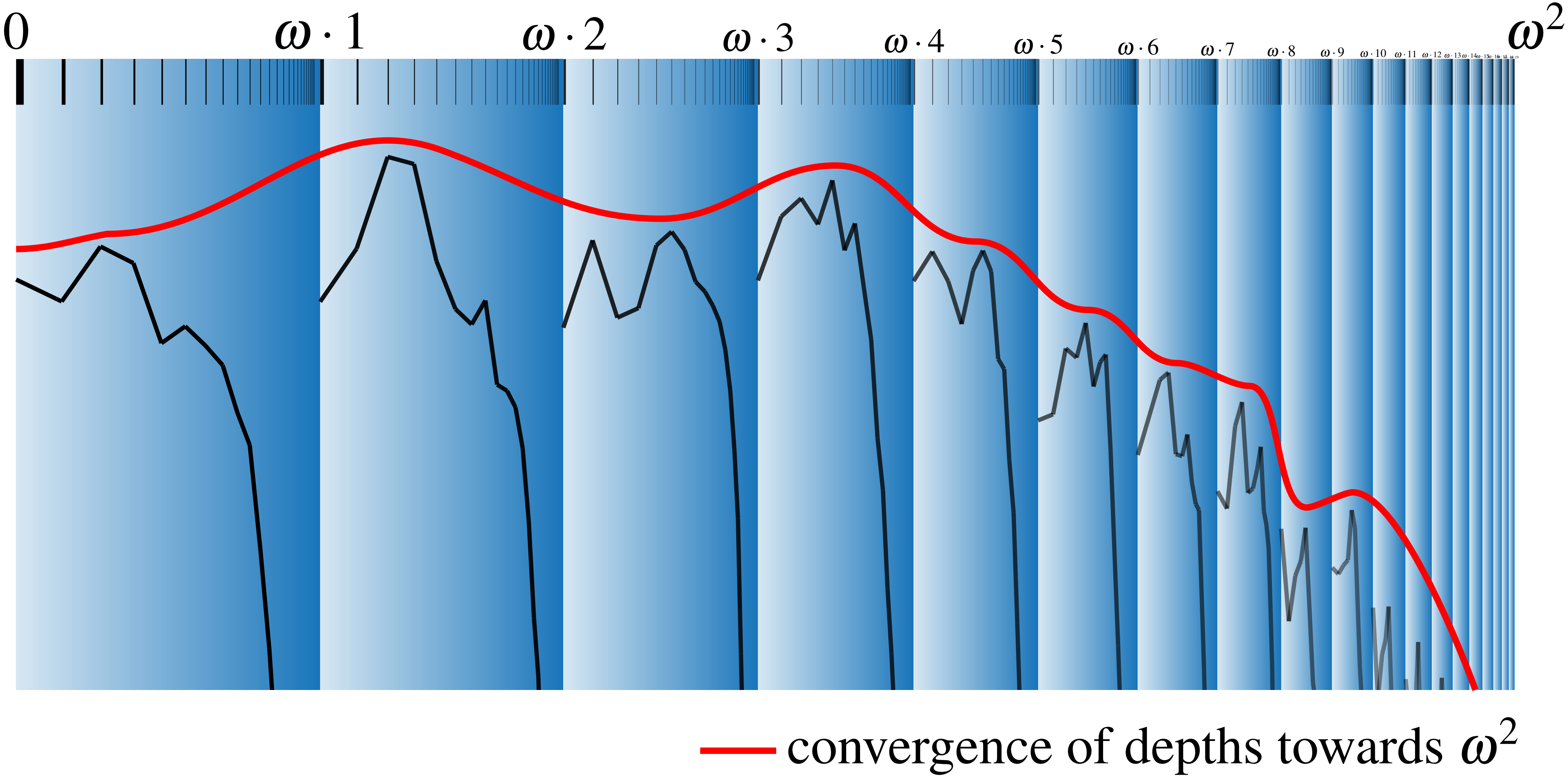
$$F(x) \rightarrow P(x, F(S(x)))$$



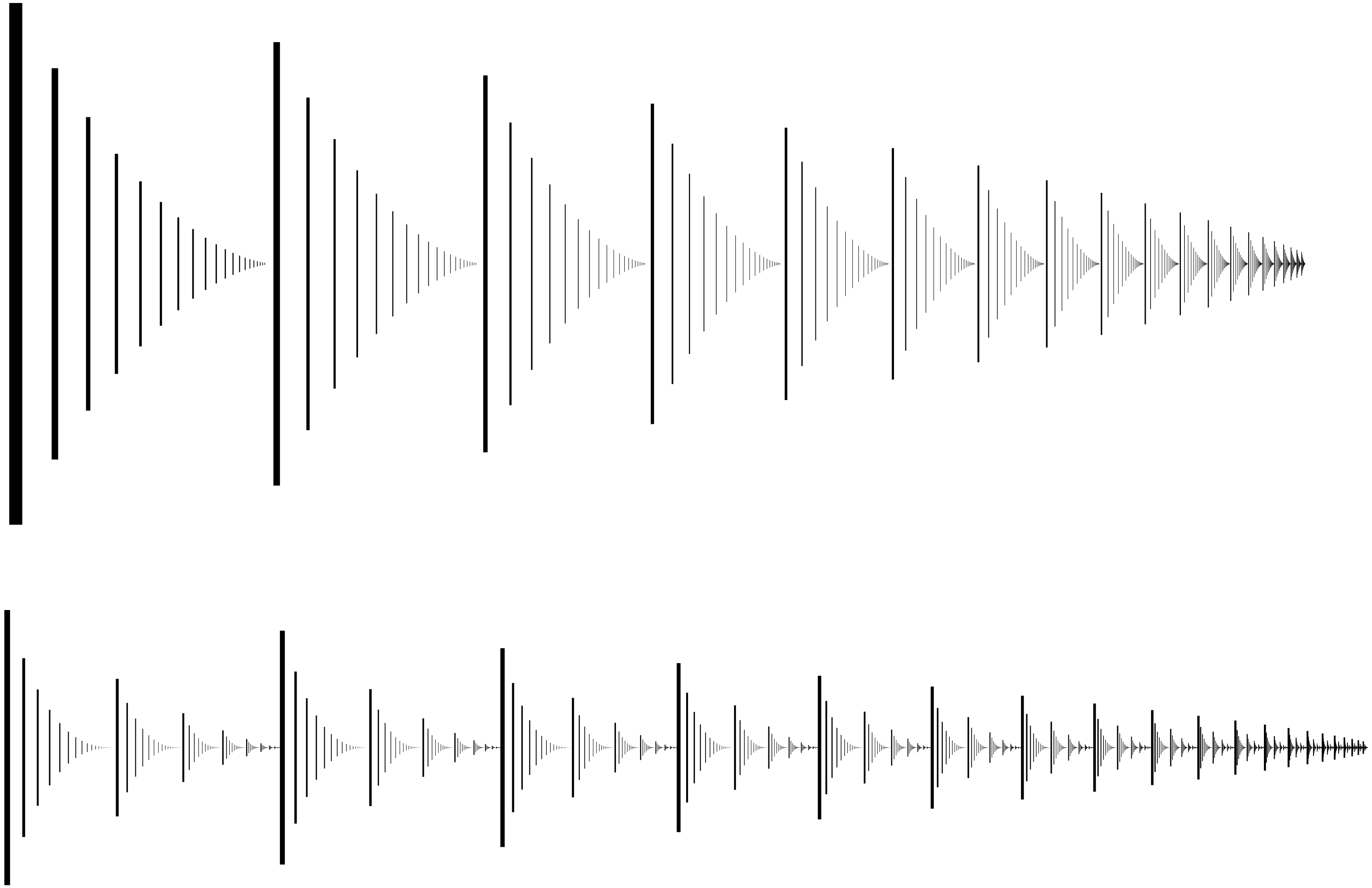
Cauchy converging reduction sequence: activity may occur everywhere



Strongly converging reduction sequence, with descendant relations



0
$\omega \cdot 1$
$\omega \cdot 2$
$\omega \cdot 3$
$\omega \cdot 4$
$\omega \cdot 5$
$\omega \cdot 6$
$\omega \cdot 7$
$\omega \cdot 8$
$\omega \cdot 9$
$\omega \cdot 10$
$\omega \cdot 11$
$\omega \cdot 12$
$\omega \cdot 13$
ω^2
convergence of depths towards ω^2



Ordinals ω^2 and ω^3 embedded in the reals, order-respecting.

Exercise: which ordinals can be embedded in the real segment [0,1]?

(i) $(\omega^{\omega} \cdot 2 + \omega^{3} \cdot 4 + \omega^{2}) + (\omega^{3} \cdot 3 + \omega^{2} \cdot 2 + 1) = \omega^{\omega} \cdot 2 + \omega^{3} \cdot 7 + \omega^{2} \cdot 2 + 1$

(ii) $(\omega^{6} \cdot 3 + \omega^{2} \cdot 4 + 2) + (\omega^{4} \cdot 5 + \omega^{2}) = \omega^{6} \cdot 3 + \omega^{4} \cdot 5 + \omega^{2}$

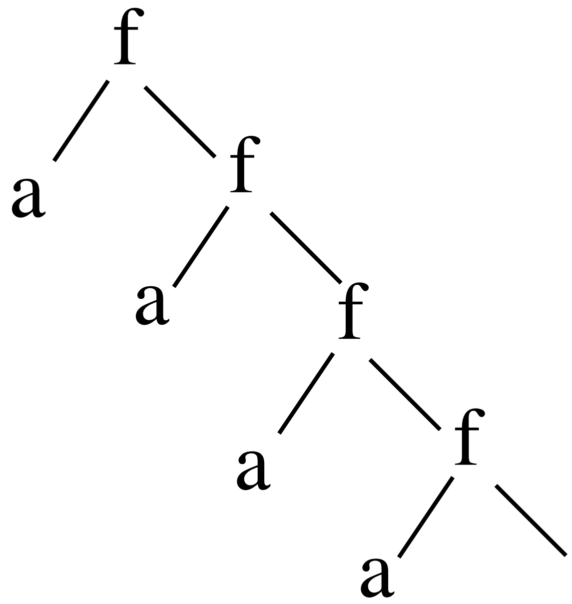
(iii) $(\omega^{\omega+2} \cdot 3 + \omega^{\omega} + \omega + 7) \cdot (\omega^{\omega+1} \cdot 2 + \omega^{\omega} + 3) = \omega^{\omega \cdot 2+1} \cdot 2 + \omega^{\omega \cdot 2} + \omega^{\omega+2} \cdot 9 + \omega^{\omega} + \omega + 7$

But if you don't like ordinals, there is for orthogonal TRSs the Compression Lemma:

every reduction of length α *can be compressed to* ω *or less.*

use dove-tailing

Every countable ordinal can be the length of an infinite reduction. Consider the TRS $\{c \longrightarrow f(a, c) \text{ and } a \longrightarrow b\}$



finitary rewriting	infinitary rewriting
finite reduction	*strongly convergent reduction*
infinite reduction	*divergent reduction*
normal form	*(poss. infinite) normal form*
CR: *finite coinitial reductions can be joined*	CR^{∞}: *infinite coinitial reductions can be joined*
UN: *coinitial reductions to nf end in same nf*	UN^{∞}: *coinitial reductions to nf end in same nf*
SN: *there are no infinite reductions*	SN^{∞}: *there are no divergent reductions*
WN: *there is a reduction to nf*	WN^{∞}: *there is a reduction to nf*

How to define SN^∞ and WN^∞?

WN^∞ is easy: There is a possibly infinite reduction to the possibly infinite normal form.

SN^∞ : all reductions will eventually terminate in the normal form. The only way such a reduction could fail to reach a normal form, is that it stagnates at some point in the tree which is developing, for infinitely many steps. Then no limit can be taken.

Good and bad reductions. In ordinary rewriting the finite reductions are good, they have an end point, and the infinite ones are bad, they have no end point.

Same in infinitary rewriting. The good reductions are the ones that are strongly convergent, they have an end point. E.g.
$a \to b(a)$ reaches after ω steps the end point b^{ω}.

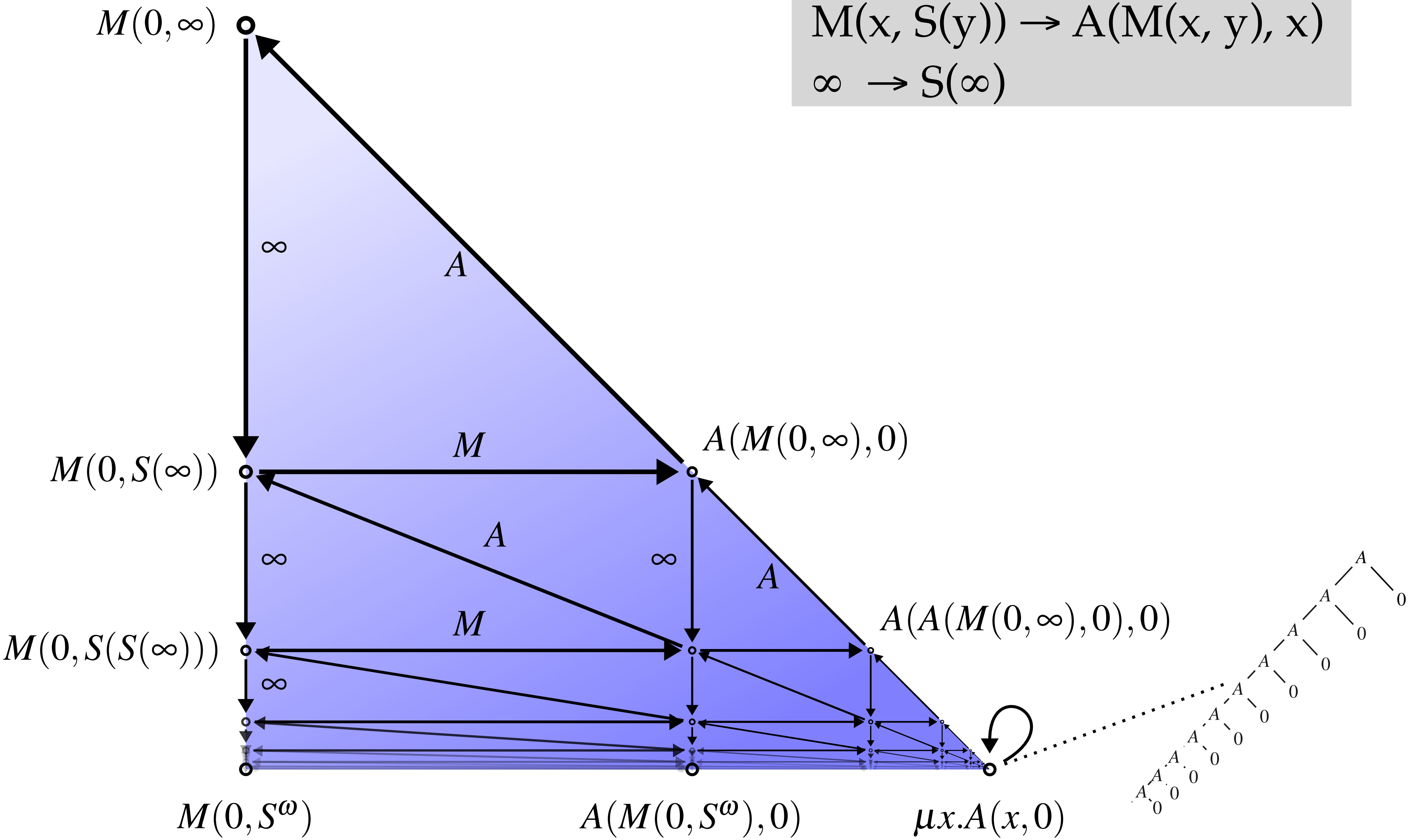
The bad reductions (divergent, stagnating) are the ones without an end point. Their reductions may be long, a limit ordinal long, but there they fail.

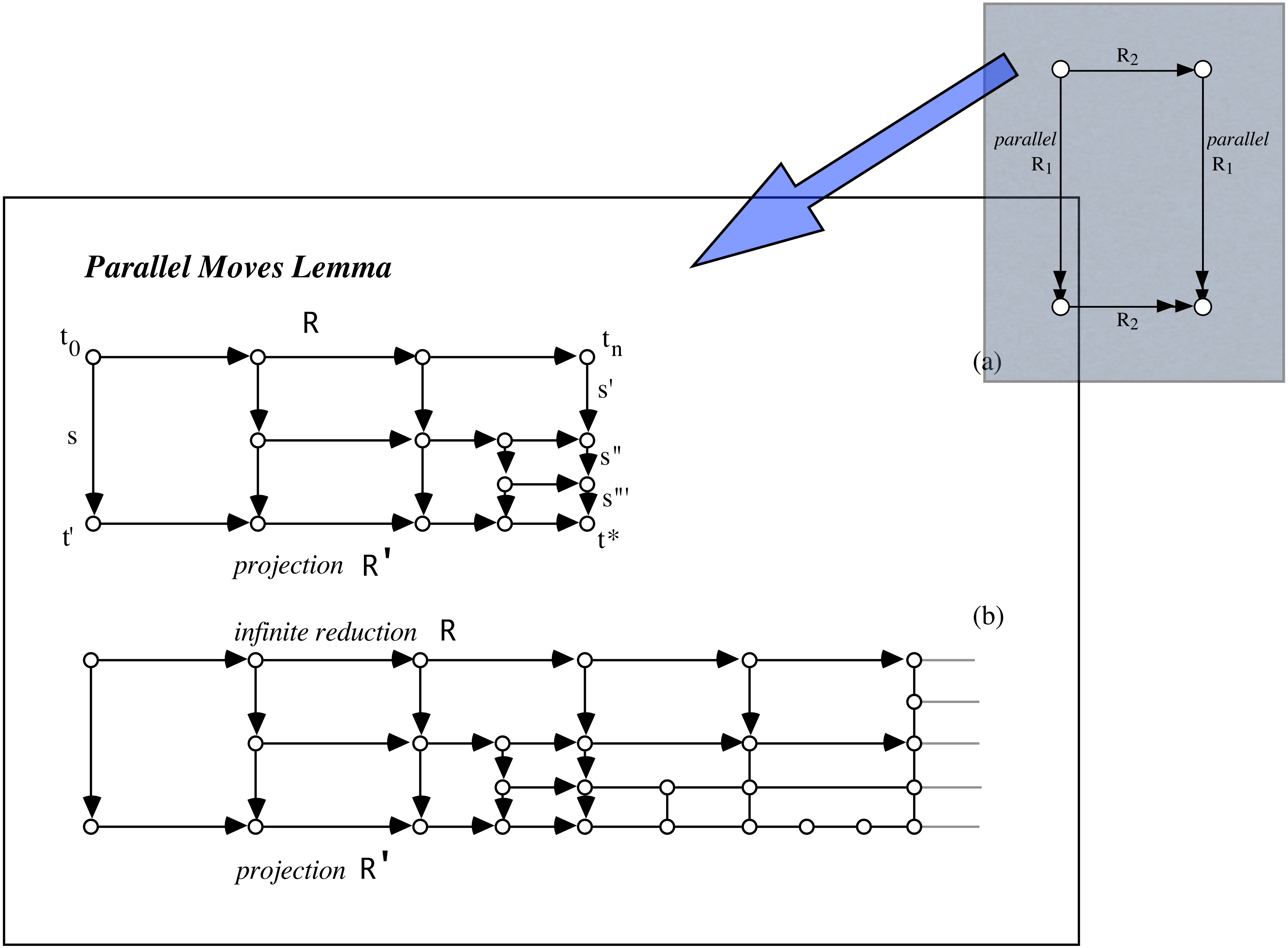
SN^{∞} states that there are no bad reductions.

In other words: say we select at random in each step a redex and perform this step. We can go on until we reach a limit ordinal. At that point we look back, and if the reduction was strongly convergent we take the limit and go on. If not, we stop there and we had a bad reduction.

CLAIM: we can then identify a stagnating term, a term where infinitely often a root step was performed.

$$A(x, 0) \to x$$
$$A(x, S(y)) \to S(A(x, y))$$
$$M(x, 0) \to 0$$
$$M(x, S(y)) \to A(M(x, y), x)$$
$$\infty \to S(\infty)$$




Parallel Moves Lemma
R
t_0
t_n
s
s'
s''
s'''
t'
t*
projection R'
infinite reduction R
projection R'
(b)
R_2
parallel
R_1
parallel
R_1
R_2
(a)

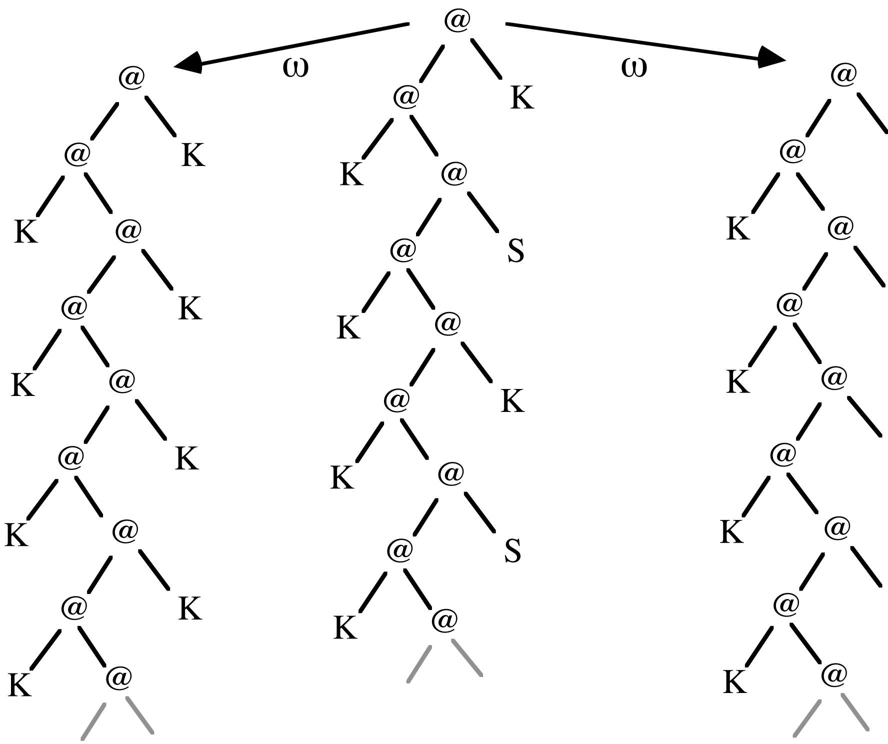
PML$^\infty$ *For first order infinitary term rewriting we have the infinitary Parallel Moves Lemma PML*$^\infty$

not CR^{∞}

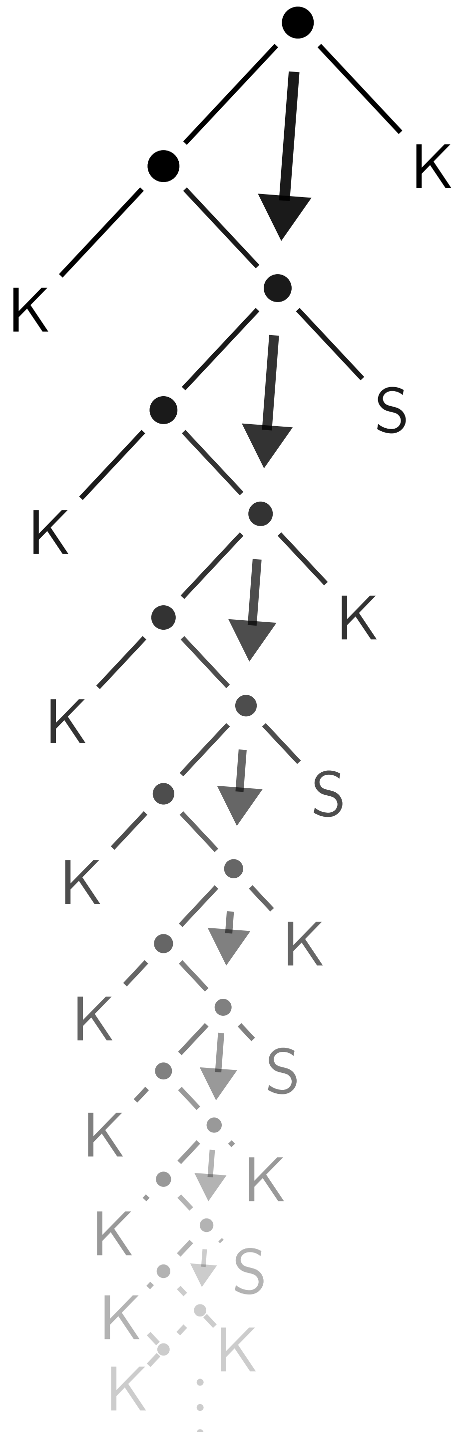
$Sxyz \rightarrow xz(yz)$ $\qquad$ $@(@(@(S, x), y), z) \rightarrow @(@(x, z), @(y, z))$

$Kxy \rightarrow x$ $\qquad$ $@(@(K, x), y) \rightarrow x$

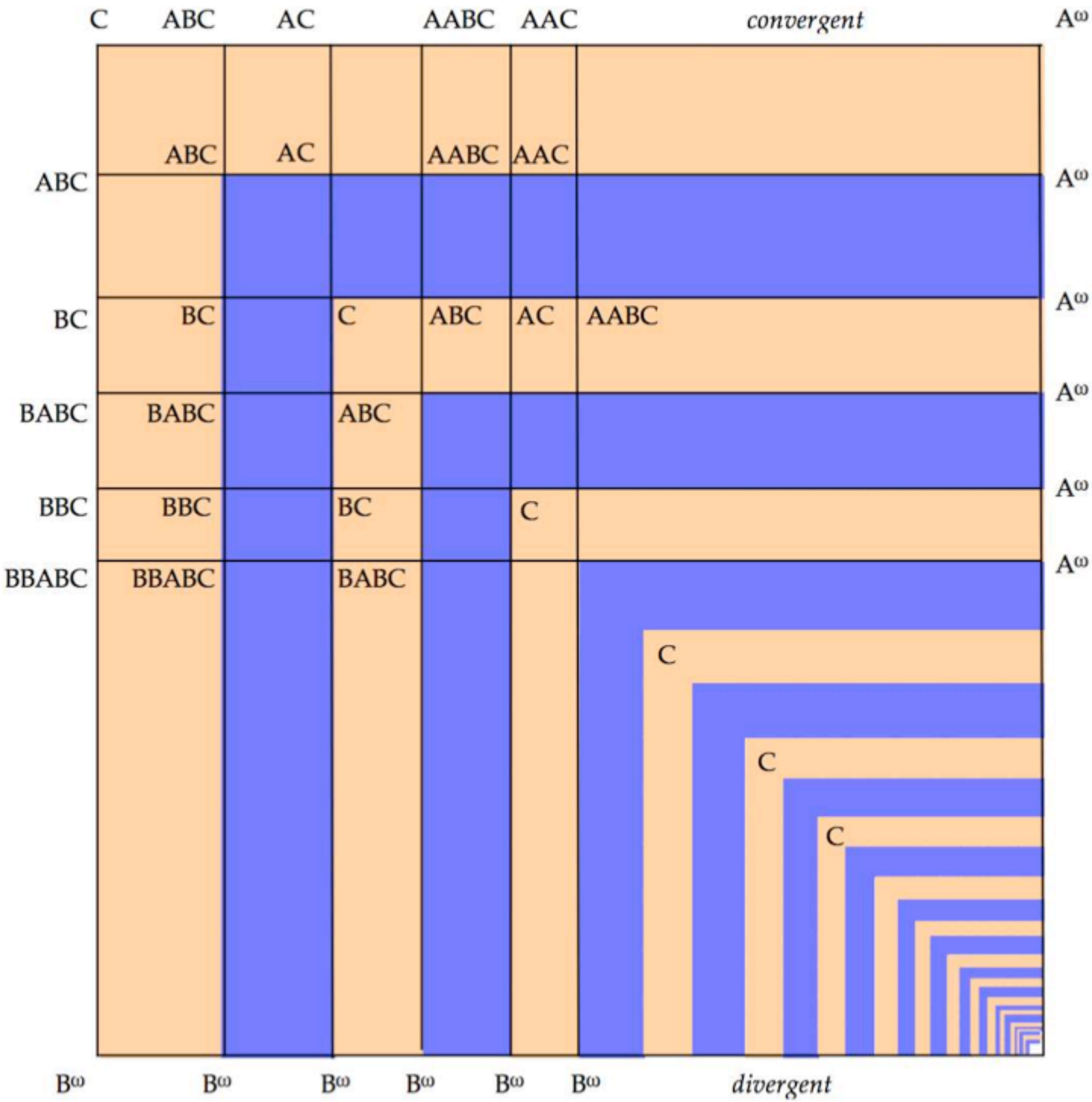
collapsing contexts



Failure of infinitary confluence for Combinatory Logic


K
K
S
K
K
K
S
K
K
K
S
K
K
K
S
K
K
K

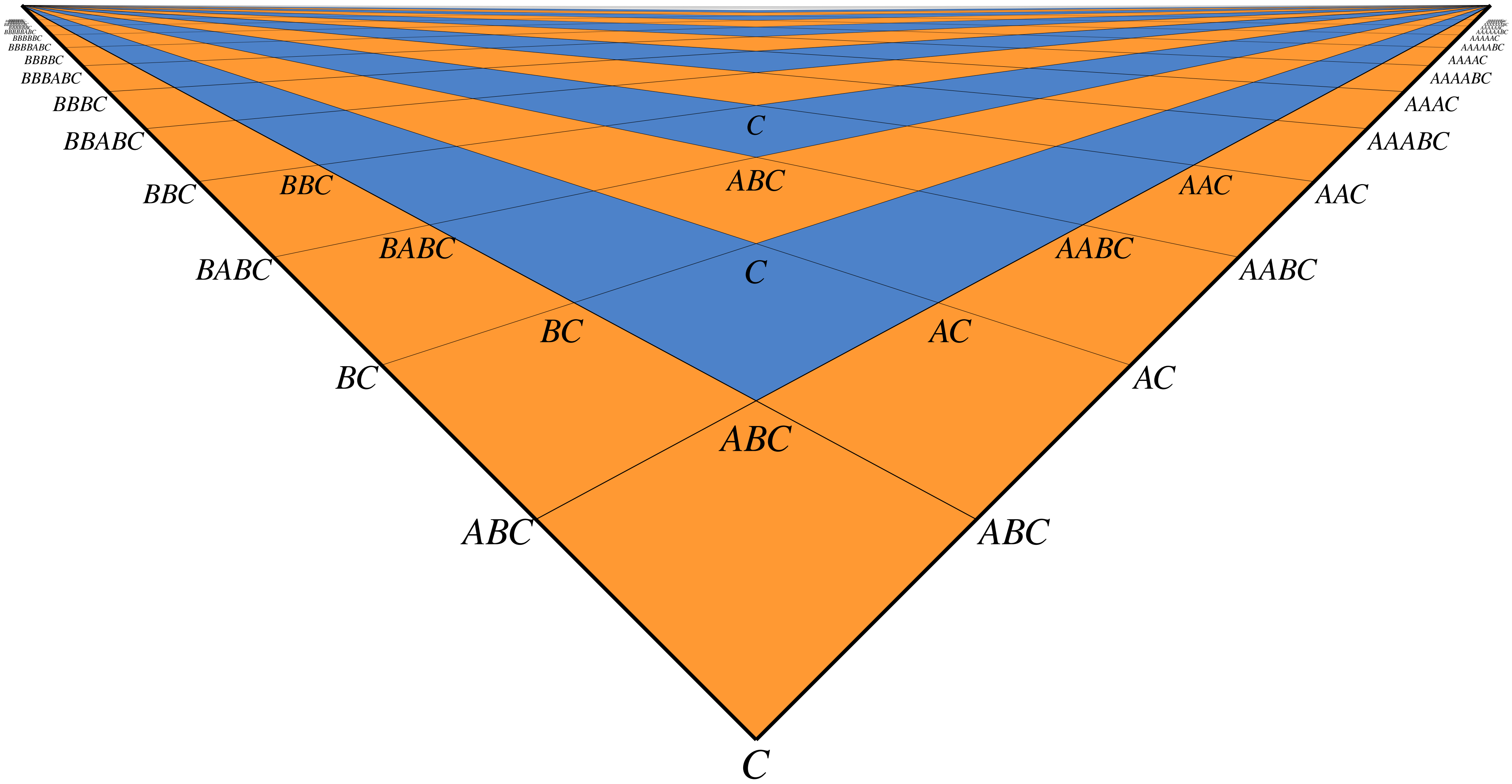
Failure of CR$^\infty$



$A(x) \to x$

$B(x) \to x$

$C \to A(B(C))$

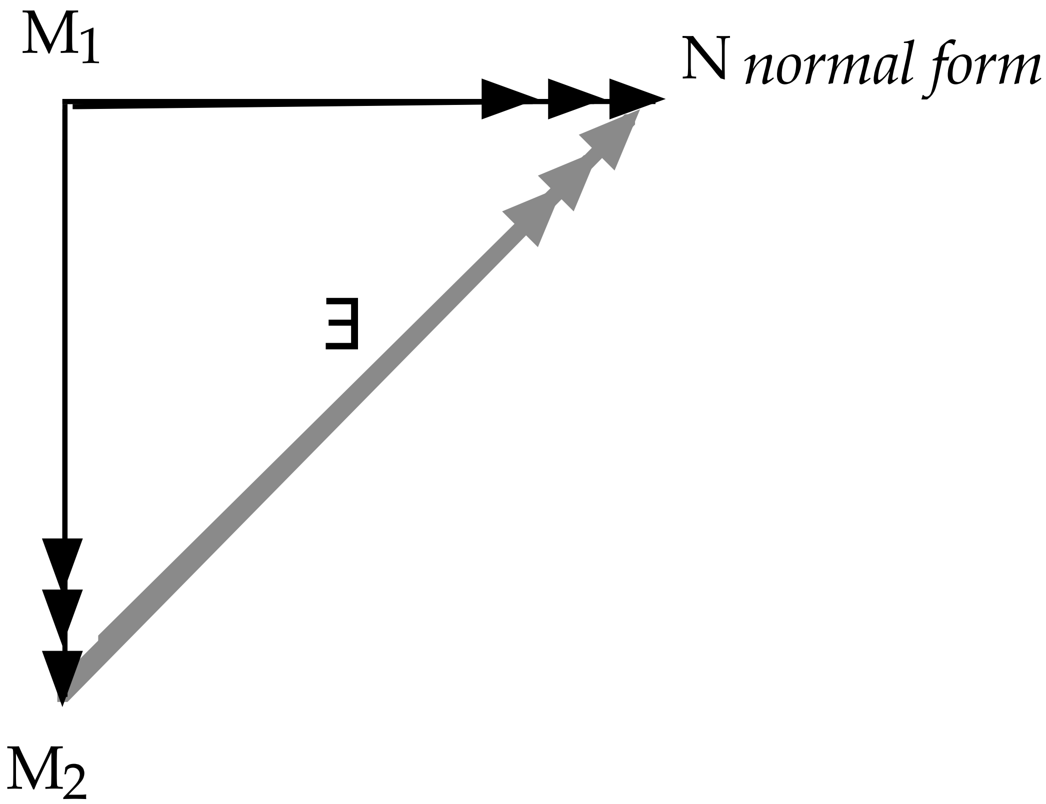


BBBABC
BBBC
BBABC
BBC
BABC
BC
ABC
C
AAAABC
AAAC
AAABC
AAC
AABC
AC
ABC
C
BBC
BABC
BC
ABC
C
ABC
C
AAC
AABC
AC
BBBBABC
BBBBC
AAAAC
AAAAABC

for OTRSs: UN^{∞}.

Corollary: Dershowitz et al:
for OTRSs SN^{∞} => CR^{∞}.

Proof: as for finite case
SN & UN => CR



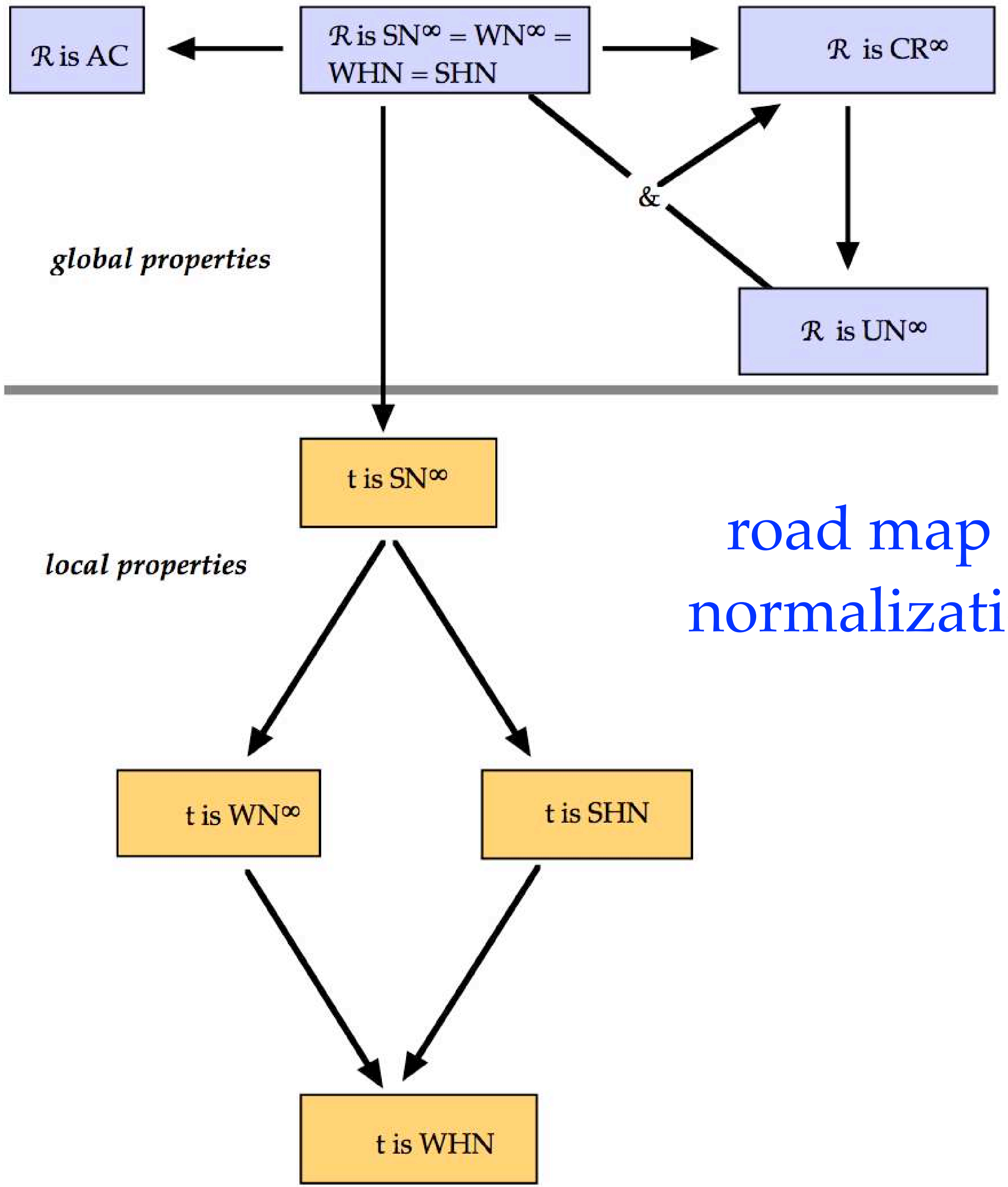
M1
N *normal form*
∃
M2

Confluence in infinitary rewriting

	PML	CR	UN	PML$^\infty$	CR$^\infty$	UN$^\infty$
OTRS	*yes*	*yes*	*yes*	*yes*	*no*	*yes*
w.o. TRS	*yes*	*yes*	*yes*	*?*	*no*	*?*
$\lambda\beta$	*yes*	*yes*	*yes*	*no*	*no*	*yes*
OCRS	*yes*	*yes*	*yes*	*no*	*no*	*?*

by CR$^\infty$ for a quotient of $\lambda\beta^\infty$, e.g. mute terms, or hypercollapsing terms, and applying an abstract lemma of de Vrijer.

Let (A, $\to_1$) and (B, $\to_2$) be two ARSs with A included in B,
reduction $\to_1$ included in $\to_2$, normal forms nf(A) included in nf(B).
Then CR for B implies UN for A.



road map of infinitary normalization properties

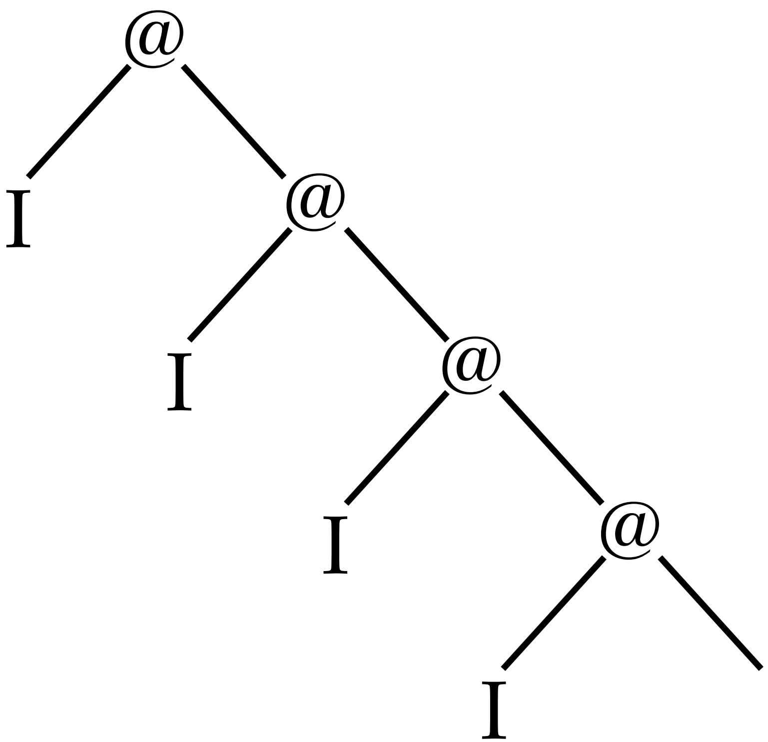
λ^∞ :not PML^∞

$$\omega_I \equiv (\lambda x.I(xx))$$

$$\omega \equiv \lambda x.xx$$

$$YI \to \omega_I\ \omega_I$$

$I^\omega \equiv$



For infinitary lambda calculus Parallel Moves Lemma PML^∞ fails, hence also CR^∞

Y_0: λf. (x.f(xx)(λx.f(xx))

Y_1: (λab. b(aab)) (λab. b(aab))

Y_0(SI) Y_1

Exercise. Prove that $Y_0 \neq_\beta Y_1$

infinitary lambda calculus subsumes scott's induction rule

$$Yx \twoheadrightarrow x(Yx) \twoheadrightarrow x^2(Yx) \twoheadrightarrow^{\omega} x^{\omega} \equiv x(x(x(x\ldots$$

$$BY \equiv (\lambda abc.a(bc))\,Y \qquad =_{\infty} \qquad BYS \equiv (\lambda abc.a(bc))\,YS$$

$$\neq_{\beta}$$

$$BY \twoheadrightarrow \lambda bc.Y(bc) \twoheadrightarrow^{\omega} \lambda bc.\,(bc)^{\omega}$$

$$BYS \twoheadrightarrow \lambda c.Y(Sc) \twoheadrightarrow \lambda c.\,Sc(Y(Sc)) \twoheadrightarrow \lambda cz.\,cz(Y(Sc)z) \twoheadrightarrow \lambda cz.\,cz(cz(Y(Sc)z)) \twoheadrightarrow^{\omega} \lambda cz.\,(cz)^{\omega}$$

$$\lambda bc.\,(bc)^{\omega} \equiv \lambda cz.\,(cz)^{\omega}$$

A simple proof

BY $\neq_\beta$? BYS

BYI

BYSI

BYI ≡ (λabc.a(bc)) YI

BYSI ≡ (λabc.a(bc))YSI

$\downarrow$

$\downarrow$

λc.Y(Ic)

$\downarrow$

$\downarrow$

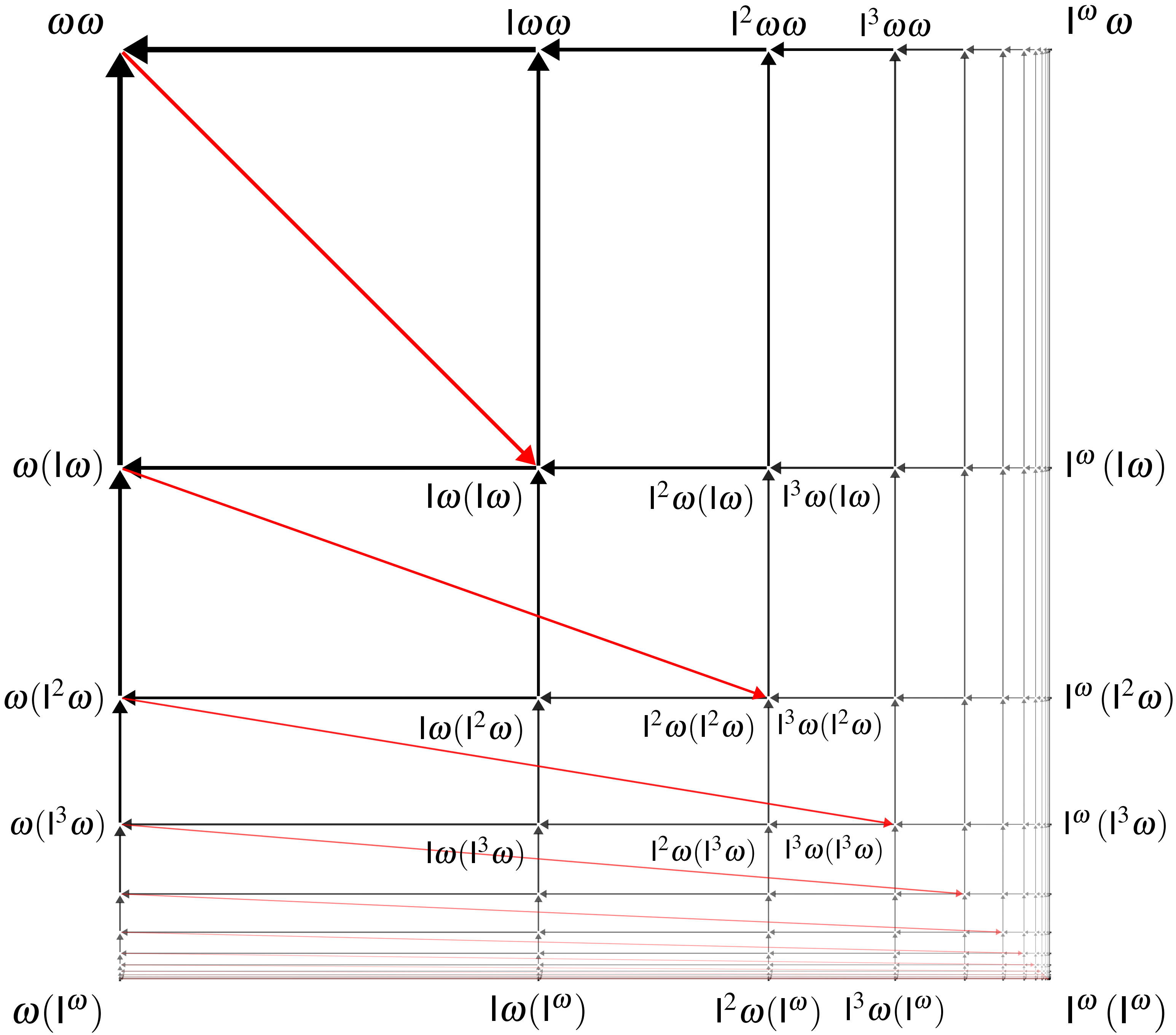
λc.Yc

$\downarrow$

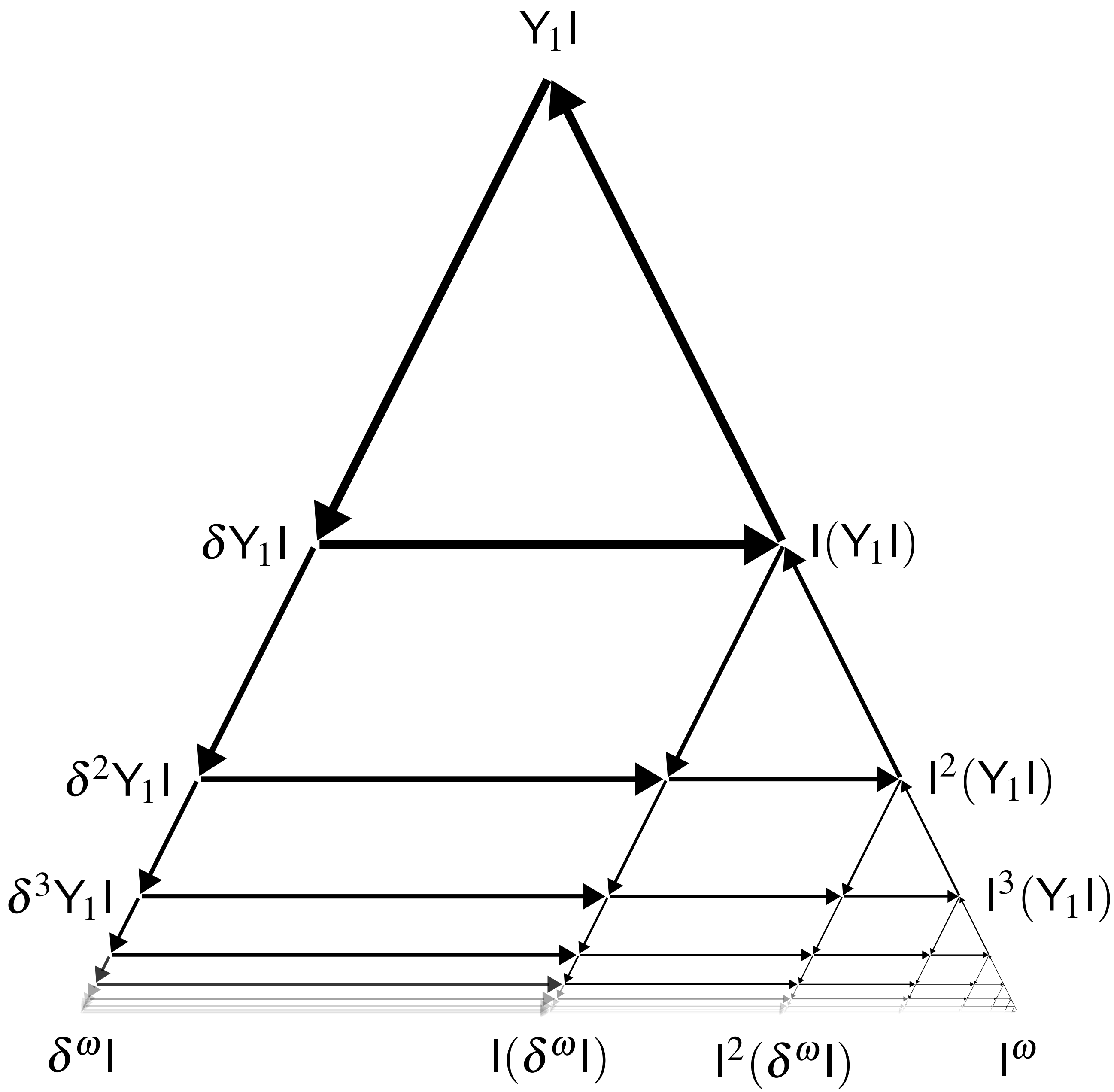
Y $\neq_\beta$! Y(SI)

Curry's fpc

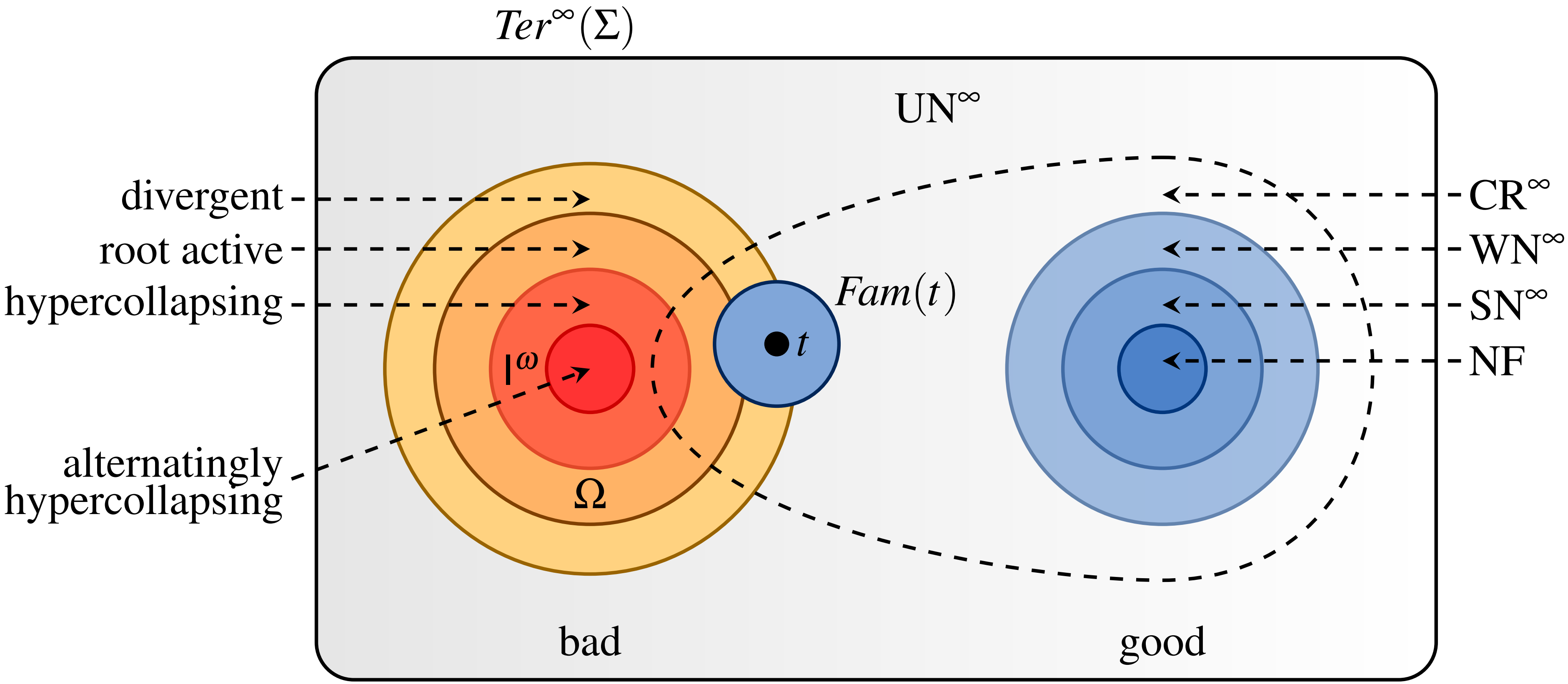
Turing's fpc



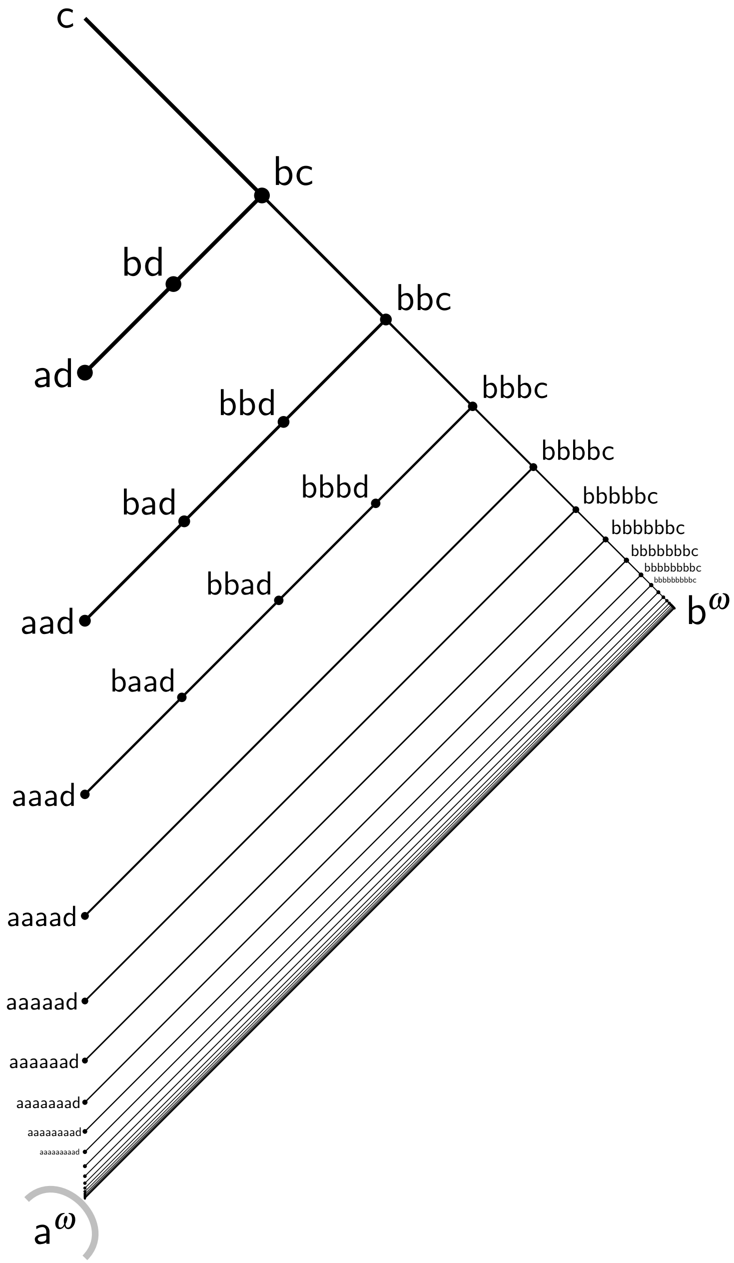
$\omega\omega$
$l\omega\omega$
$l^2\omega\omega$
$l^3\omega\omega$
$l^\omega\,\omega$
$\omega(l\omega)$
$l\omega(l\omega)$
$l^2\omega(l\omega)$
$l^3\omega(l\omega)$
$l^\omega\,(l\omega)$
$\omega(l^2\omega)$
$l\omega(l^2\omega)$
$l^2\omega(l^2\omega)$
$l^3\omega(l^2\omega)$
$l^\omega\,(l^2\omega)$
$\omega(l^3\omega)$
$l\omega(l^3\omega)$
$l^2\omega(l^3\omega)$
$l^3\omega(l^3\omega)$
$l^\omega\,(l^3\omega)$
$\omega(l^\omega)$
$l\omega(l^\omega)$
$l^2\omega(l^\omega)$
$l^3\omega(l^\omega)$
$l^\omega\,(l^\omega)$



Y1I
δY1I
I(Y1I)
δ2Y1I
I2(Y1I)
δ3Y1I
I3(Y1I)
δωI
I(δωI)
I2(δωI)
Iω



Ter∞(Σ)
UN∞
divergent
root active
hypercollapsing
alternatingly hypercollapsing
I^ω
Ω
Fam(t)
t
CR∞
WN∞
SN∞
NF
bad
good



$$\mathsf{c} \to \mathsf{b}(\mathsf{c})$$
$$\mathsf{b}(\mathsf{c}) \to \mathsf{a}(\mathsf{d})$$
$$\mathsf{b}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{a}(x))$$

0. A few words on history

1. rewriting dictionary

2. two theorems in abstract rewriting

3. word rewriting: monoids and braids

tea, coffee

4. term rewriting: divide et impera; termination by stars

5. Lambda calculus and combinatory logic

6. Infinitary rewriting

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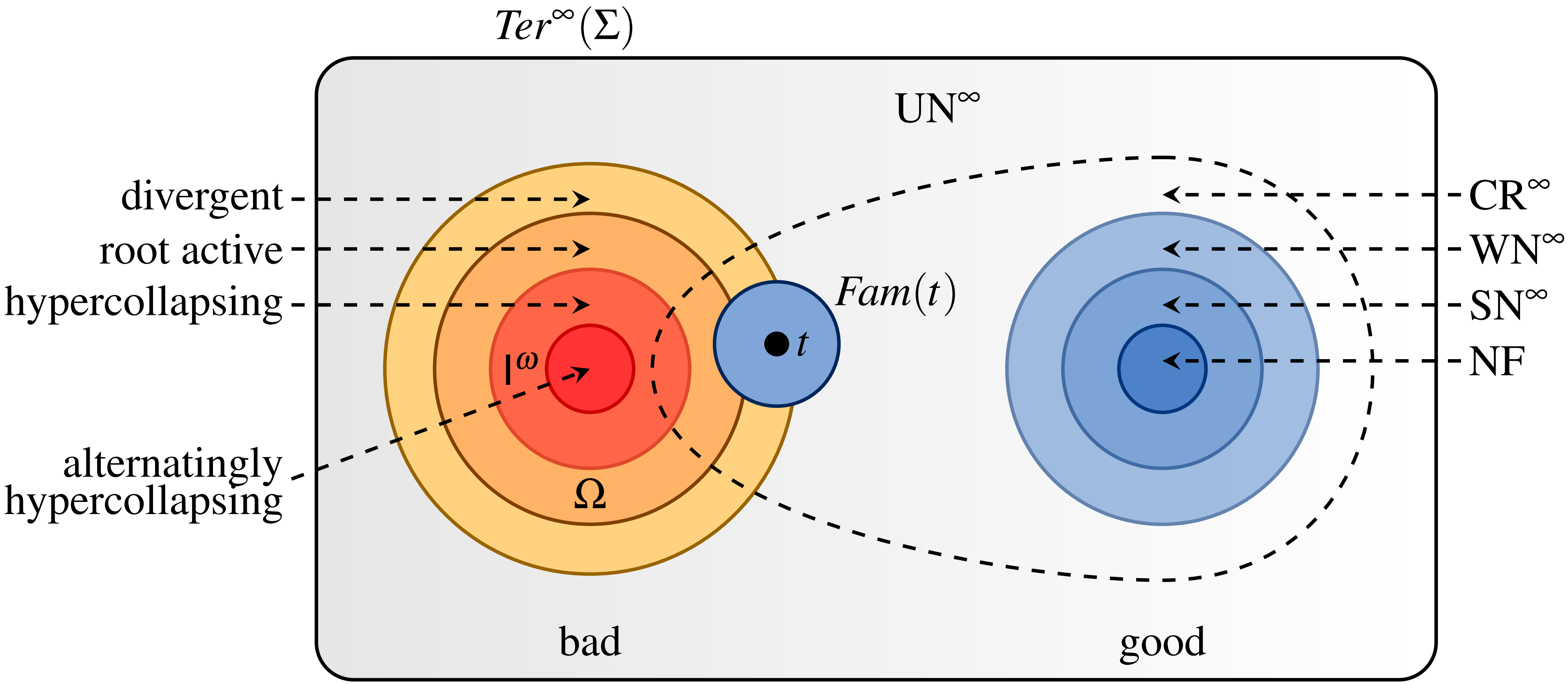
tea, coffee

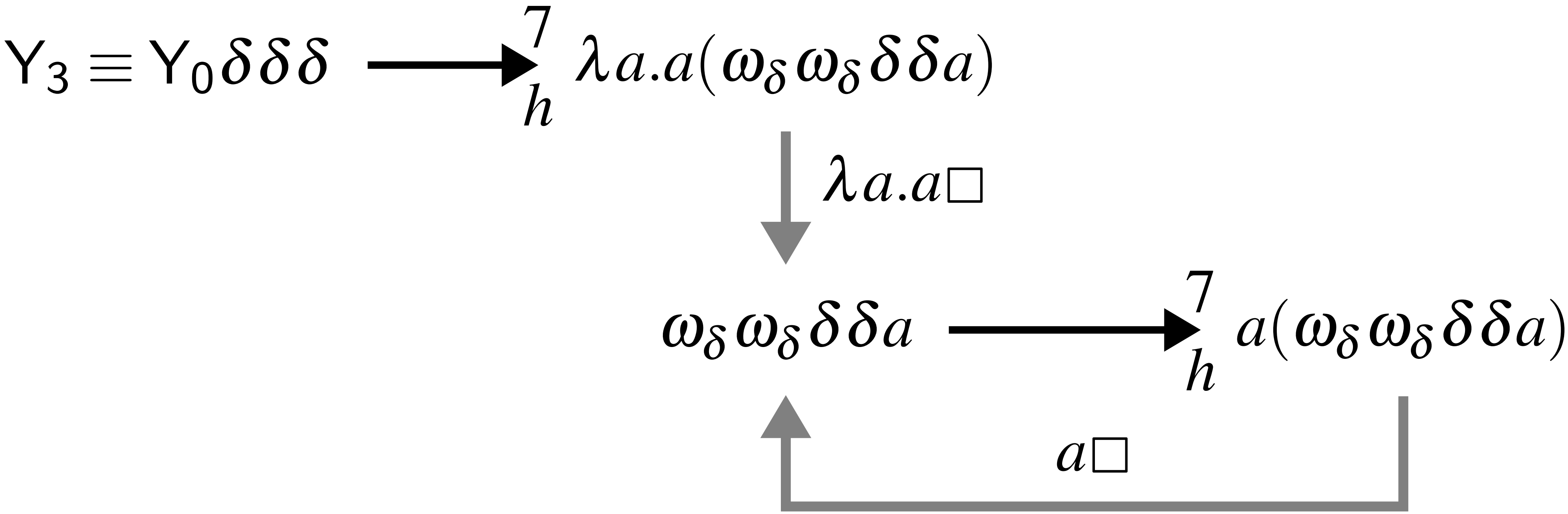
7.infinitary lambda calculus and the threefold path

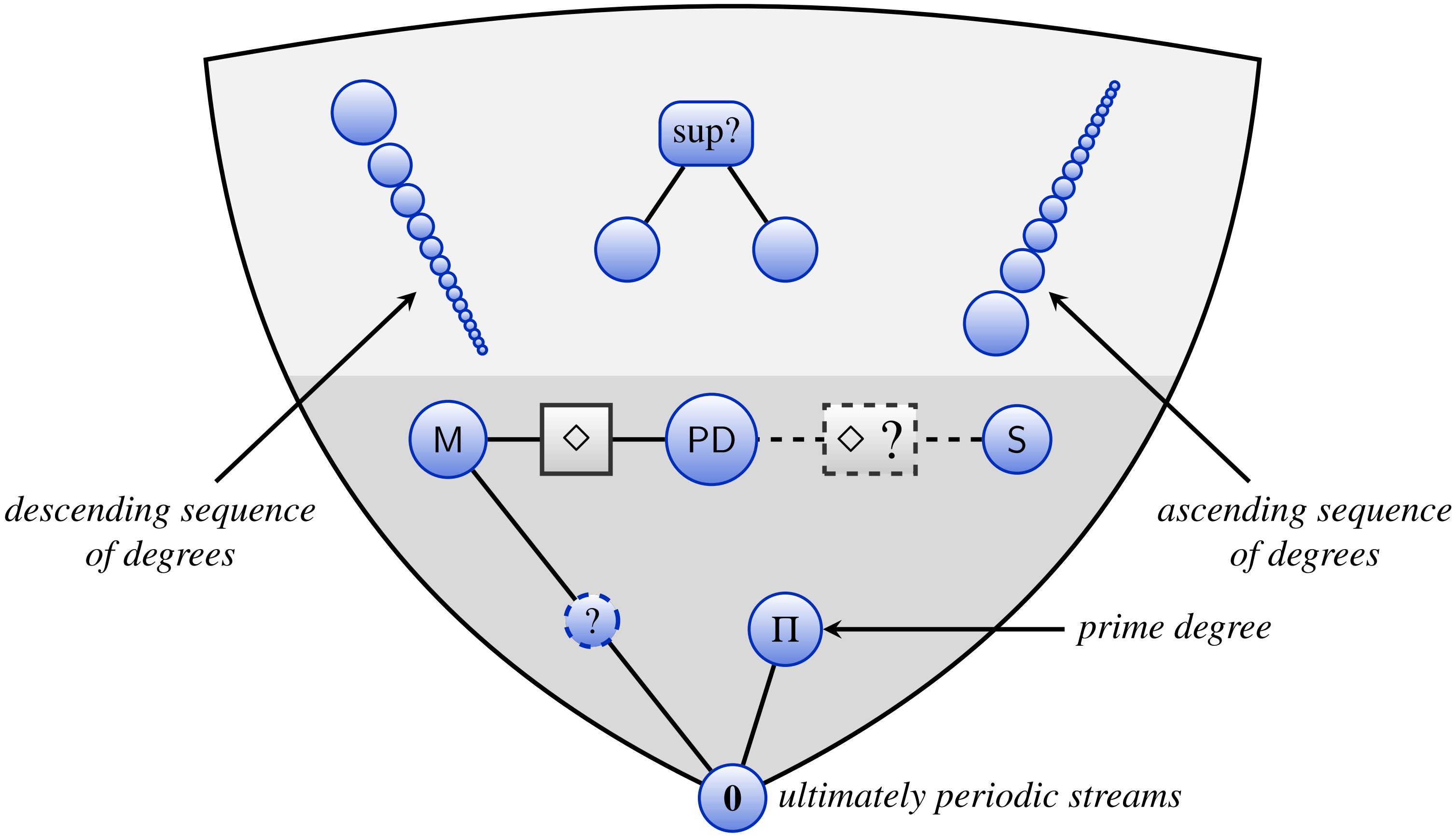
8.clocked semantics of lambda calculus

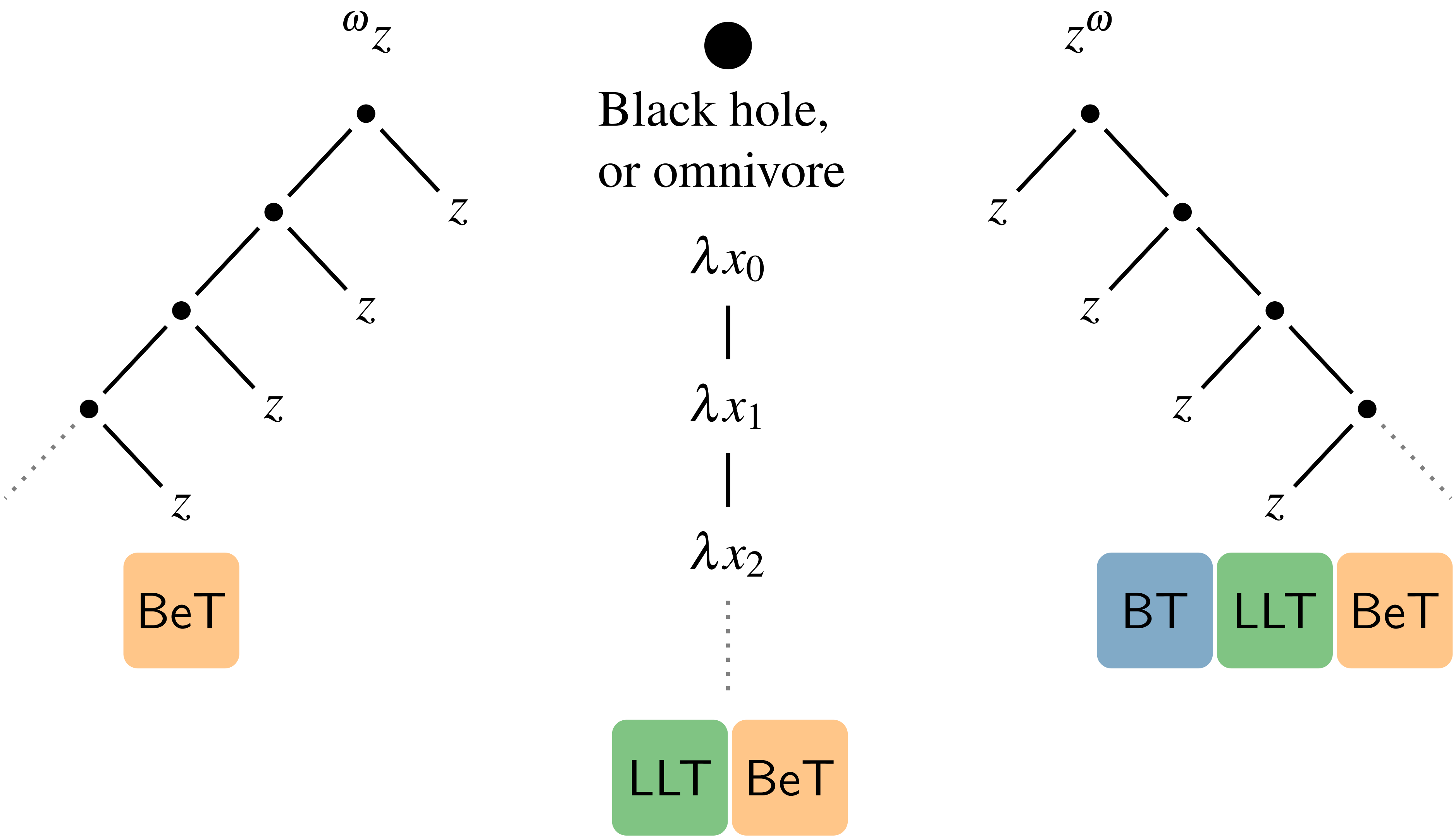
9.streams running forever



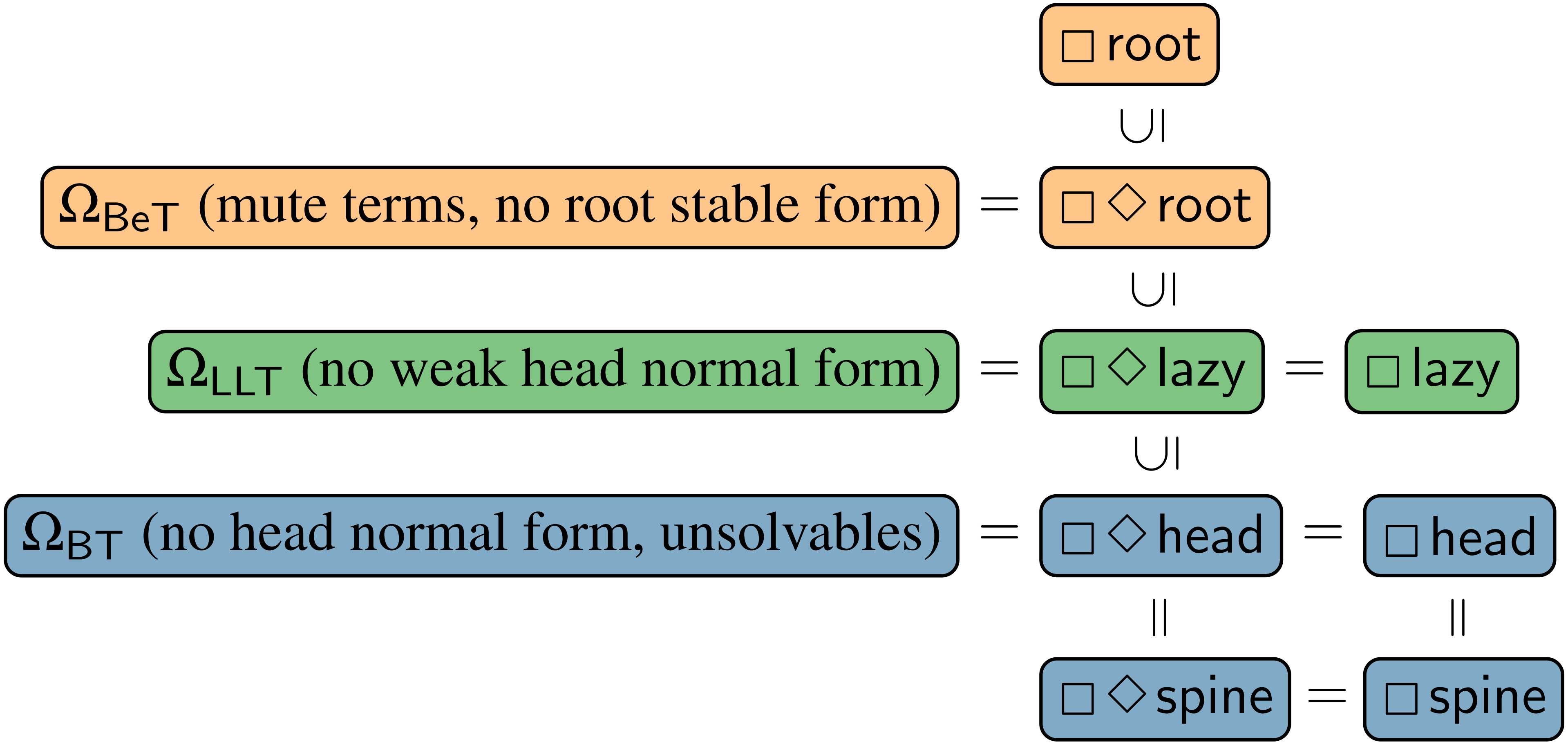
$Ter^\infty(\Sigma)$
UN^∞
divergent
root active
hypercollapsing
alternatingly hypercollapsing
I^ω
Ω
t
$Fam(t)$
CR^∞
WN^∞
SN^∞
NF
bad
good


$\mathsf{Y}_3 \equiv \mathsf{Y}_0 \delta\delta\delta$
$\overset{7}{\underset{h}{\longrightarrow}}$
$\lambda a.a(\omega_\delta \omega_\delta \delta\delta a)$
$\lambda a.a\square$
$\omega_\delta \omega_\delta \delta\delta a$
$\overset{7}{\underset{h}{\longrightarrow}}$
$a(\omega_\delta \omega_\delta \delta\delta a)$
$a\square$

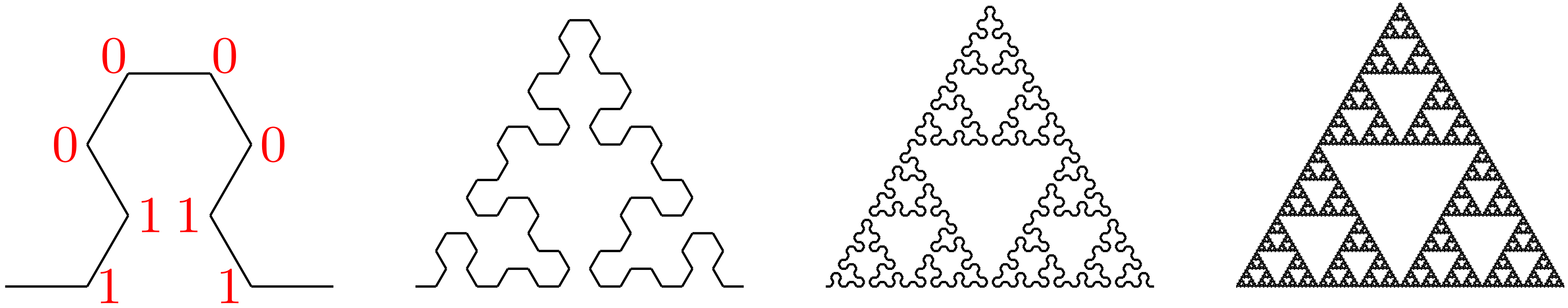

sup?
M
◇
PD
◇ ?
S
?
Π
0
descending sequence of degrees
ascending sequence of degrees
prime degree
ultimately periodic streams

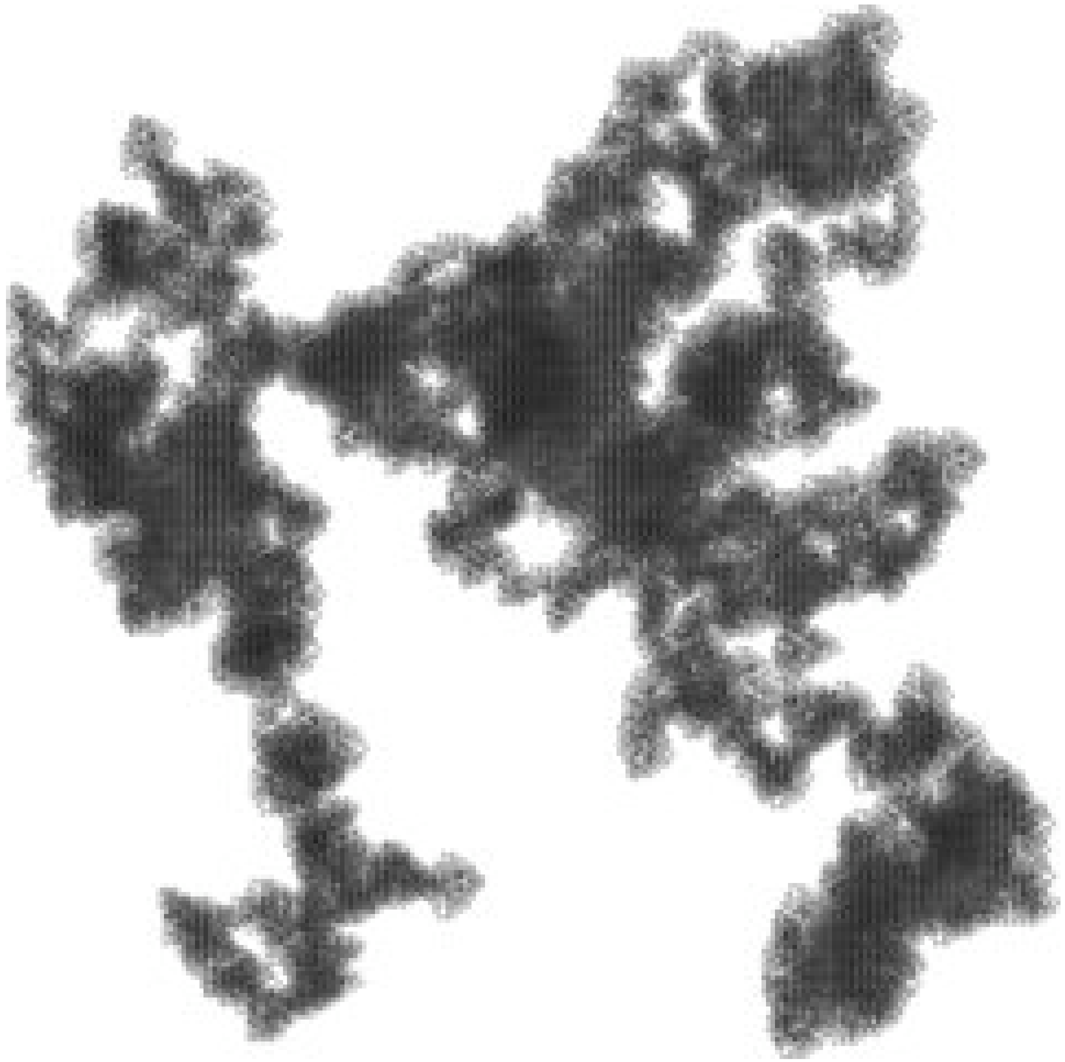


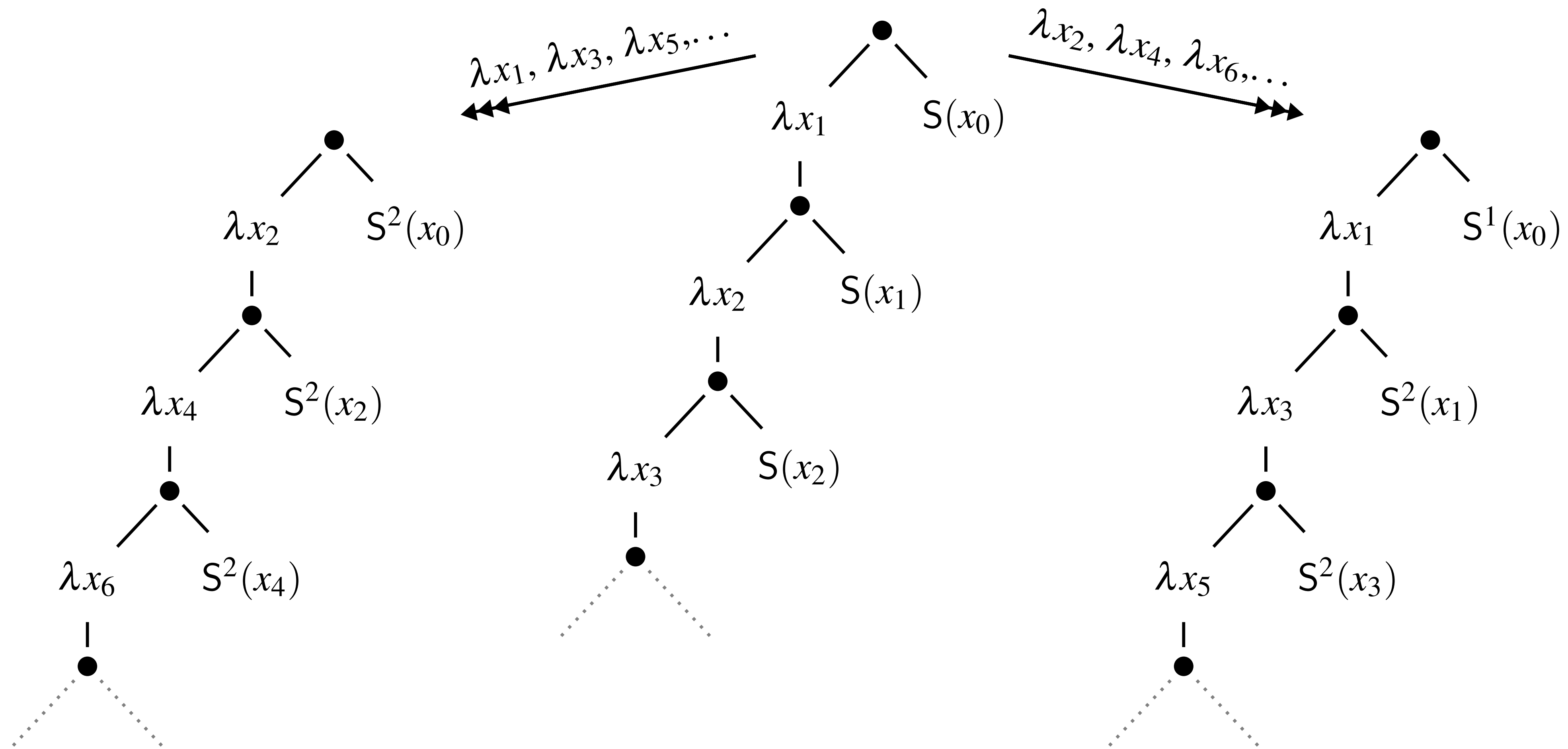
ωz
z
z
z
z
BeT
Black hole,
or omnivore
λx0
λx1
λx2
LLT
BeT
zω
z
z
z
z
BT
LLT
BeT



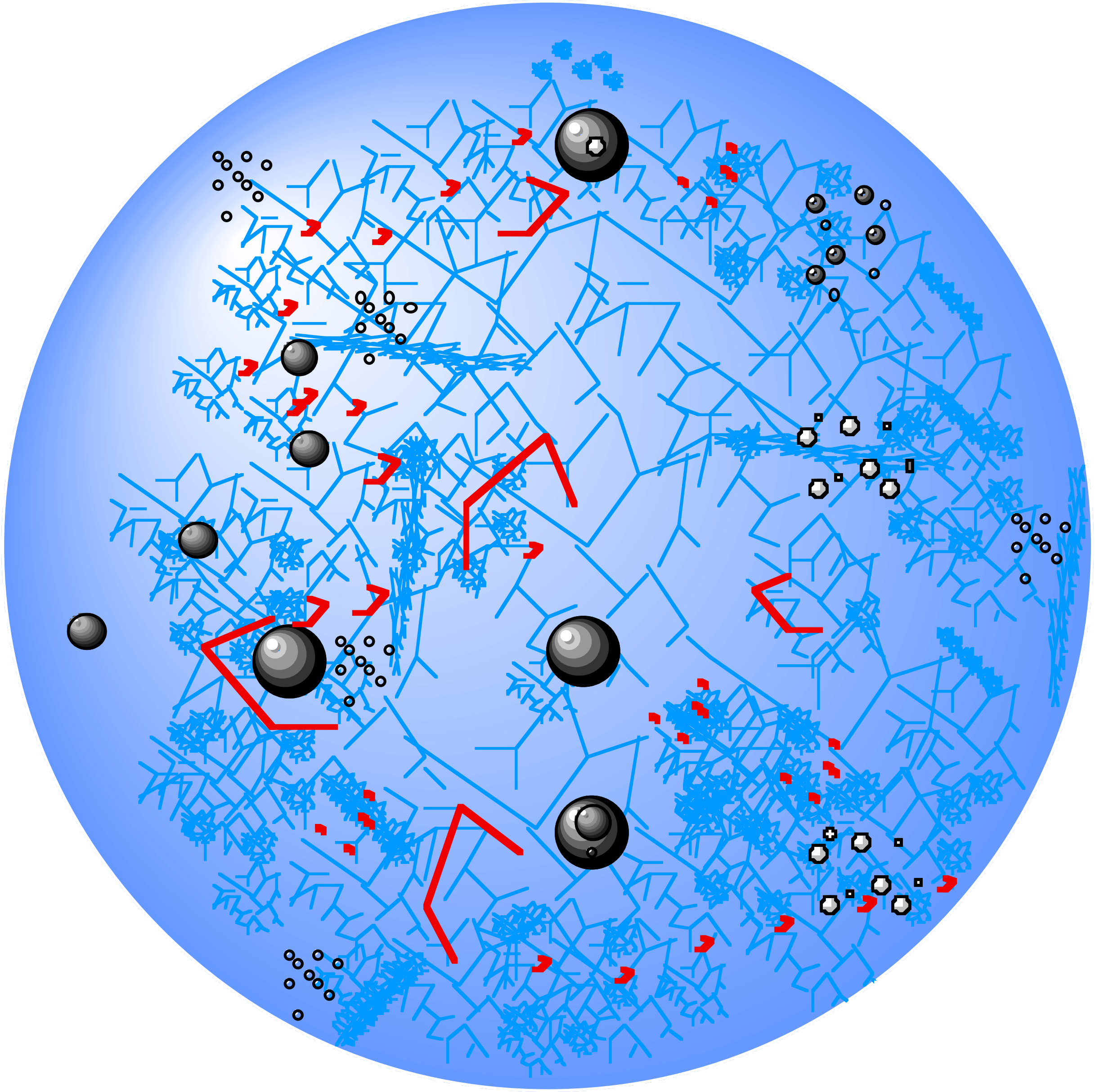
□ root
⊆
Ω_BeT (mute terms, no root stable form)
=
□ ◇ root
⊆
Ω_LLT (no weak head normal form)
=
□ ◇ lazy
=
□ lazy
⊆
Ω_BT (no head normal form, unsolvables)
=
□ ◇ head
=
□ head
||
□ ◇ spine
=
□ spine

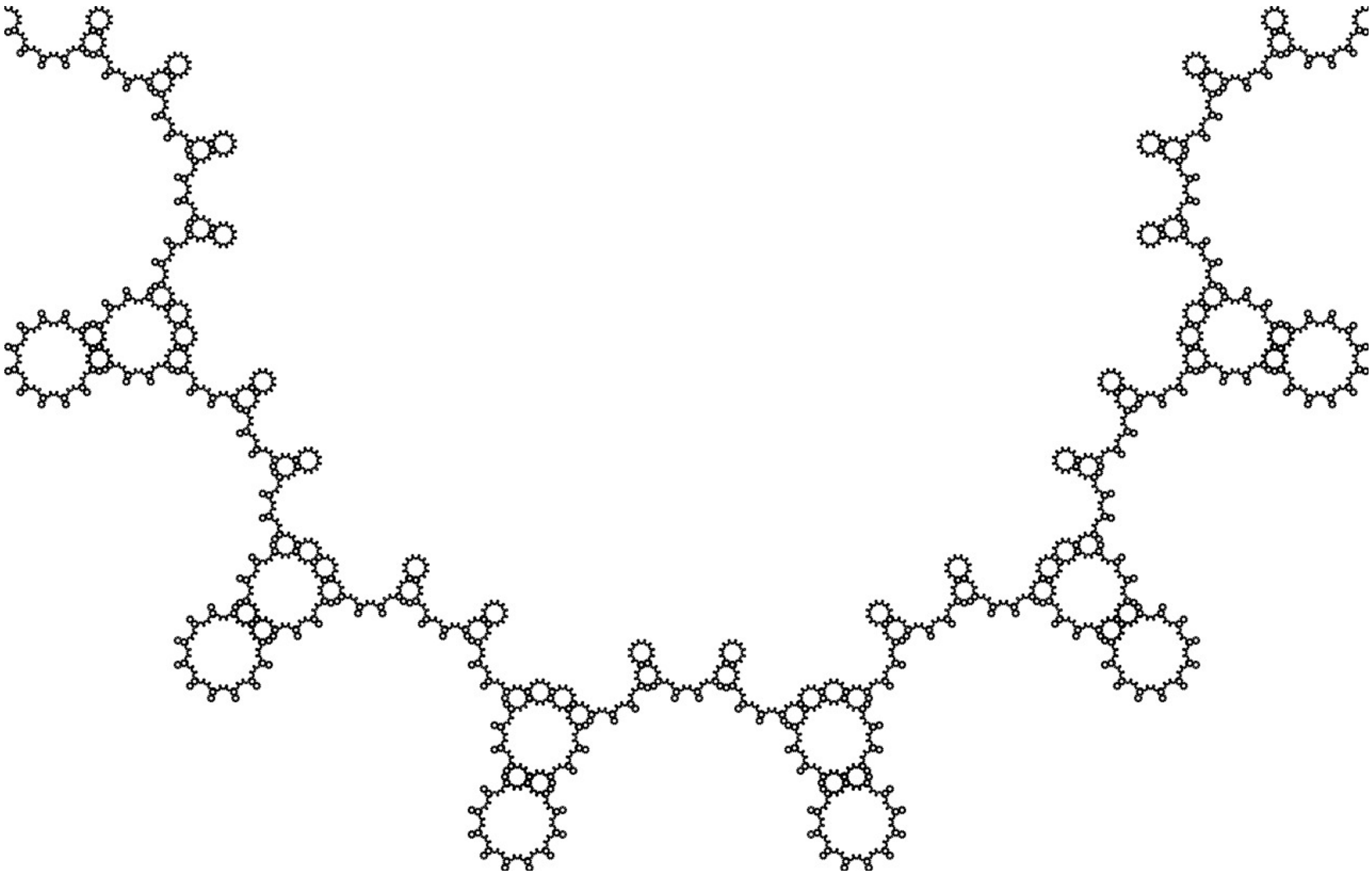

0
0
0
0
1
1
1
1





$\lambda x_1, \lambda x_3, \lambda x_5, \ldots$
$\lambda x_2, \lambda x_4, \lambda x_6, \ldots$
λx_1
$\mathsf{S}(x_0)$
λx_2
$\mathsf{S}(x_1)$
λx_3
$\mathsf{S}(x_2)$
λx_2
$\mathsf{S}^2(x_0)$
λx_4
$\mathsf{S}^2(x_2)$
λx_6
$\mathsf{S}^2(x_4)$
λx_1
$\mathsf{S}^1(x_0)$
λx_3
$\mathsf{S}^2(x_1)$
λx_5
$\mathsf{S}^2(x_3)$





0. A few words on history

1. rewriting dictionary

2. two theorems in abstract rewriting

3. word rewriting: monoids and braids

tea, coffee

4. term rewriting: divide et impera; termination by stars

5. Lambda calculus and combinatory logic

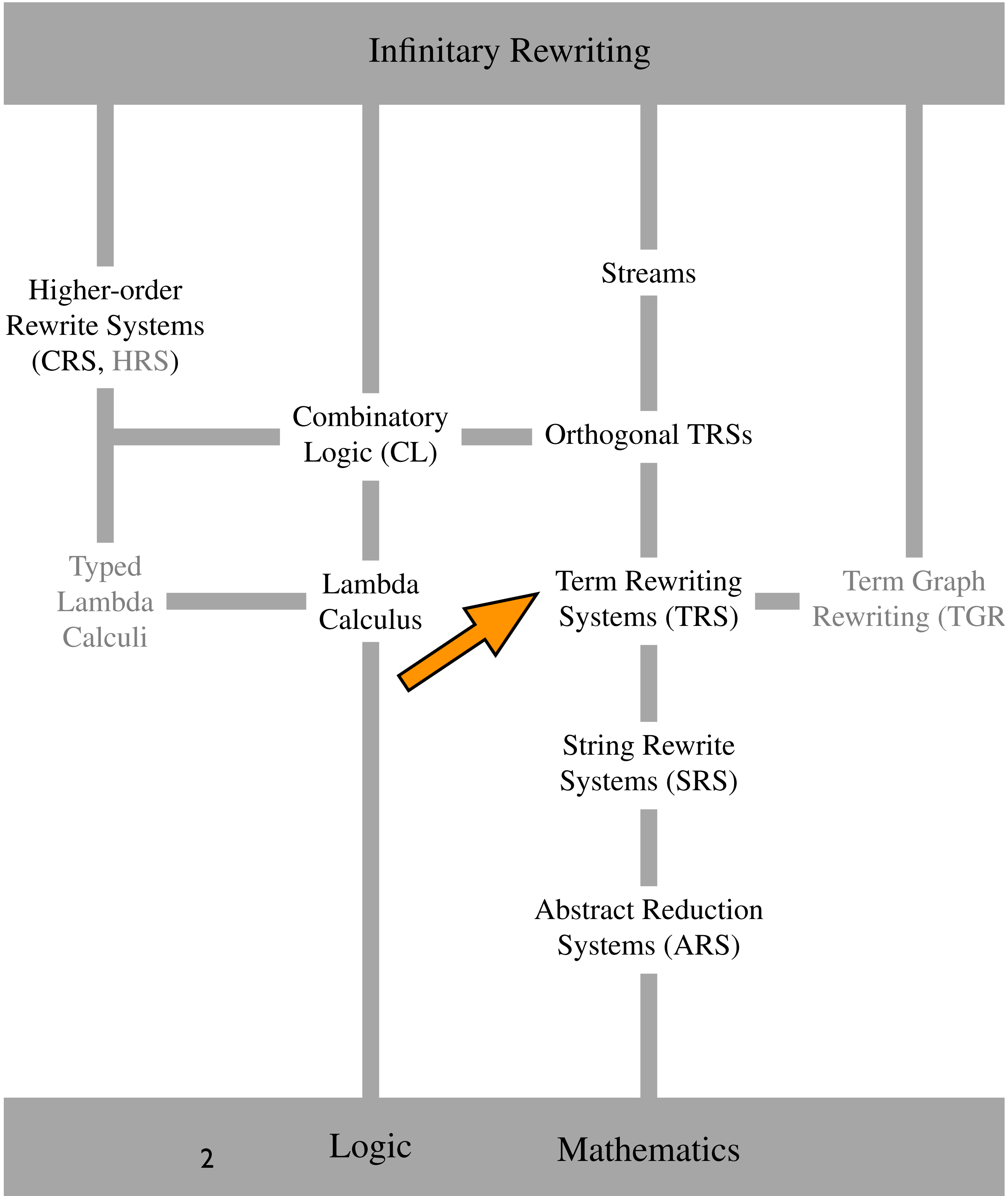
6. Infinitary rewriting

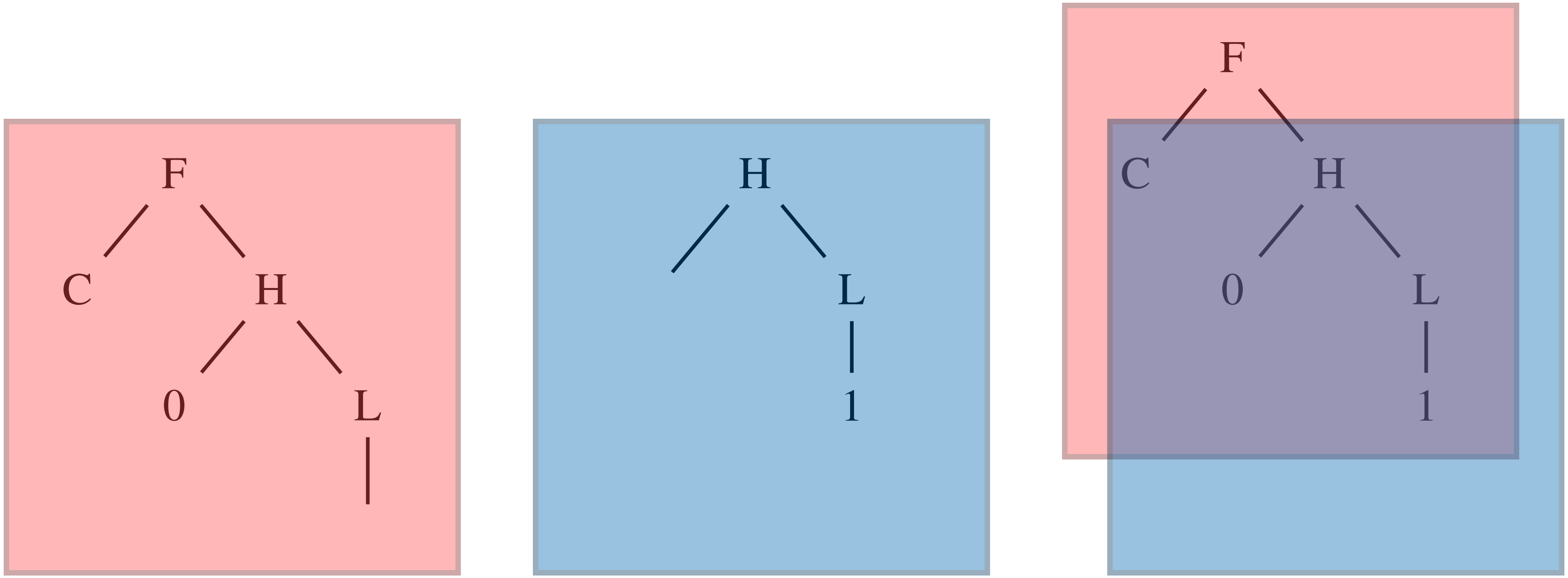
tea, coffee

7.infinitary lambda calculus and the threefold path

8.clocked semantics of lambda calculus

9.streams running forever

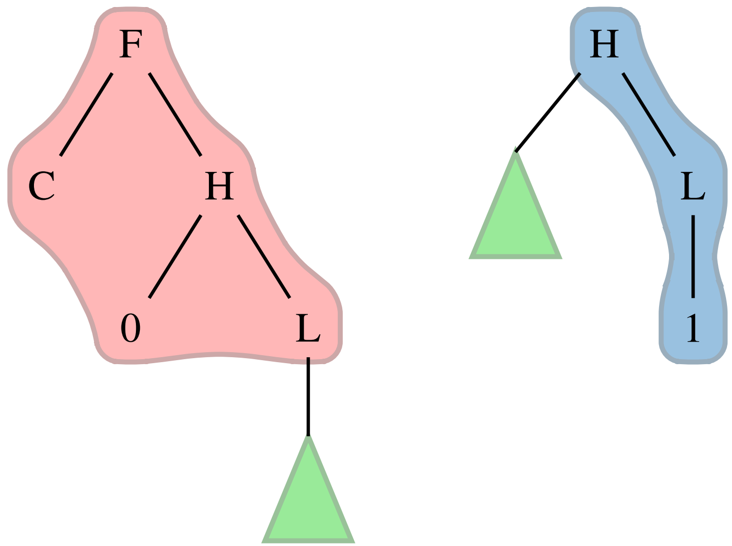

Infinitary Rewriting
Higher-order Rewrite Systems (CRS, HRS)
Streams
Combinatory Logic (CL)
Orthogonal TRSs
Typed Lambda Calculi
Lambda Calculus
Term Rewriting Systems (TRS)
Term Graph Rewriting (TGR
String Rewrite Systems (SRS)
Abstract Reduction Systems (ARS)
Logic
Mathematics

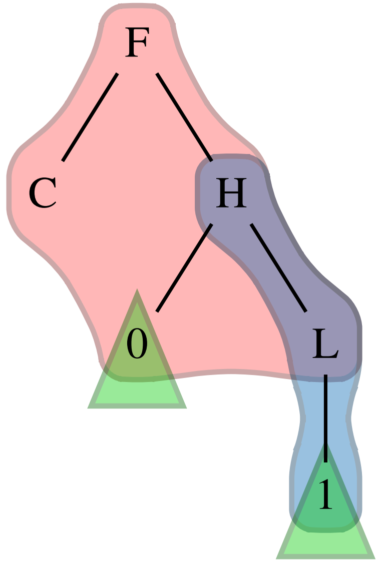


slide 1 slide 2 overlap

$$r_1 : F(C, H(0, L(x))) \rightarrow L(x)$$
$$r_2 : H(y, L(1)) \qquad \rightarrow H(y, y)$$

The term arising from this superposition, $F(C, H(0, L(1)))$, is now subject to two rewritings, as follows.

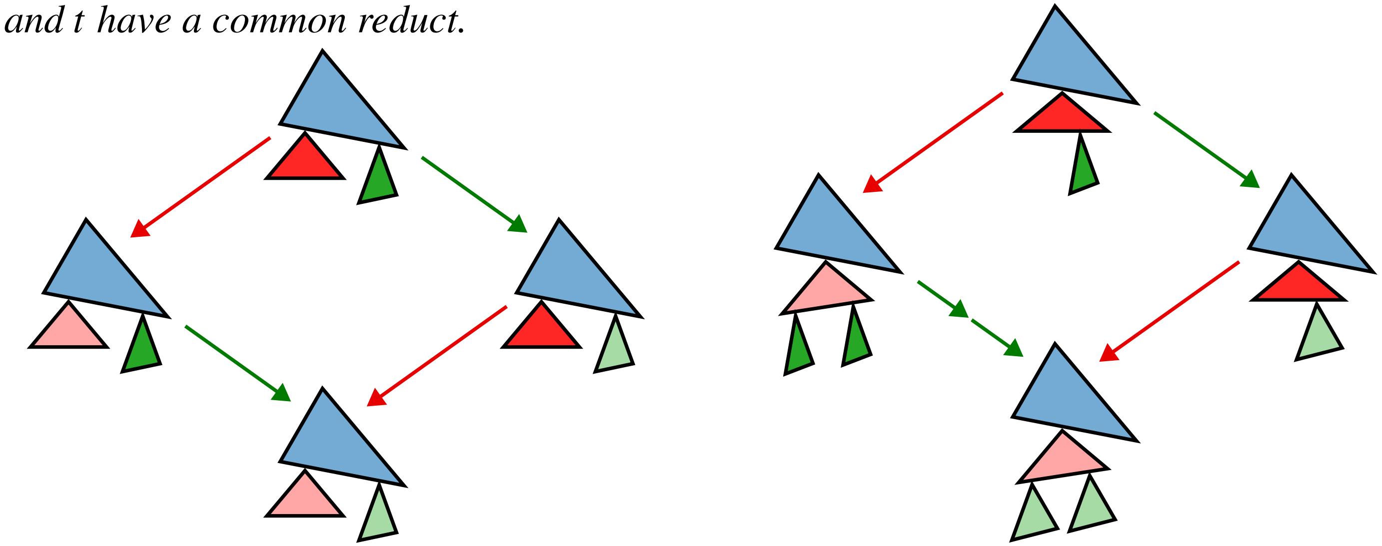




$$F(C,H(0,L(1))) \rightarrow_{r_1} L(1)$$
$$F(C,H(0,L(1))) \rightarrow_{r_2} F(C,H(0,0))$$

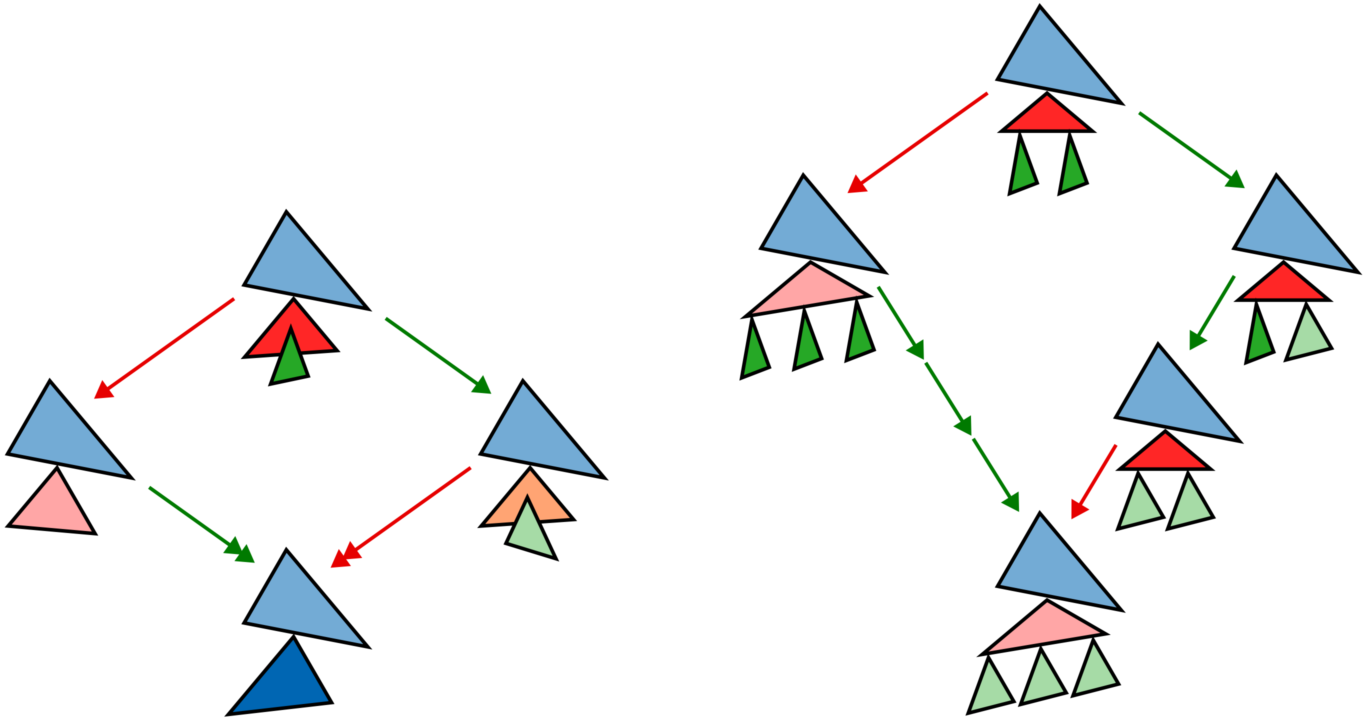
Now $\langle L(1), F(C,H(0,0))\rangle$ is the critical pair generated by this overlapping between r_1 and r_2.

Theorem 5.3. *(Huet [Hue80]) A TRS is weakly confluent iff all its critical pairs* $\langle s,t \rangle$ *are convergent, i.e.* $s \downarrow t$*, in words:* s *and* t *have a common reduct.*



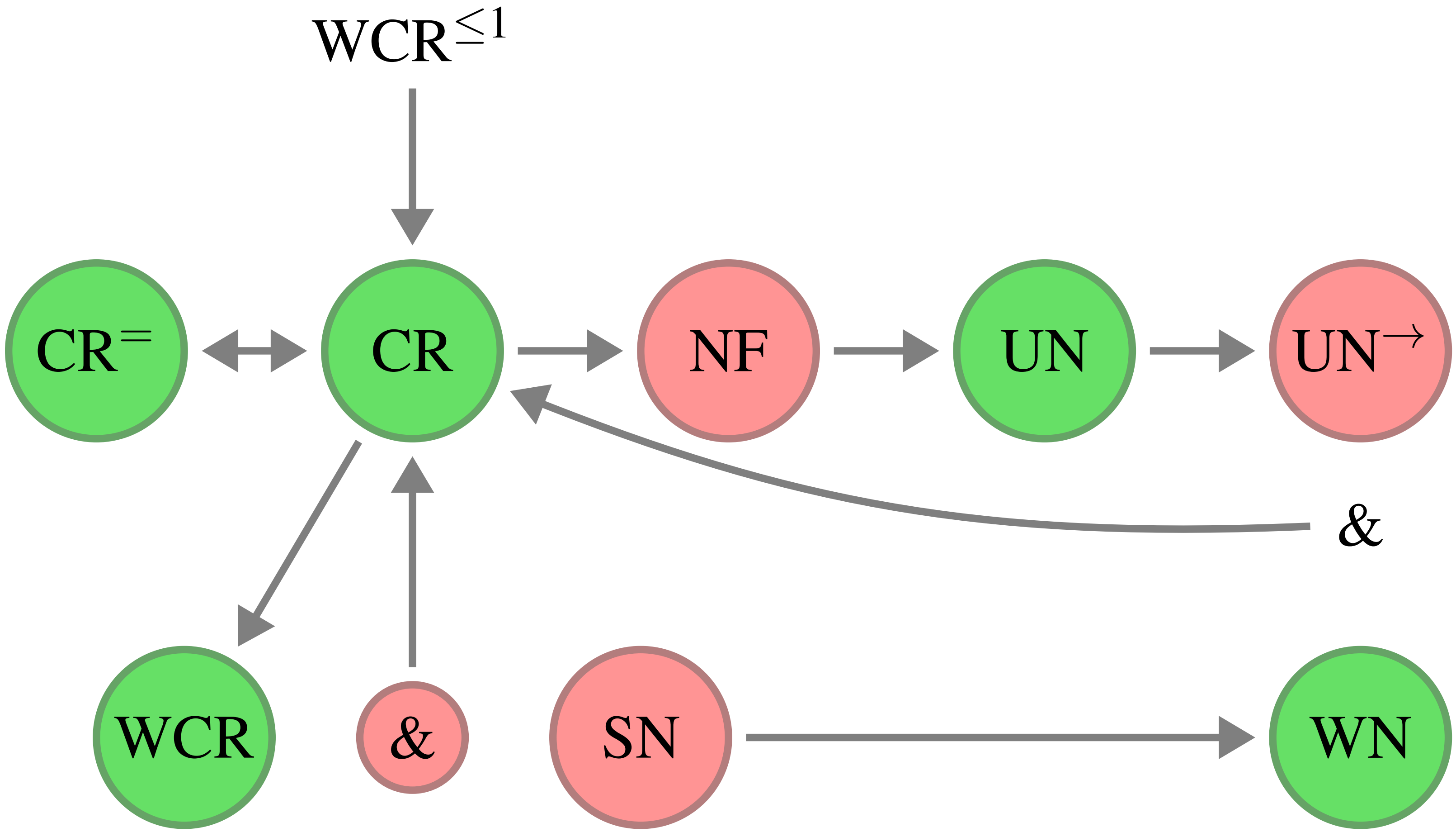
(a) Disjoint redexes

(b) Nested redexes

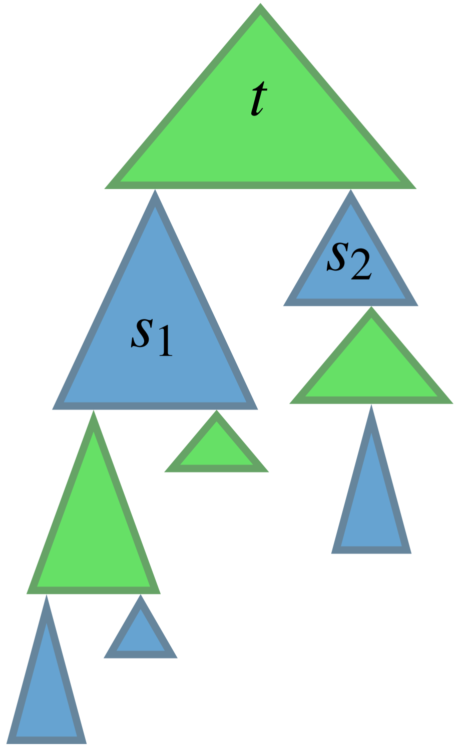


(c) Overlapping redexes

(d) Non-left-linear redexes



WCR$^{\leq 1}$
CR$^{=}$
CR
NF
UN
UN$^{\rightarrow}$
&
WCR
&
SN
WN



t
s1
s2

$A(x,0) \rightarrow x$

$A(x,S(y)) \rightarrow S(A(x,y))$

$F(0) \rightarrow 0$

$F(S(x)) \rightarrow A(F(x),S(x))$

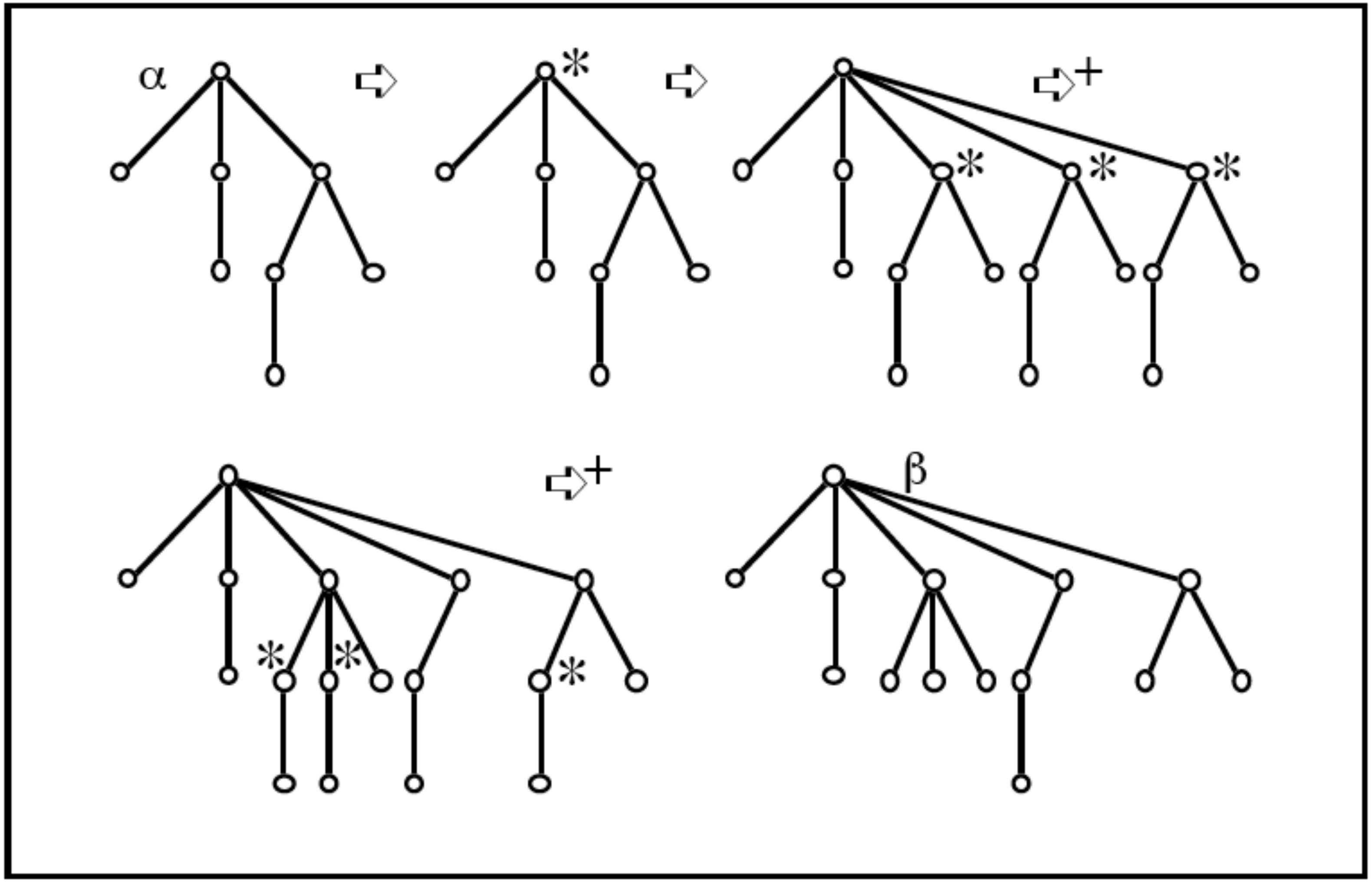
$\mathcal{R}_2$

$M(x,0) \rightarrow 0$

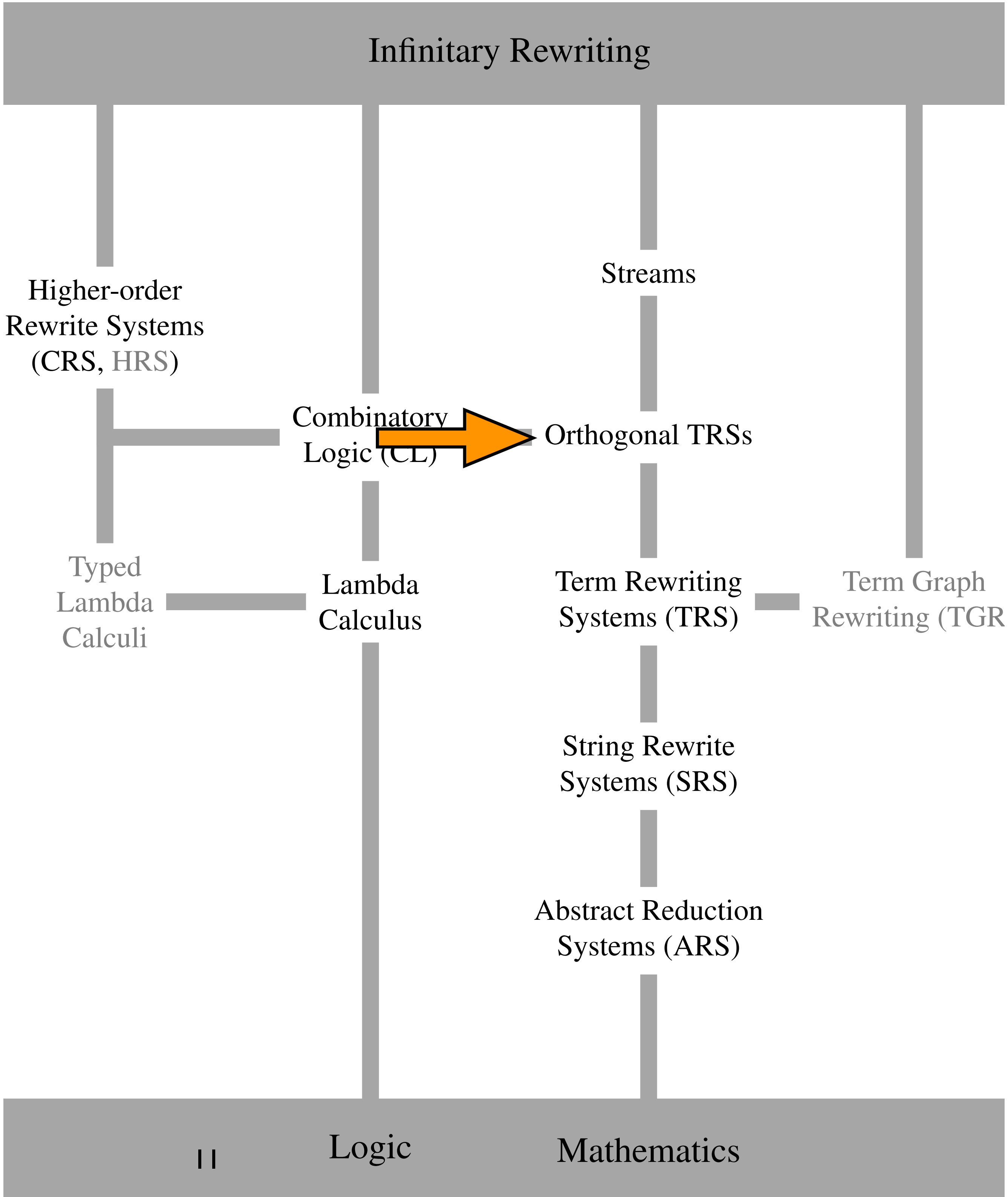
$M(x,S(y)) \rightarrow A(M(x,y),x)$

$\mathcal{D} = \mathcal{R}_1$

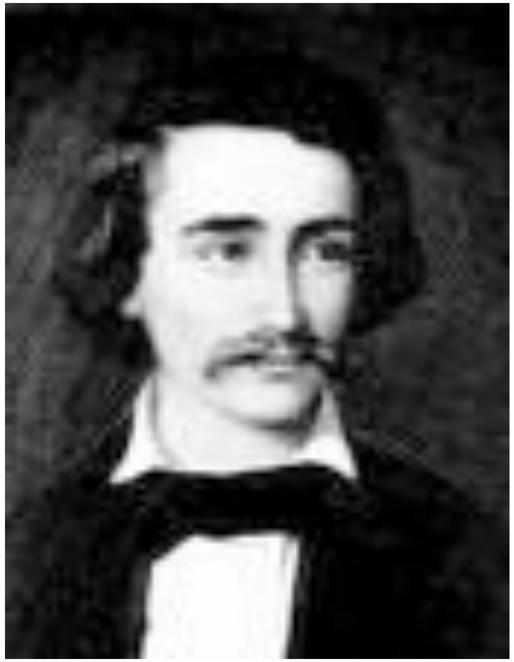
4. term rewriting: divide et impera; termination by stars



Some streets we want to walk



Grassmann 1861, Dedekind 1888



$$A(x, 0) \to x$$

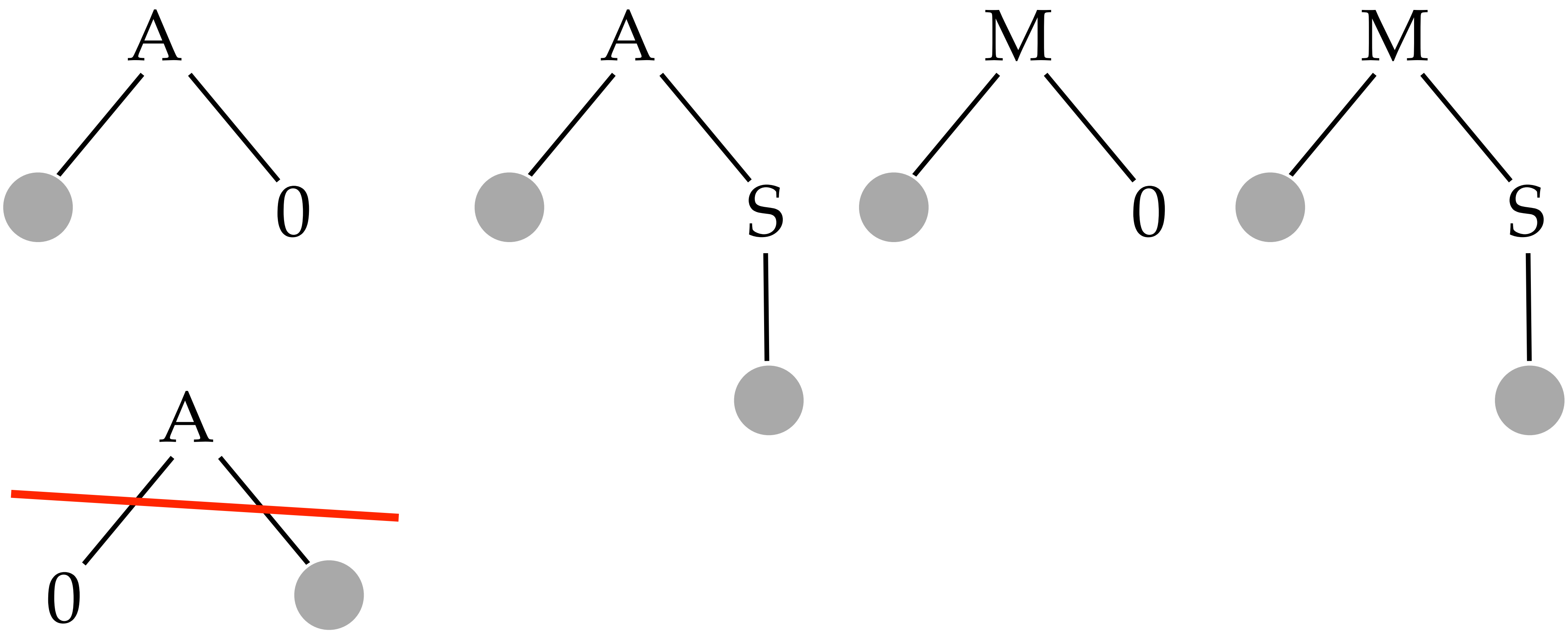
$$A(x, S(y)) \to S(A(x, y))$$

$$M(x, 0) \to 0$$

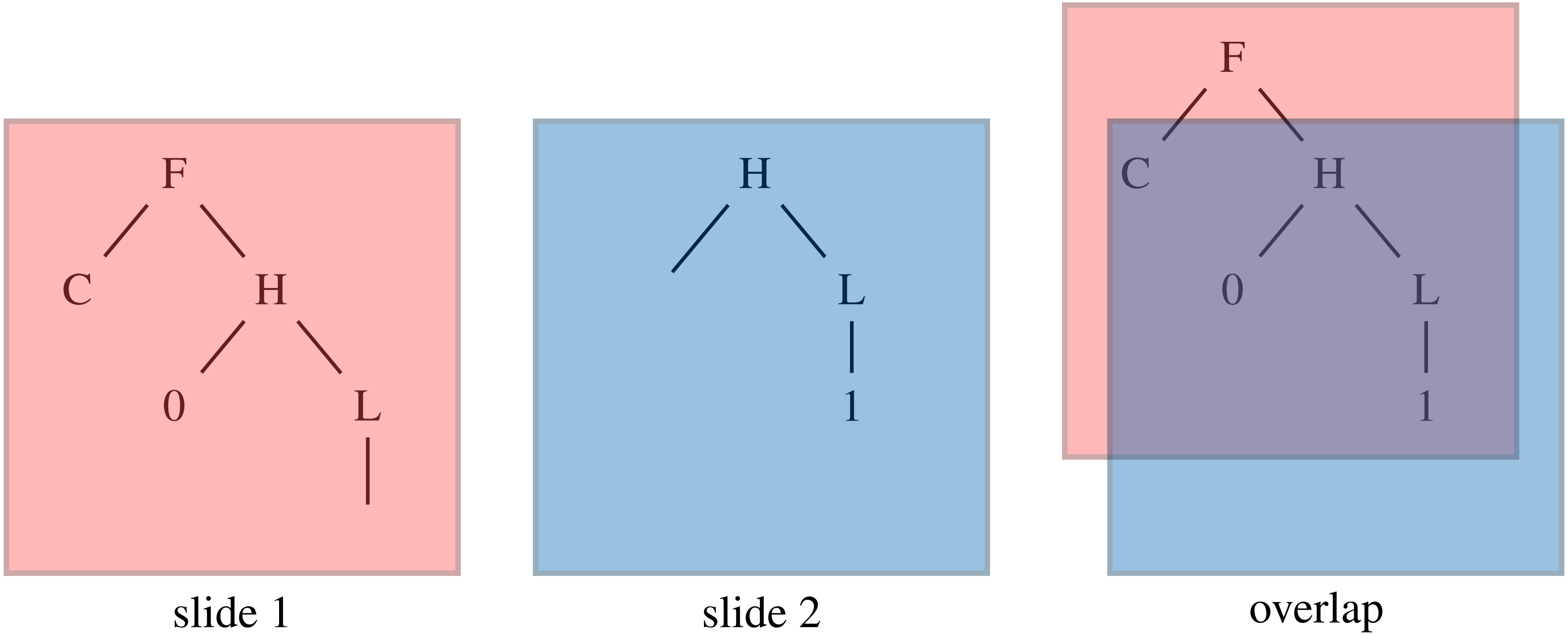
$$M(x, S(y)) \to A(M(x, y), x)$$

$$A(x, 0) \to x$$
$$A(x, S(y)) \to S(A(x, y))$$
$$M(x, 0) \to 0$$
$$M(x, S(y)) \to A(M(x, y), x)$$

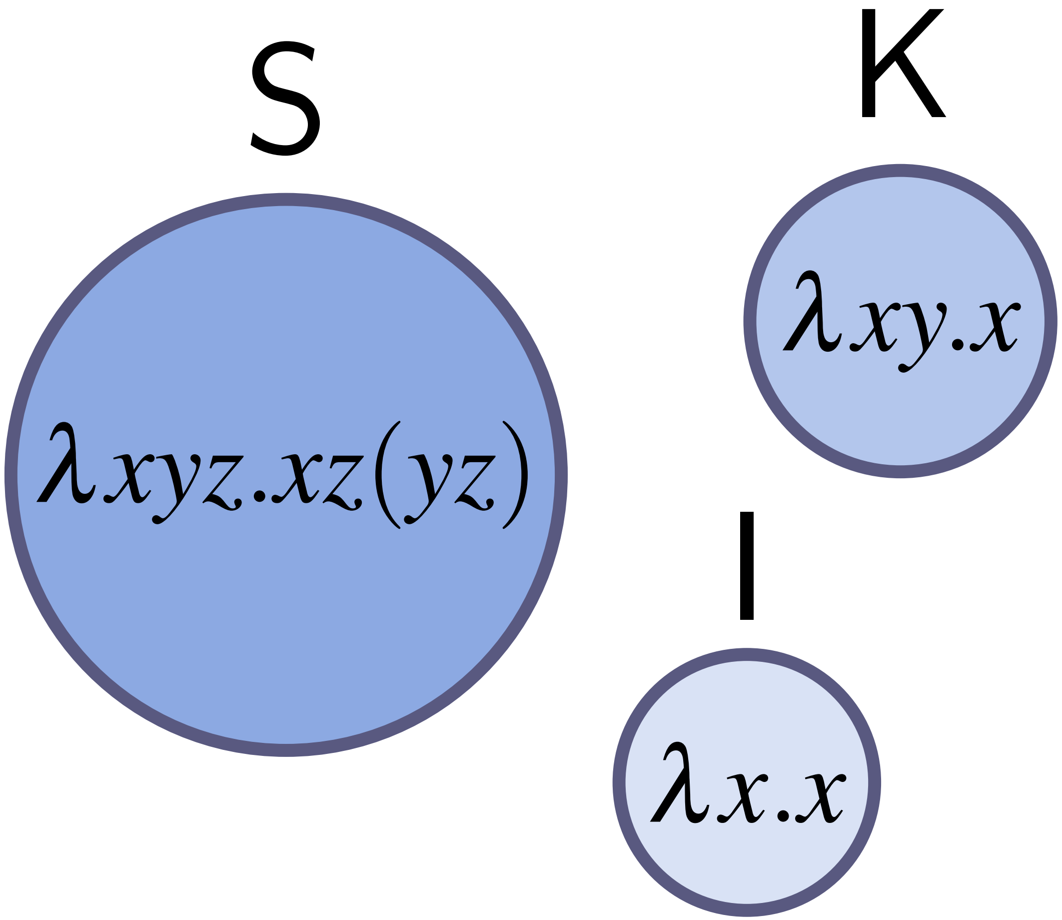
left linear
non-overlapping rules



orthogonal TRSs: no overlaps



and no repeated variables

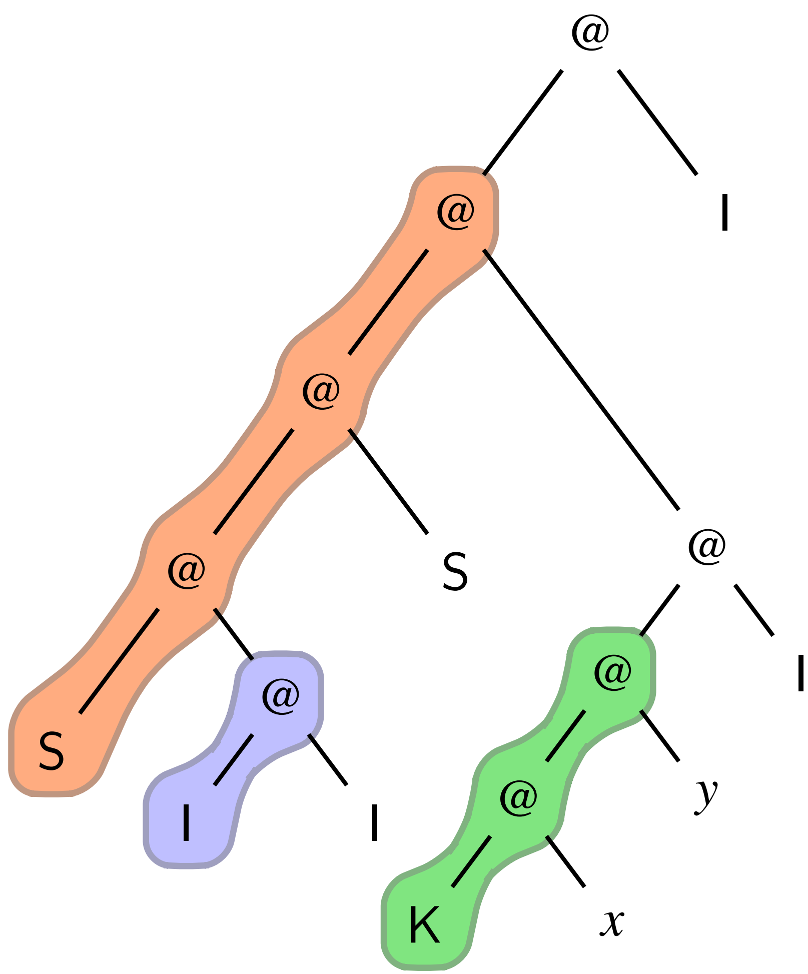


1924. "Über die Bausteine der mathematischen Logik"

Moses Schönfinkel

Combinatory Logic
Turing complete

$$Ix \longrightarrow x$$
$$Kxy \longrightarrow x$$
$$Sxyz \longrightarrow xz(yz)$$

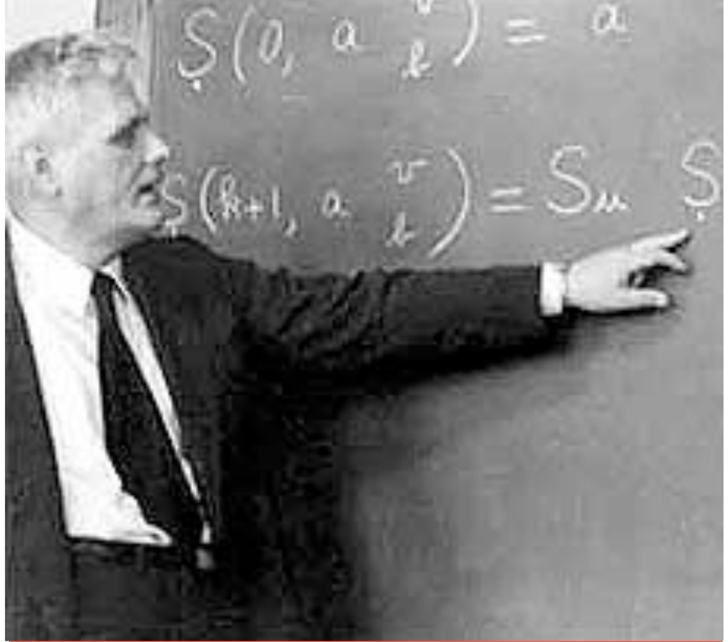


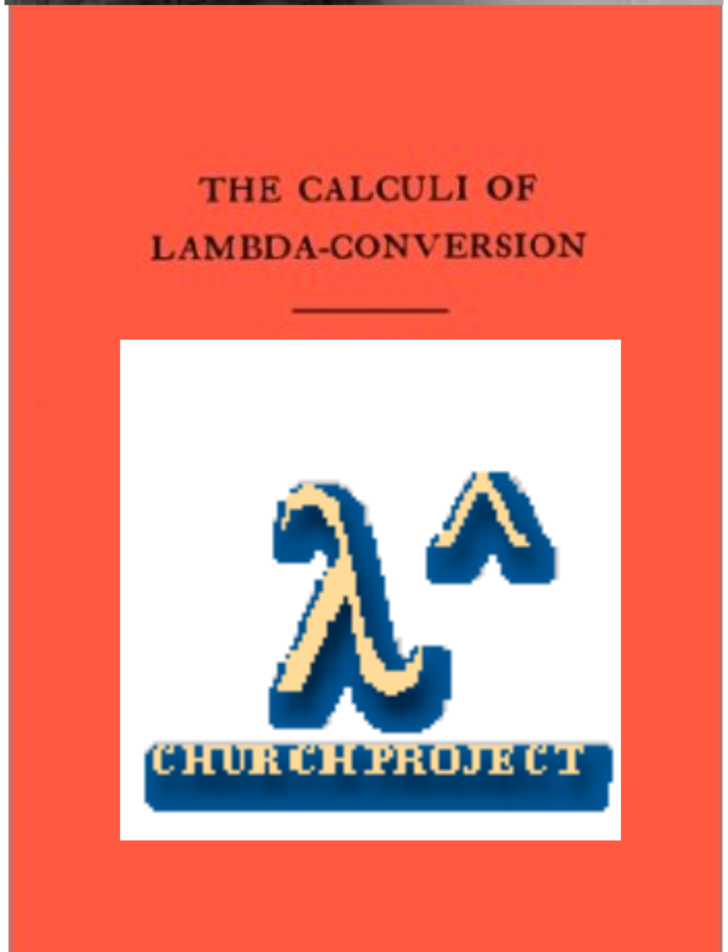
orthogonal, hence confluent

Alonzo Church

1903- 1995

At the time of his death, Church was widely regarded as the greatest living logician in the world

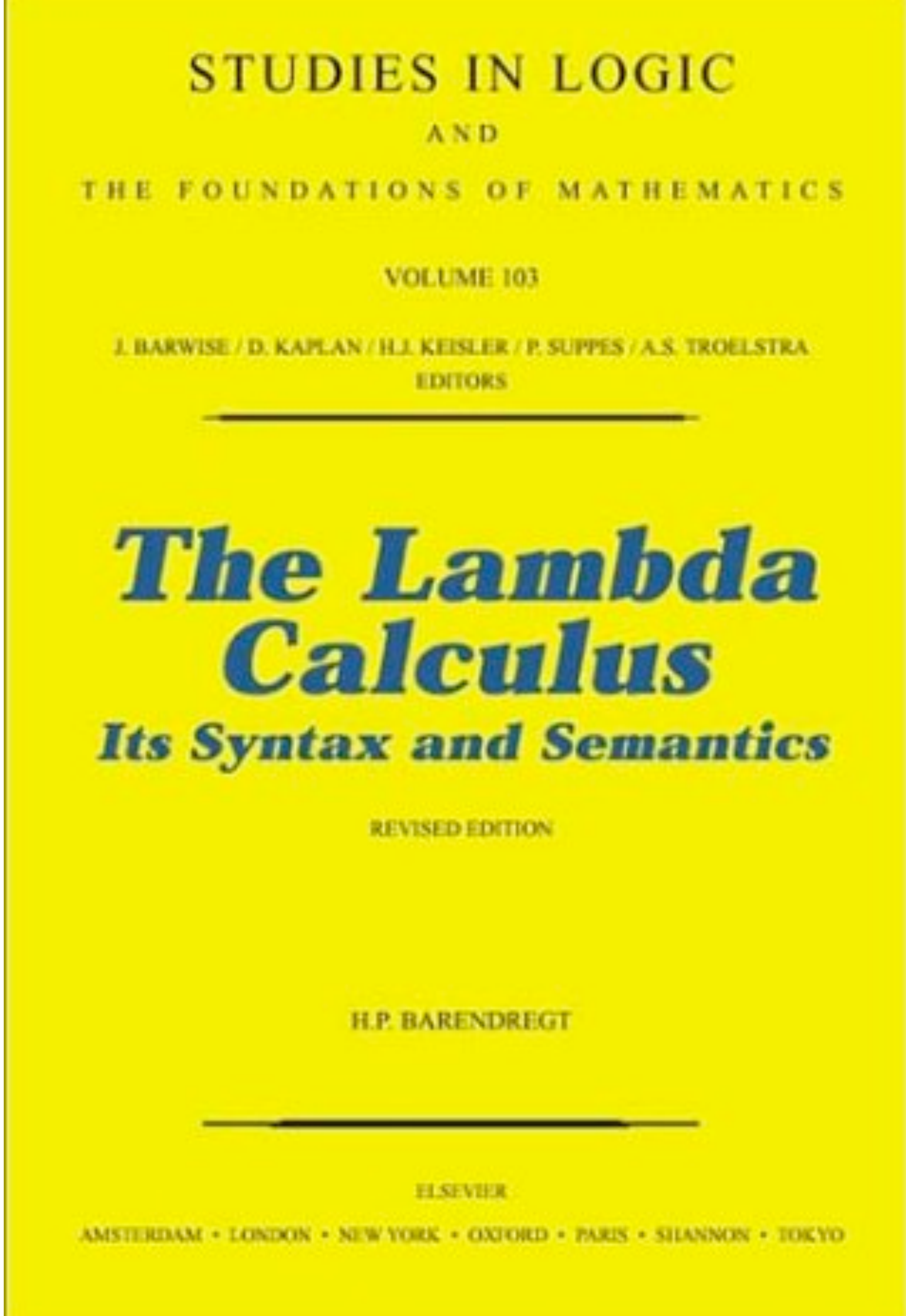


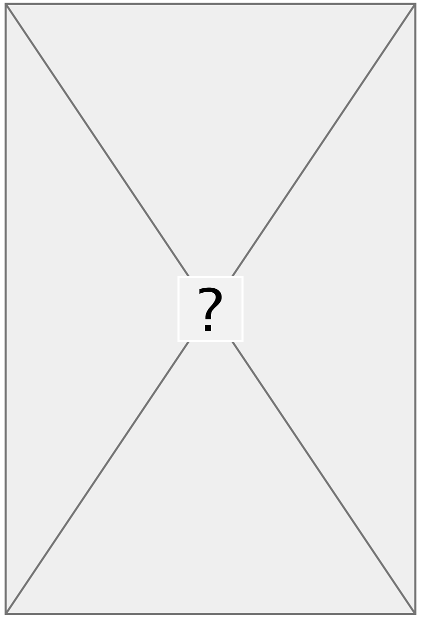


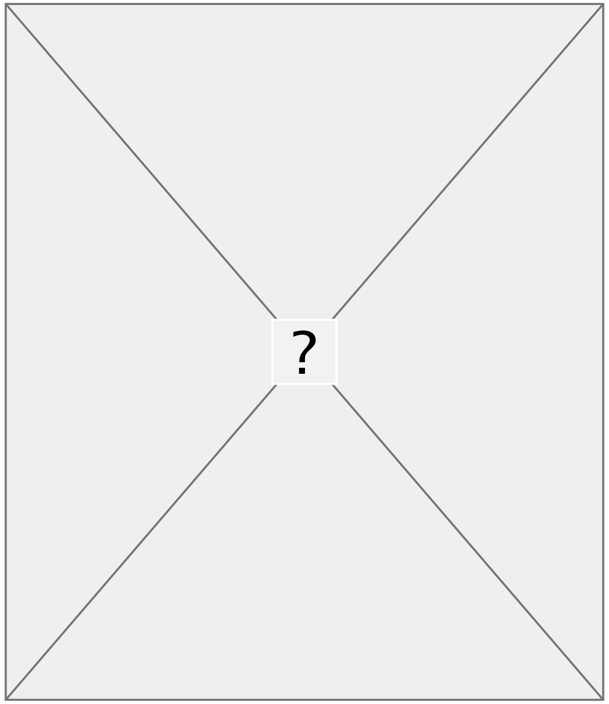
Lambda Calculus

$$(\lambda x.Z(x))Y \longrightarrow Z(Y)$$

Turing complete







(λx.xx)(λx.xx)

ur-cycle

pure 3-cycle

Not in CL!

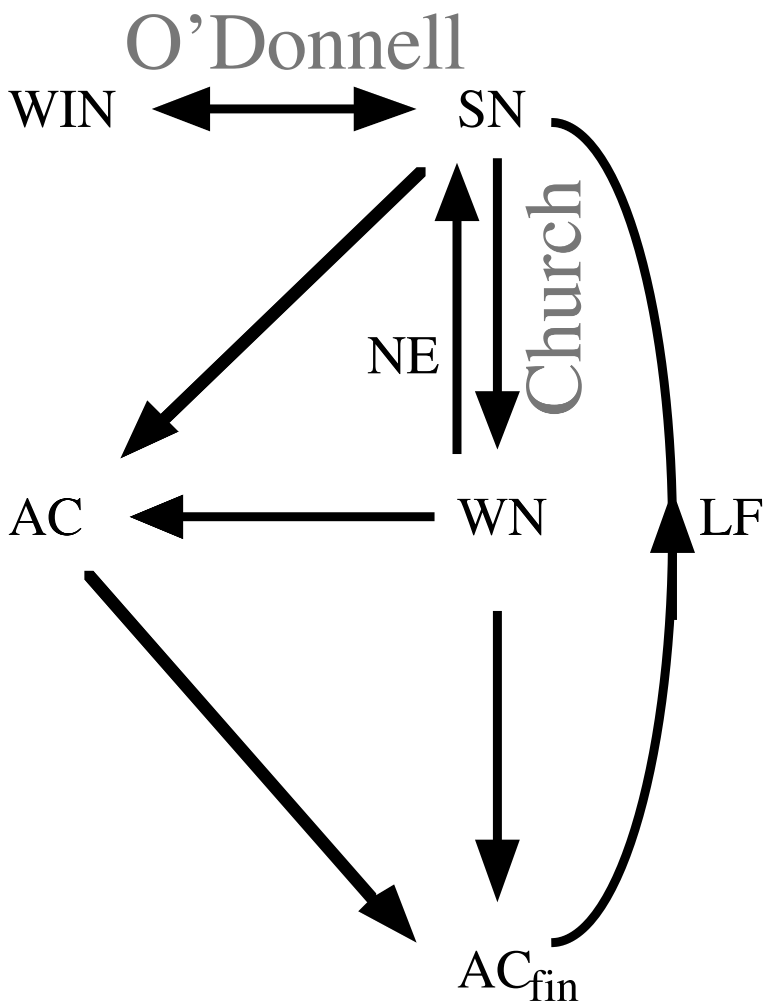
M.H. Sorensen:

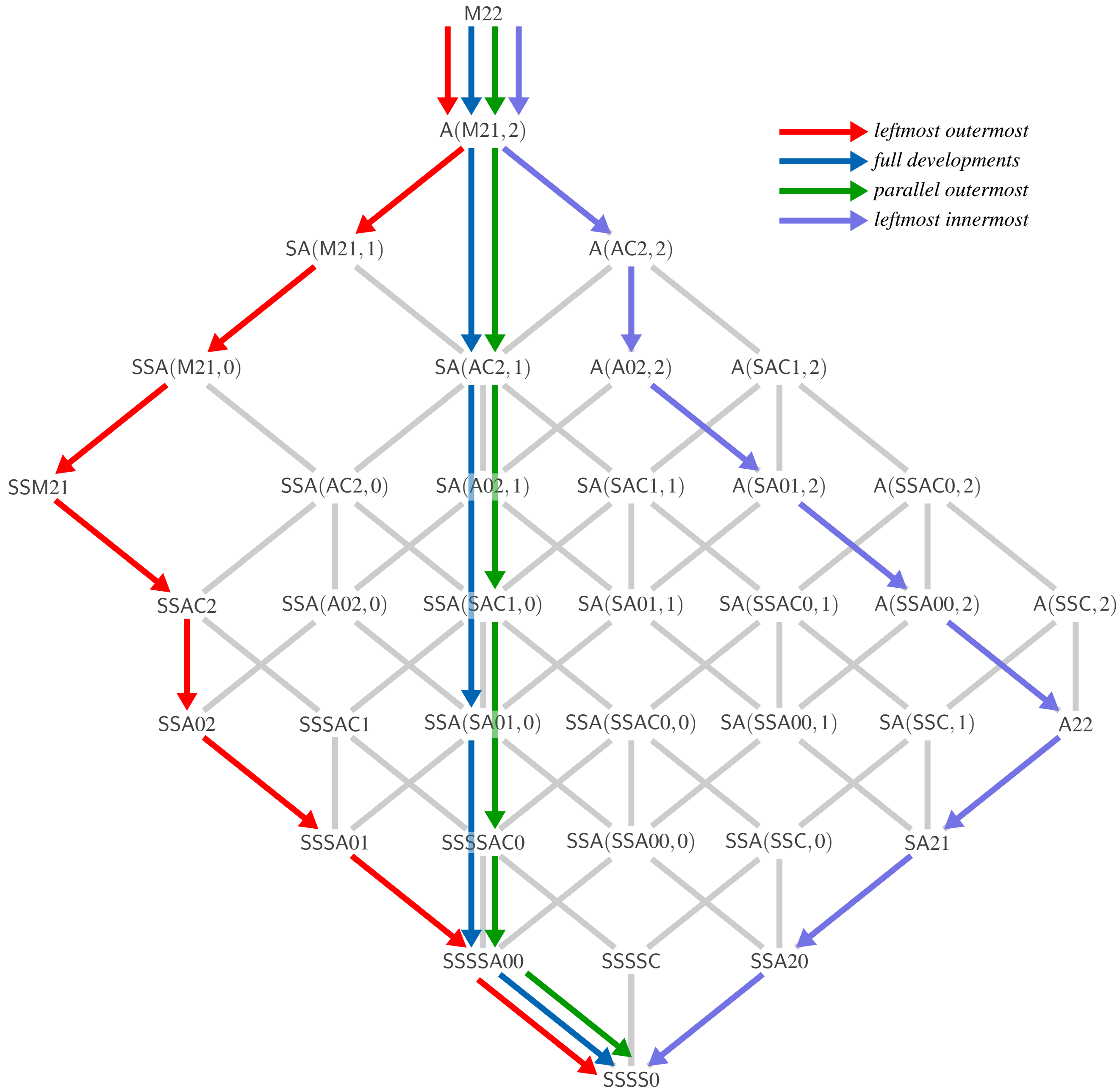
λ-term has infinite reduction ⇒
(λx.xx)(λx.xx) *is a subword*

$$(\lambda xy.y(xxy))(\lambda xy.y(xxy))$$

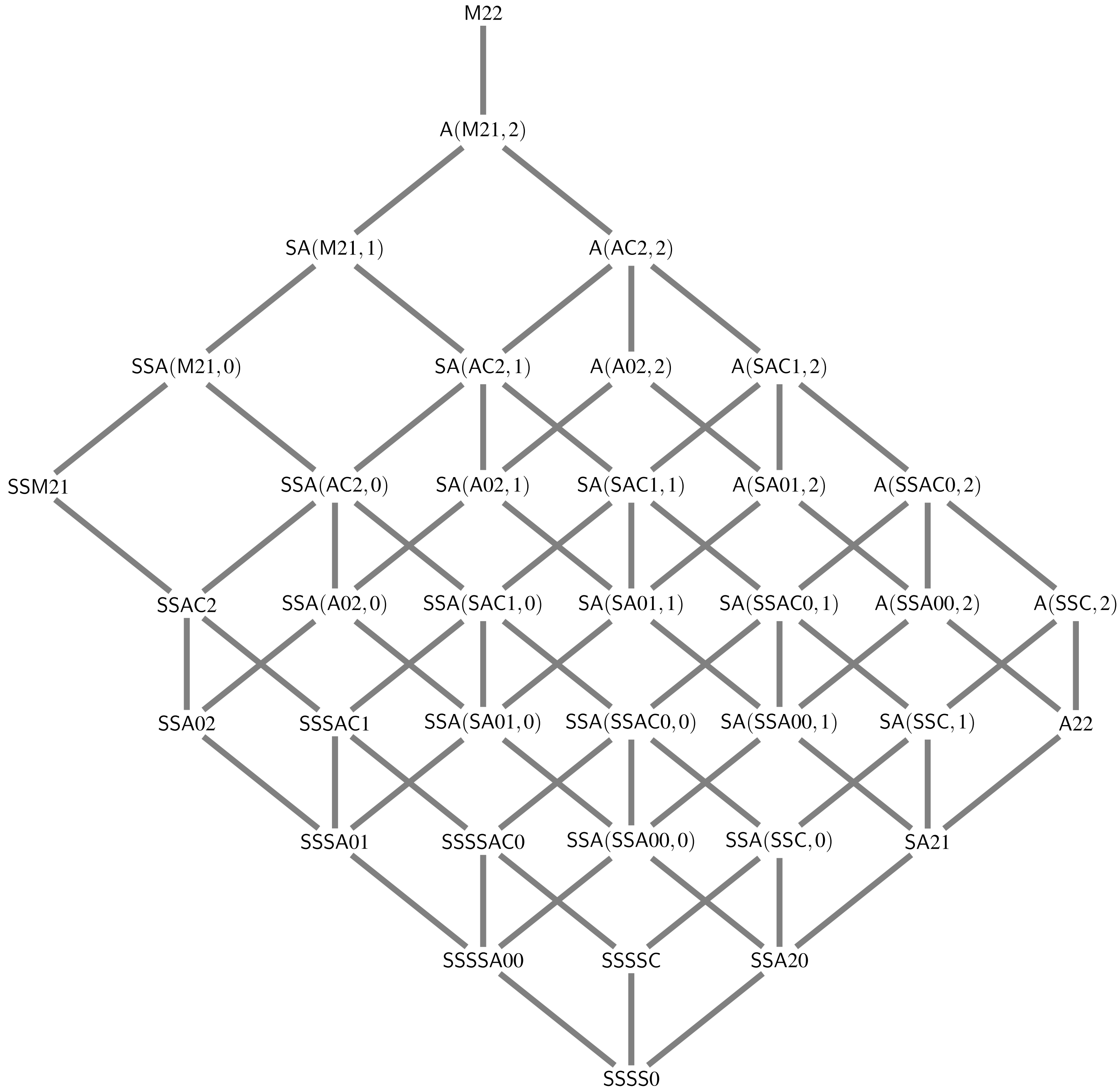
The TRS of S-terms, fragment of CL was another favourite passtime

- *is not SN:* SSS(SSS)(SSS) has infinite reduction (Barendregt earns 25 guilders)
- *has no cycles* (Bergstra)
- *is top terminating* (Waldmann)

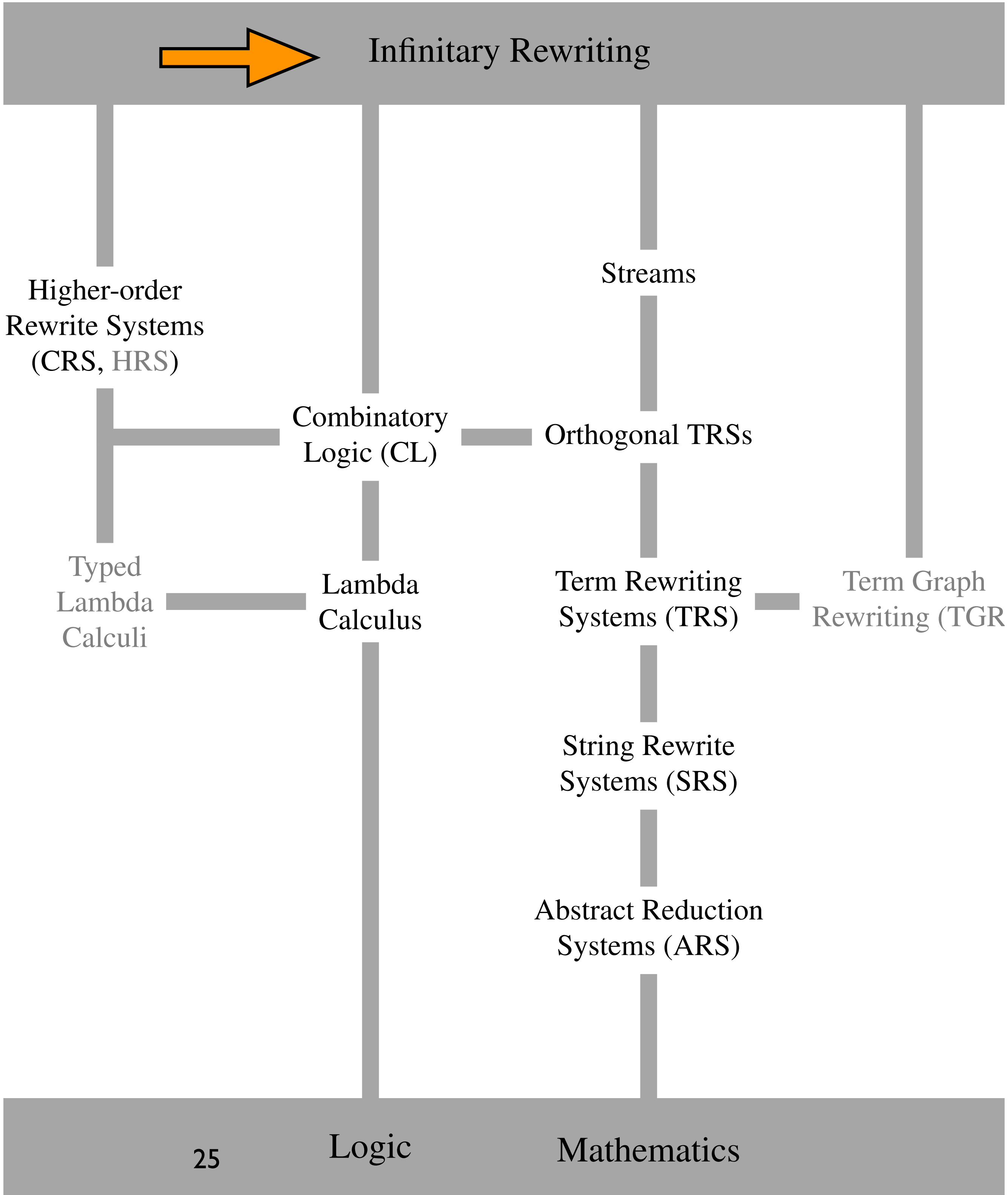

O'Donnell
WIN
SN
Church
NE
AC
WN
LF
AC$_{fin}$



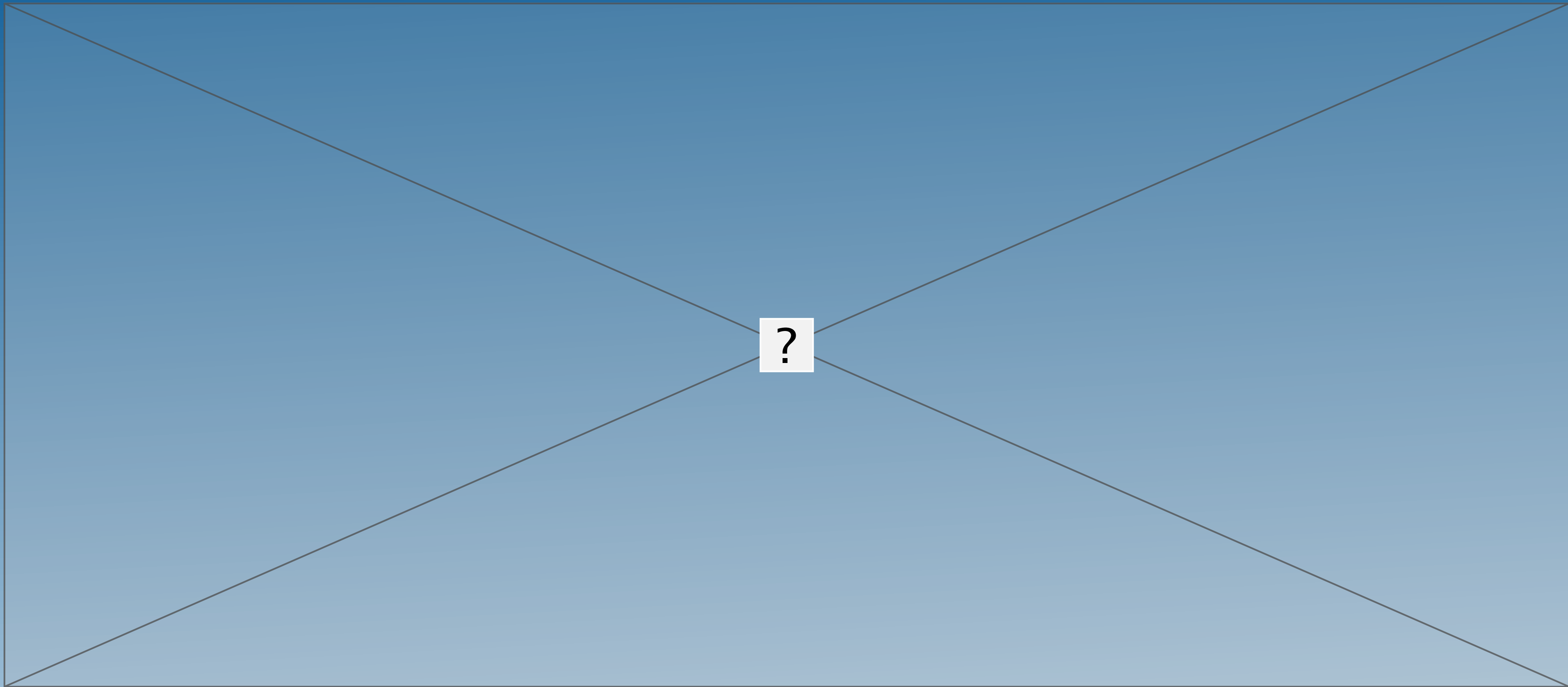
leftmost outermost
full developments
parallel outermost
leftmost innermost
M22
A(M21,2)
SA(M21,1)
A(AC2,2)
SSA(M21,0)
SA(AC2,1)
A(A02,2)
A(SAC1,2)
SSM21
SSA(AC2,0)
SA(A02,1)
SA(SAC1,1)
A(SA01,2)
A(SSAC0,2)
SSAC2
SSA(A02,0)
SSA(SAC1,0)
SA(SA01,1)
SA(SSAC0,1)
A(SSA00,2)
A(SSC,2)
SSA02
SSSAC1
SSA(SA01,0)
SSA(SSAC0,0)
SA(SSA00,1)
SA(SSC,1)
A22
SSSA01
SSSSAC0
SSA(SSA00,0)
SSA(SSC,0)
SA21
SSSSA00
SSSSC
SSA20
SSSS0



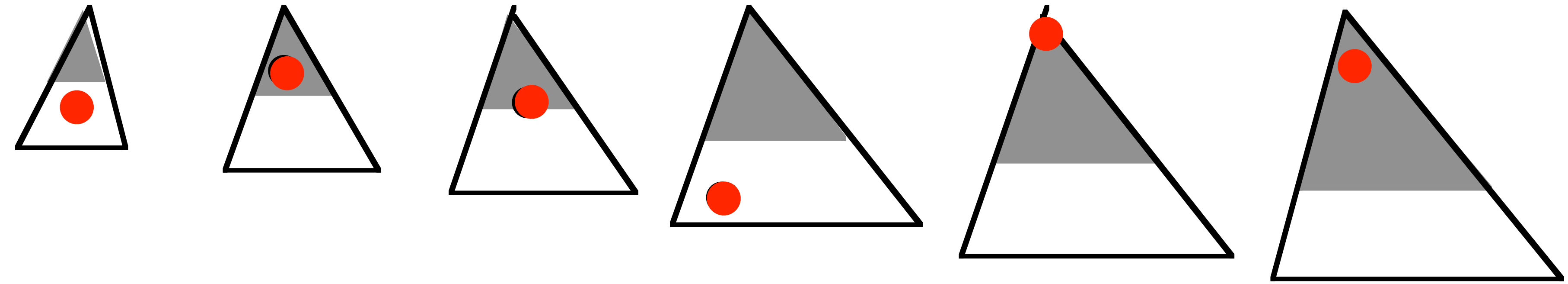
M22
A(M21, 2)
SA(M21, 1)
A(AC2, 2)
SSA(M21, 0)
SA(AC2, 1)
A(A02, 2)
A(SAC1, 2)
SSM21
SSA(AC2, 0)
SA(A02, 1)
SA(SAC1, 1)
A(SA01, 2)
A(SSAC0, 2)
SSAC2
SSA(A02, 0)
SSA(SAC1, 0)
SA(SA01, 1)
SA(SSAC0, 1)
A(SSA00, 2)
A(SSC, 2)
SSA02
SSSAC1
SSA(SA01, 0)
SSA(SSAC0, 0)
SA(SSA00, 1)
SA(SSC, 1)
A22
SSSA01
SSSSAC0
SSA(SSA00, 0)
SSA(SSC, 0)
SA21
SSSSA00
SSSSC
SSA20
SSSS0


Infinitary Rewriting
Higher-order
Rewrite Systems
(CRS, HRS)
Streams
Combinatory
Logic (CL)
Orthogonal TRSs
Typed
Lambda
Calculi
Lambda
Calculus
Term Rewriting
Systems (TRS)
Term Graph
Rewriting (TGR
String Rewrite
Systems (SRS)
Abstract Reduction
Systems (ARS)
Logic
Mathematics

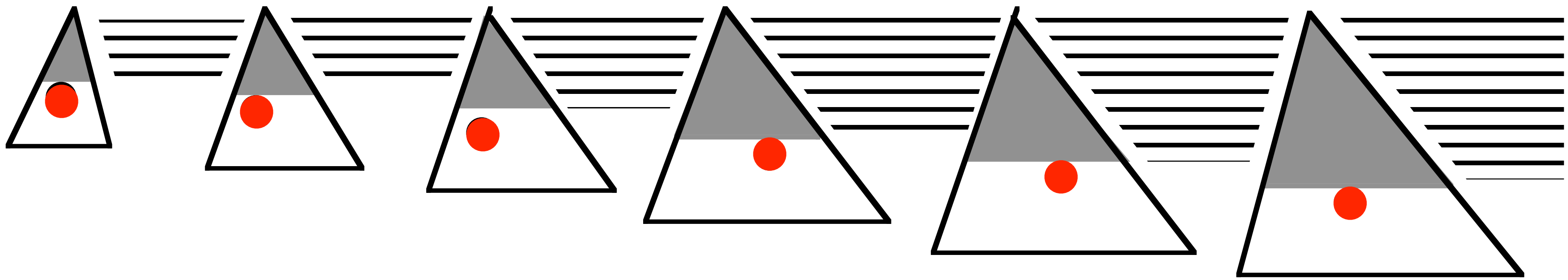
$$F(x) \to P(x, F(S(x)))$$



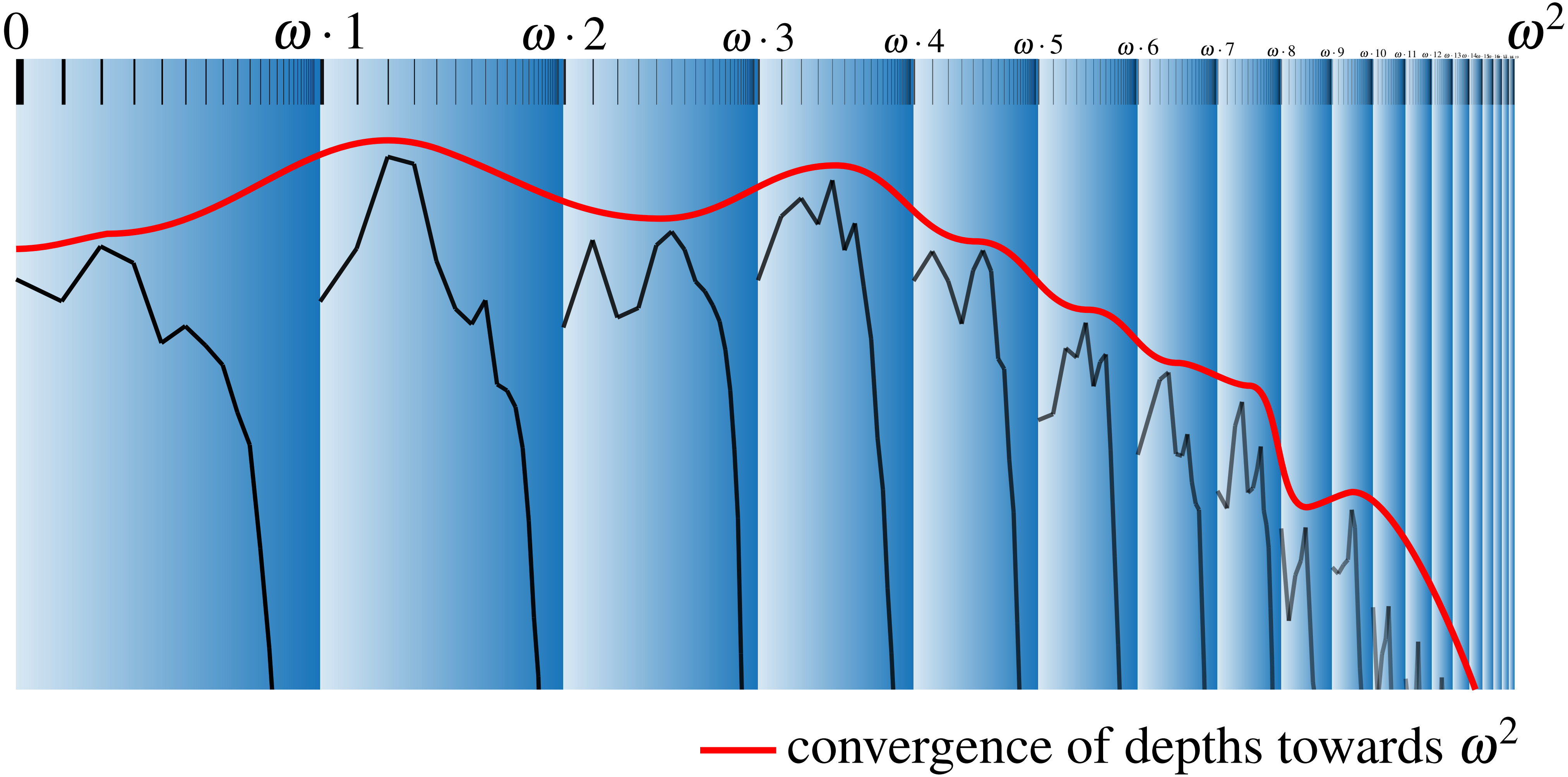
$$F(x) \rightarrow P(x, F(S(x)))$$

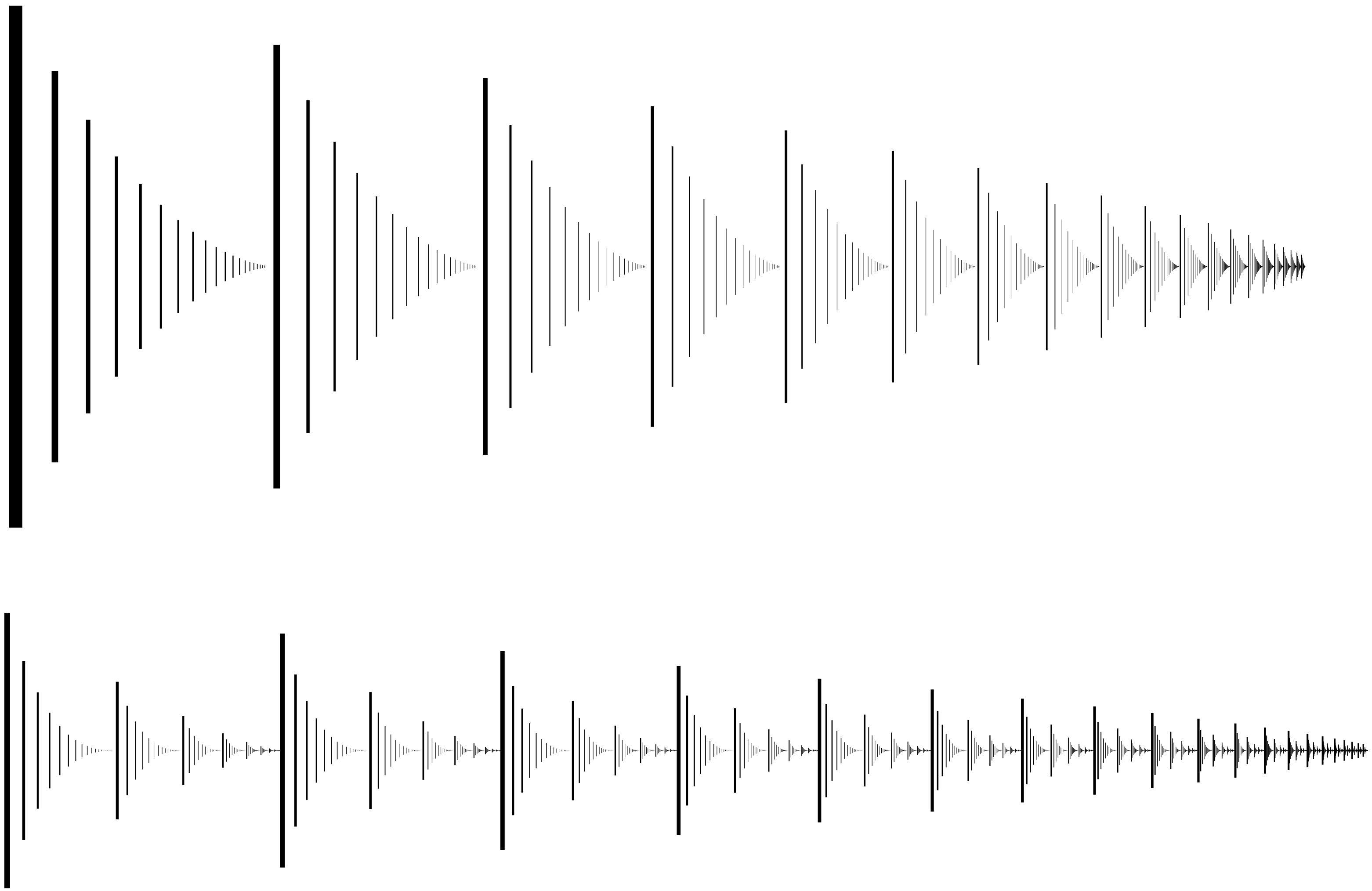


Cauchy converging reduction sequence: activity may occur everywhere



Strongly converging reduction sequence, with descendant relations


0
$\omega \cdot 1$
$\omega \cdot 2$
$\omega \cdot 3$
$\omega \cdot 4$
$\omega \cdot 5$
$\omega \cdot 6$
$\omega \cdot 7$
$\omega \cdot 8$
$\omega \cdot 9$
$\omega \cdot 10$
$\omega \cdot 11$
$\omega \cdot 12$
$\omega \cdot 13$
ω^2
convergence of depths towards ω^2



Ordinals ω^2 and ω^3 embedded in the reals, order-respecting.

Exercise: which ordinals can be embedded in the real segment [0,1]?

(i) $(\omega^\omega \cdot 2 + \omega^3 \cdot 4 + \omega^2) + (\omega^3 \cdot 3 + \omega^2 \cdot 2 + 1) = \omega^\omega \cdot 2 + \omega^3 \cdot 7 + \omega^2 \cdot 2 + 1$

(ii) $(\omega^6 \cdot 3 + \omega^2 \cdot 4 + 2) + (\omega^4 \cdot 5 + \omega^2) = \omega^6 \cdot 3 + \omega^4 \cdot 5 + \omega^2$

(iii) $(\omega^{\omega+2} \cdot 3 + \omega^\omega + \omega + 7) \cdot (\omega^{\omega+1} \cdot 2 + \omega^\omega + 3) = \omega^{\omega \cdot 2+1} \cdot 2 + \omega^{\omega \cdot 2} + \omega^{\omega+2} \cdot 9 + \omega^\omega + \omega + 7$

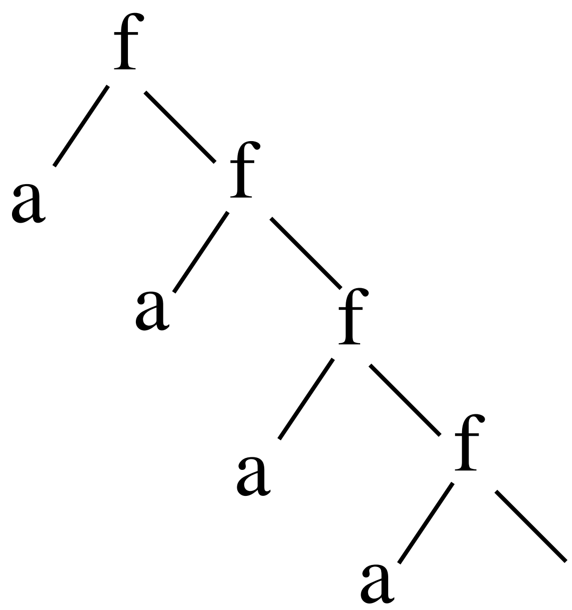
But if you don't like ordinals, there is for orthogonal TRSs the Compression Lemma:

every reduction of length α *can be compressed to* ω *or less.*

use dove-tailing

Every countable ordinal can be the length of an infinite reduction. Consider the TRS

$\{c \longrightarrow f(a, c) \text{ and } a \longrightarrow b\}$



finitary rewriting	infinitary rewriting
finite reduction	*strongly convergent reduction*
infinite reduction	*divergent reduction*
normal form	*(poss. infinite) normal form*
CR: *finite coinitial reductions can be joined*	CR^{∞}: *infinite coinitial reductions can be joined*
UN: *coinitial reductions to nf end in same nf*	UN^{∞}: *coinitial reductions to nf end in same nf*
SN: *there are no infinite reductions*	SN^{∞}: *there are no divergent reductions*
WN: *there is a reduction to nf*	WN^{∞}: *there is a reduction to nf*

How to define SN^{∞} and WN^{∞}?

WN^{∞} is easy: There is a possibly infinite reduction to the possibly infinite normal form.

SN^{∞} : all reductions will eventually terminate in the normal form. The only way such a reduction could fail to reach a normal form, is that it stagnates at some point in the tree which is developing, for infinitely many steps. Then no limit can be taken.

Good and bad reductions. In ordinary rewriting the finite reductions are good, they have an end point, and the infinite ones are bad, they have no end point.

Same in infinitary rewriting. The good reductions are the ones that are strongly convergent, they have an end point. E.g.
$a \to b(a)$ reaches after ω steps the end point b^{ω}.

The bad reductions (divergent, stagnating) are the ones without an end point. Their reductions may be long, a limit ordinal long, but there they fail.

SN^{∞} states that there are no bad reductions.

In other words: say we select at random in each step a redex and perform this step. We can go on until we reach a limit ordinal. At that point we look back, and if the reduction was strongly convergent we take the limit and go on. If not, we stop there and we had a bad reduction.

CLAIM: we can then identify a stagnating term, a term where infinitely often a root step was performed.

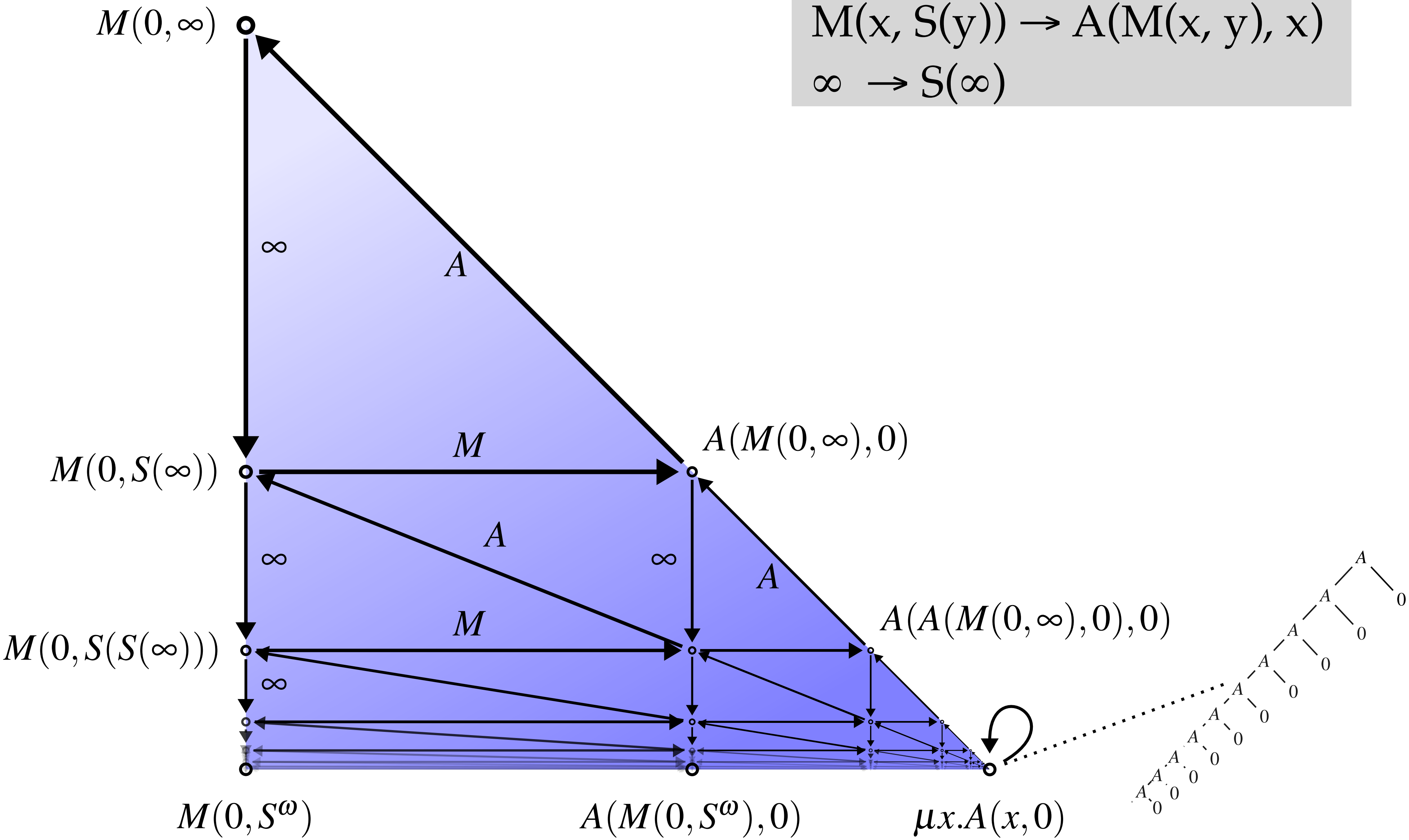
$A(x, 0) \to x$

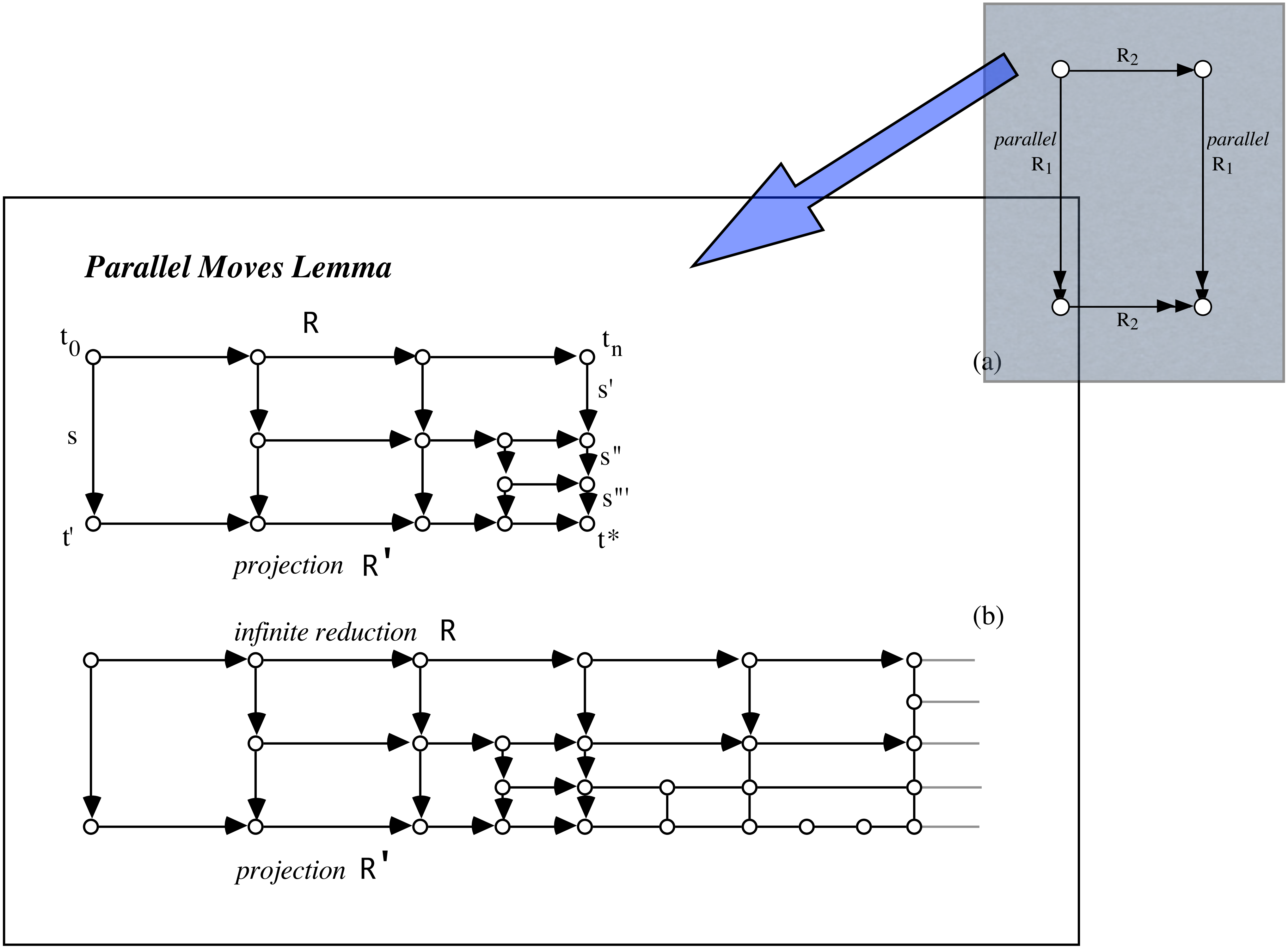
$A(x, S(y)) \to S(A(x, y))$

$M(x, 0) \to 0$

$M(x, S(y)) \to A(M(x, y), x)$

$\infty \to S(\infty)$



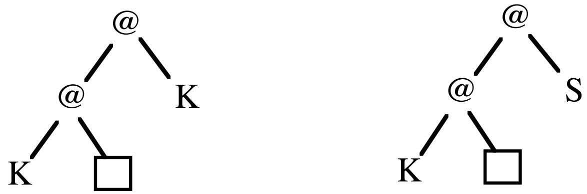

Parallel Moves Lemma
R
t_0
t_n
s
s'
s"
s"'
t'
t*
projection R'
infinite reduction R
projection R'
(a)
(b)
R_2
parallel R_1
parallel R_1
R_2

PML$^\infty$ *For first order infinitary term rewriting we have the infinitary Parallel Moves Lemma PML*$^\infty$

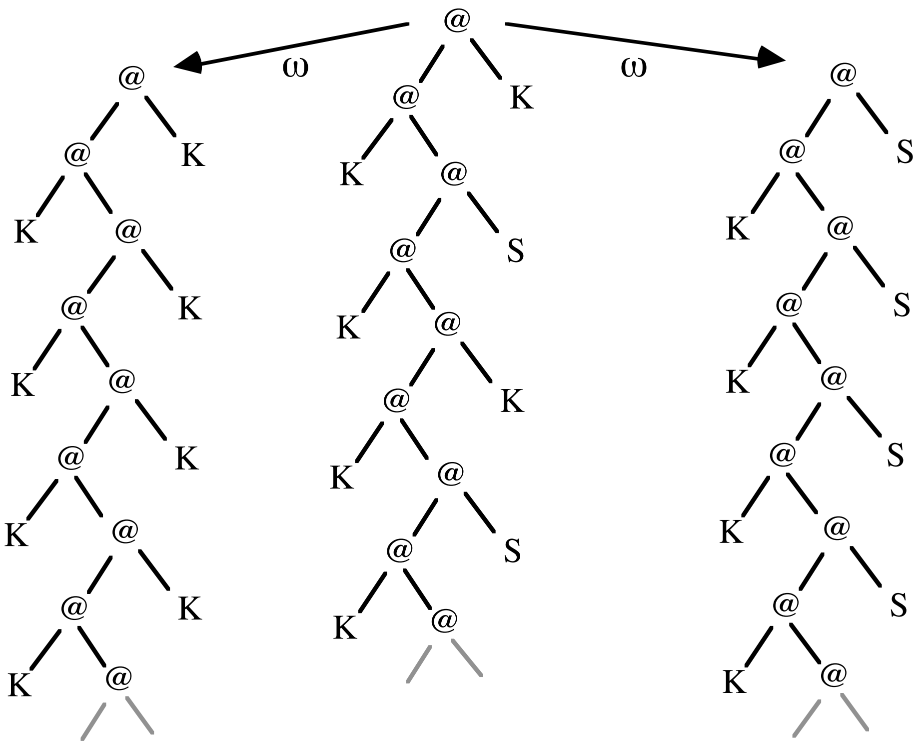
not CR^{∞}

?

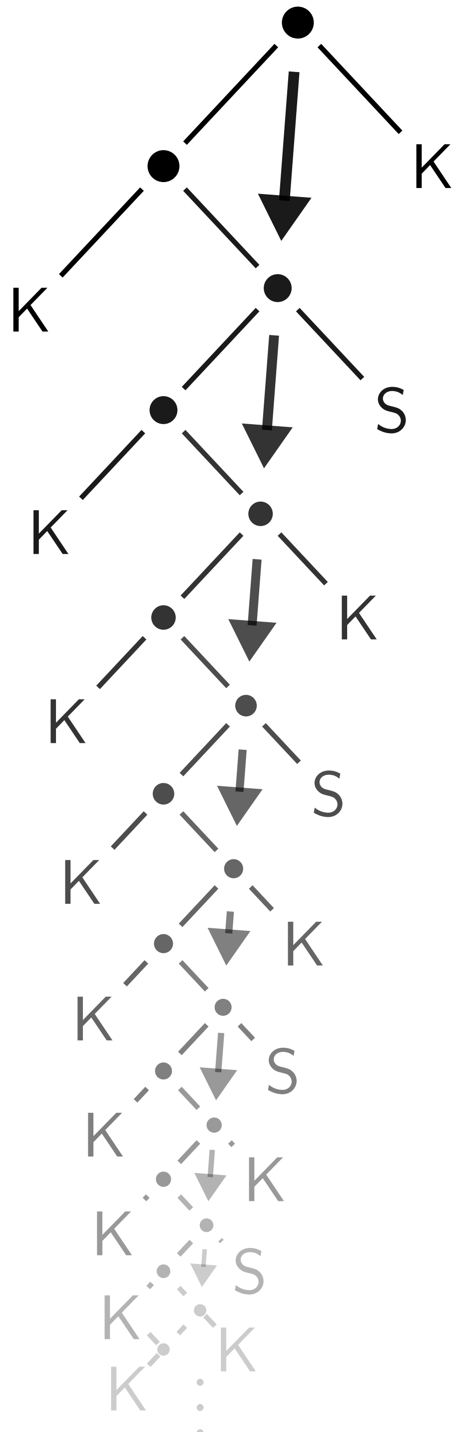
Sxyz $\rightarrow$	xz(yz)	@(@(@(S, x), y), z) $\rightarrow$	@(@(x, z), @(y, z))
Kxy $\rightarrow$	x	@(@(K, x), y) $\rightarrow$	x



collapsing contexts

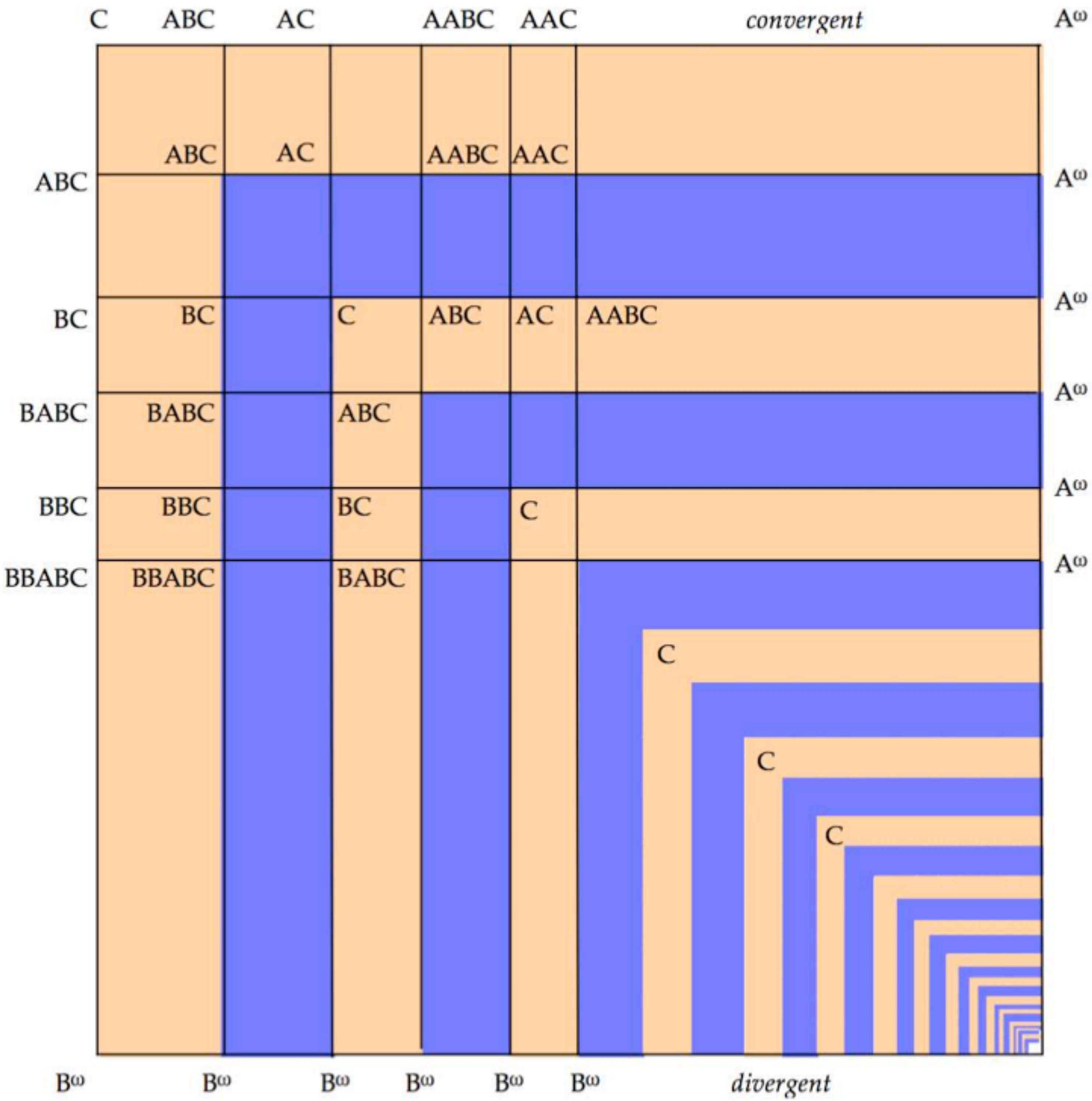


Failure of infinitary confluence for Combinatory Logic



K
K
S
K
K
K
S
K
K
K
S
K
K
K
S
K
K
K

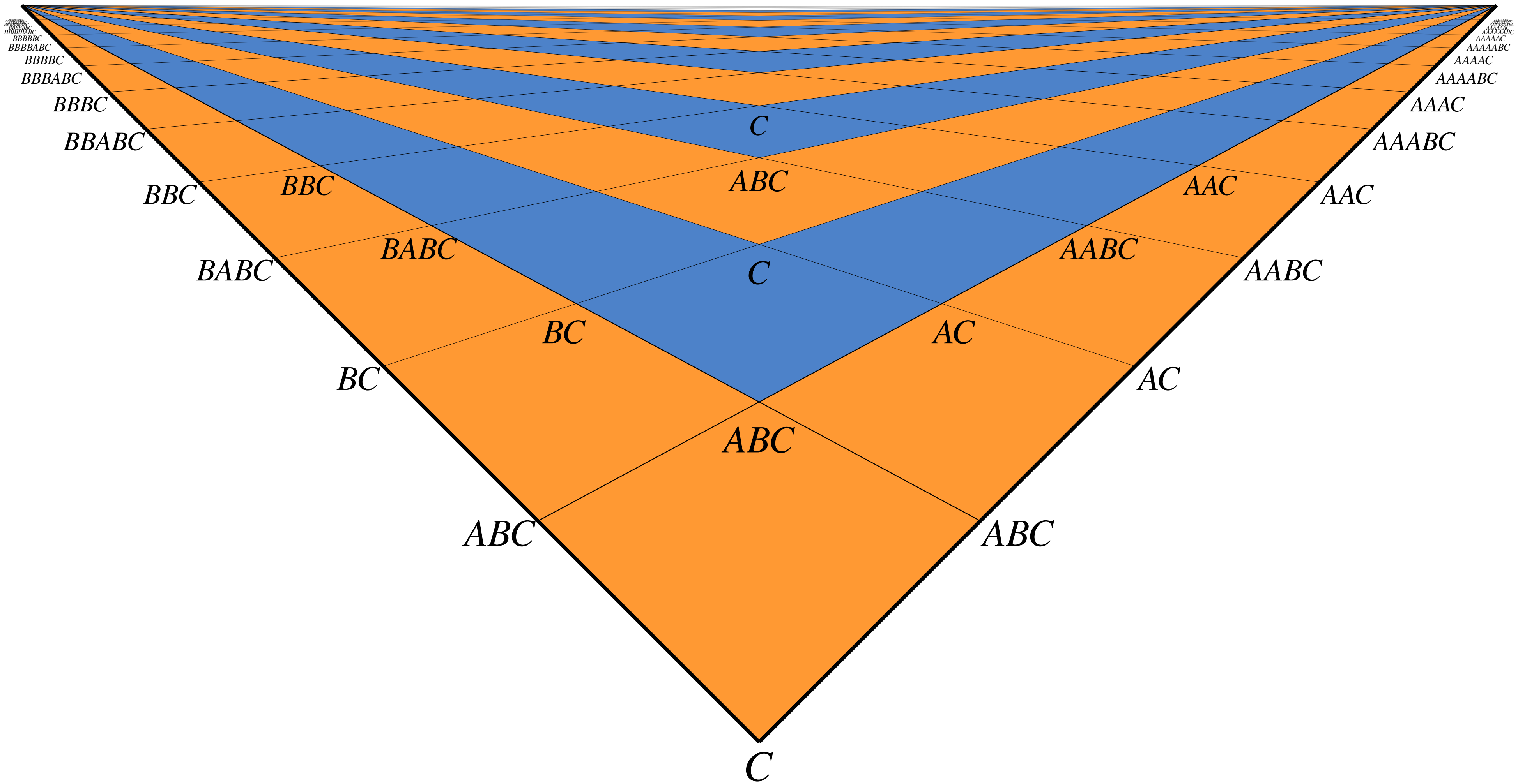
Failure of CR^{∞}



$A(x) \to x$

$B(x) \to x$

$C \to A(B(C))$

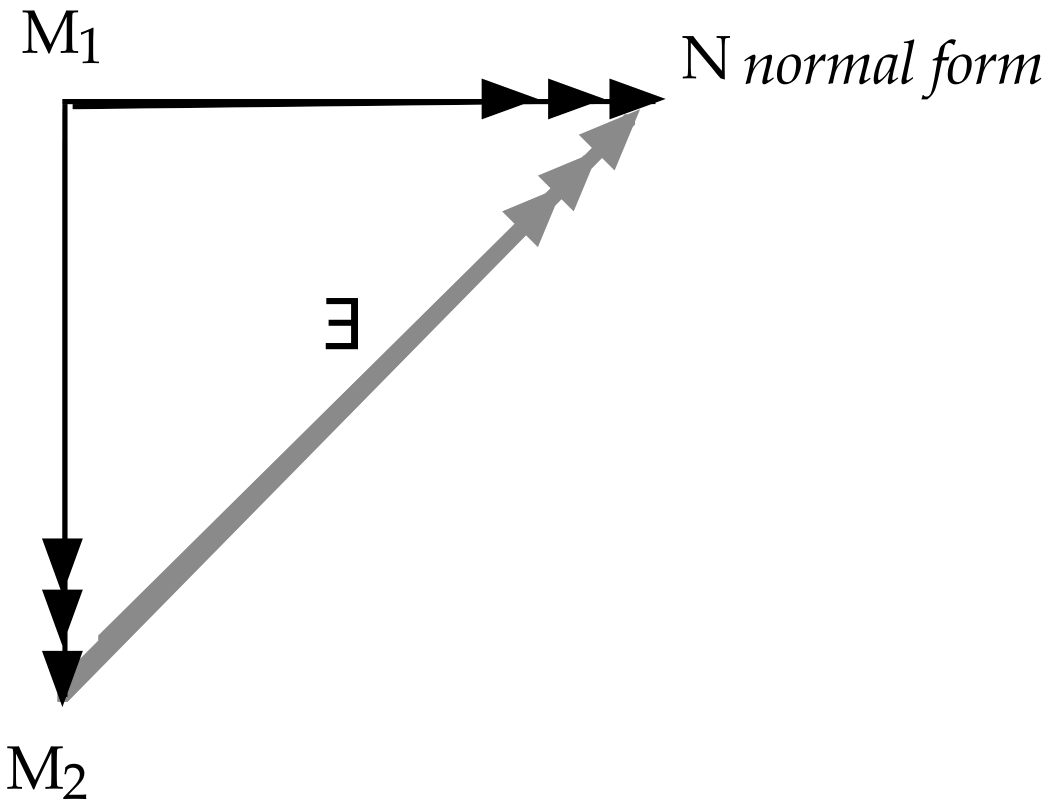


BBBBABC
BBBBC
BBBABC
BBBC
BBABC
BBC
BABC
BC
ABC
C
BBC
ABC
AAC
BABC
AABC
C
BC
AC
ABC
AAAAABC
AAAAC
AAAABC
AAAC
AAABC
AAC
AABC
AC
ABC
C

for OTRSs: UN^{∞}.

Corollary: Dershowitz et al:
for OTRSs SN^{∞} => CR^{∞}.

Proof: as for finite case
SN & UN => CR



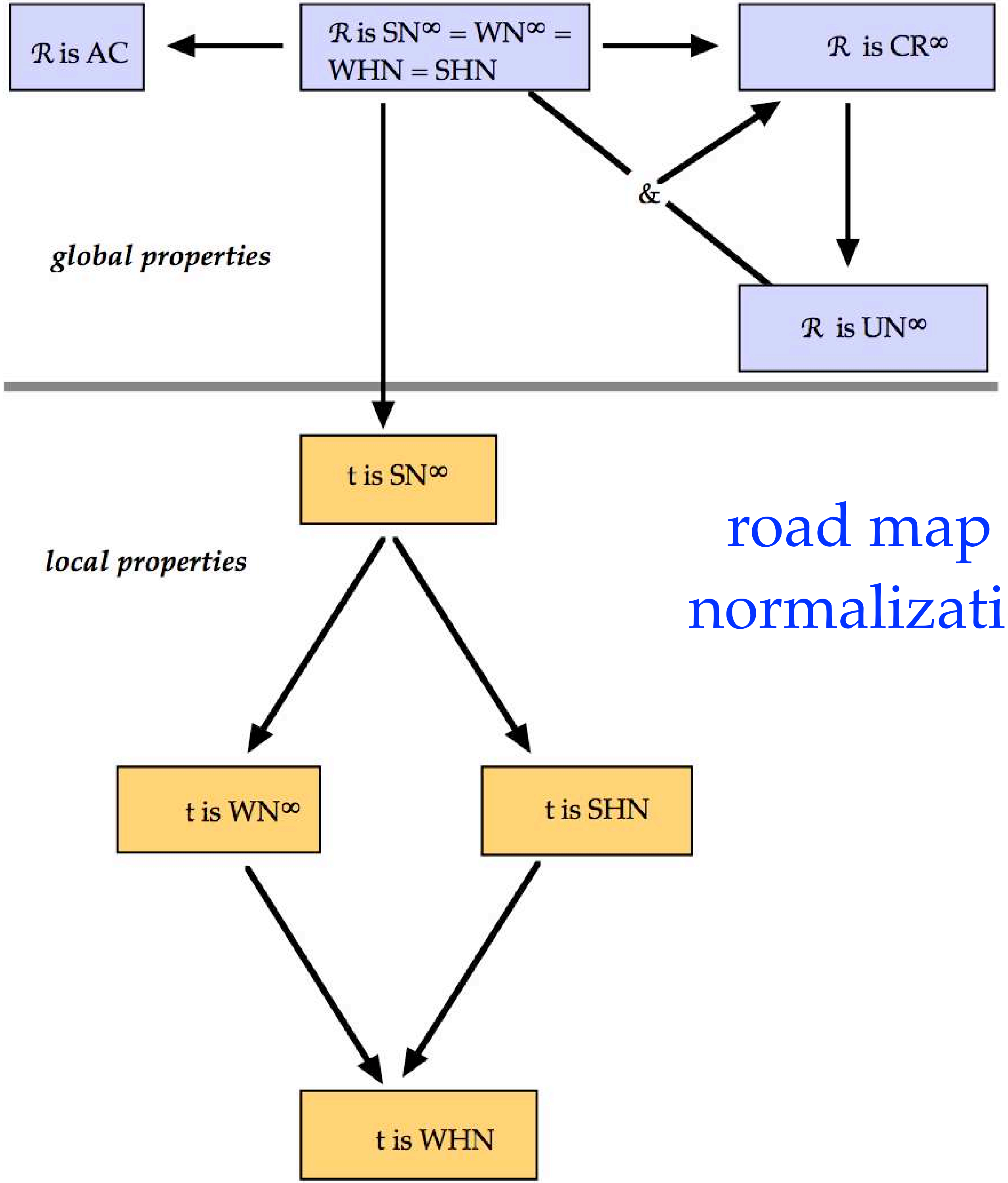
M1
N *normal form*
∃
M2

Confluence in infinitary rewriting

	PML	CR	UN	PML^{∞}	CR^{∞}	UN^{∞}
OTRS	*yes*	*yes*	*yes*	*yes*	*no*	*yes*
w.o. TRS	*yes*	*yes*	*yes*	*?*	*no*	*?*
$\lambda\beta$	*yes*	*yes*	*yes*	*no*	*no*	*yes*
OCRS	*yes*	*yes*	*yes*	*no*	*no*	*?*

by CR^{∞} for a quotient of $\lambda\beta^{\infty}$, e.g. mute terms, or hypercollapsing terms, and applying an abstract lemma of de Vrijer.

Let $(A, \to_1)$ and $(B, \to_2)$ be two ARSs with A included in B,
reduction $\to_1$ included in $\to_2$, normal forms nf(A) included in nf(B).
Then CR for B implies UN for A.



road map of infinitary normalization properties

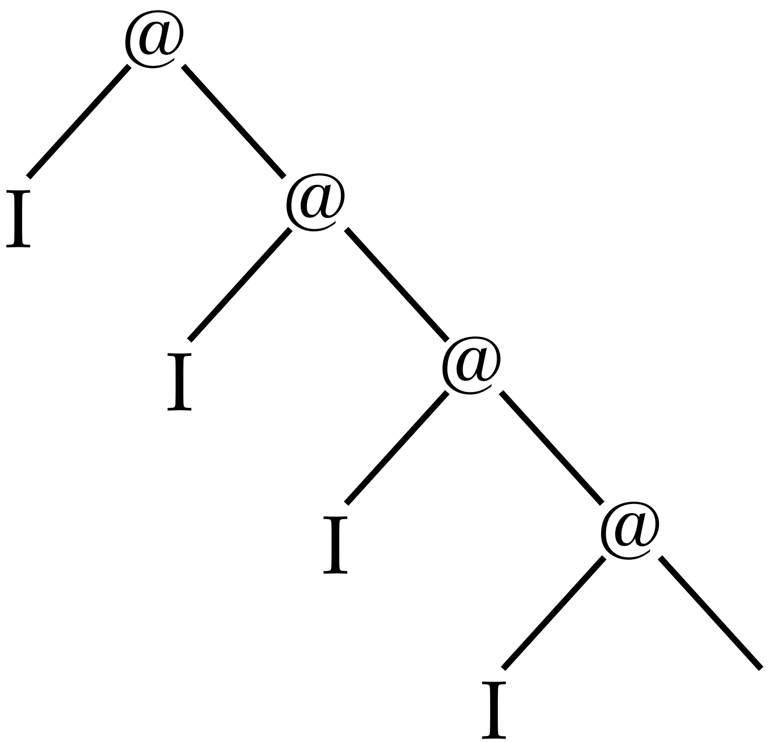
λ^∞ :not PML$^\infty$

$\omega_I \equiv (\lambda x.I(xx))$

$\omega \equiv \lambda x.xx$

$YI \rightarrow \omega_I\ \omega_I$

$I^\omega \equiv$



For infinitary lambda calculus Parallel Moves Lemma PML$^\infty$ fails, hence also CR$^\infty$

Y_0: λf. (x.f(xx)(λx.f(xx))

Y_1:   (λab. b(aab)) (λab. b(aab))

Y_0(SI) Y_1

Exercise. Prove that $Y_0 \neq_\beta Y_1$

infinitary lambda calculus subsumes scott's induction rule

$$Yx \twoheadrightarrow x(Yx) \twoheadrightarrow x^2\,(Yx) \to^{\omega} x^{\omega} \equiv x(x(x(x\ldots$$

$$BY \equiv (\lambda abc.a(bc))\,Y \qquad =_{\infty} \qquad BYS \equiv (\lambda abc.a(bc))\,YS$$

$$\neq_{\beta}$$

$$BY \to \lambda bc.Y(bc) \to^{\omega} \lambda bc.\,(bc)^{\omega}$$

$$BYS \to \lambda c.Y(Sc) \to \lambda c.\,Sc(Y(Sc)) \to \lambda cz.\,cz(Y(Sc)z) \to \lambda cz.\,cz(cz(Y(Sc)z)) \to^{\omega} \lambda cz.\,(cz)^{\omega}$$

$$\lambda bc.\,(bc)^{\omega} \equiv \lambda cz.\,(cz)^{\omega}$$

A simple proof

BY $\neq_\beta$? BYS

BYI $BYSI$

$BYI \equiv (\lambda abc.a(bc))\,YI$ $BYSI \equiv (\lambda abc.a(bc))YSI$

$\downarrow$ $\downarrow$

$\lambda c.Y(Ic)$

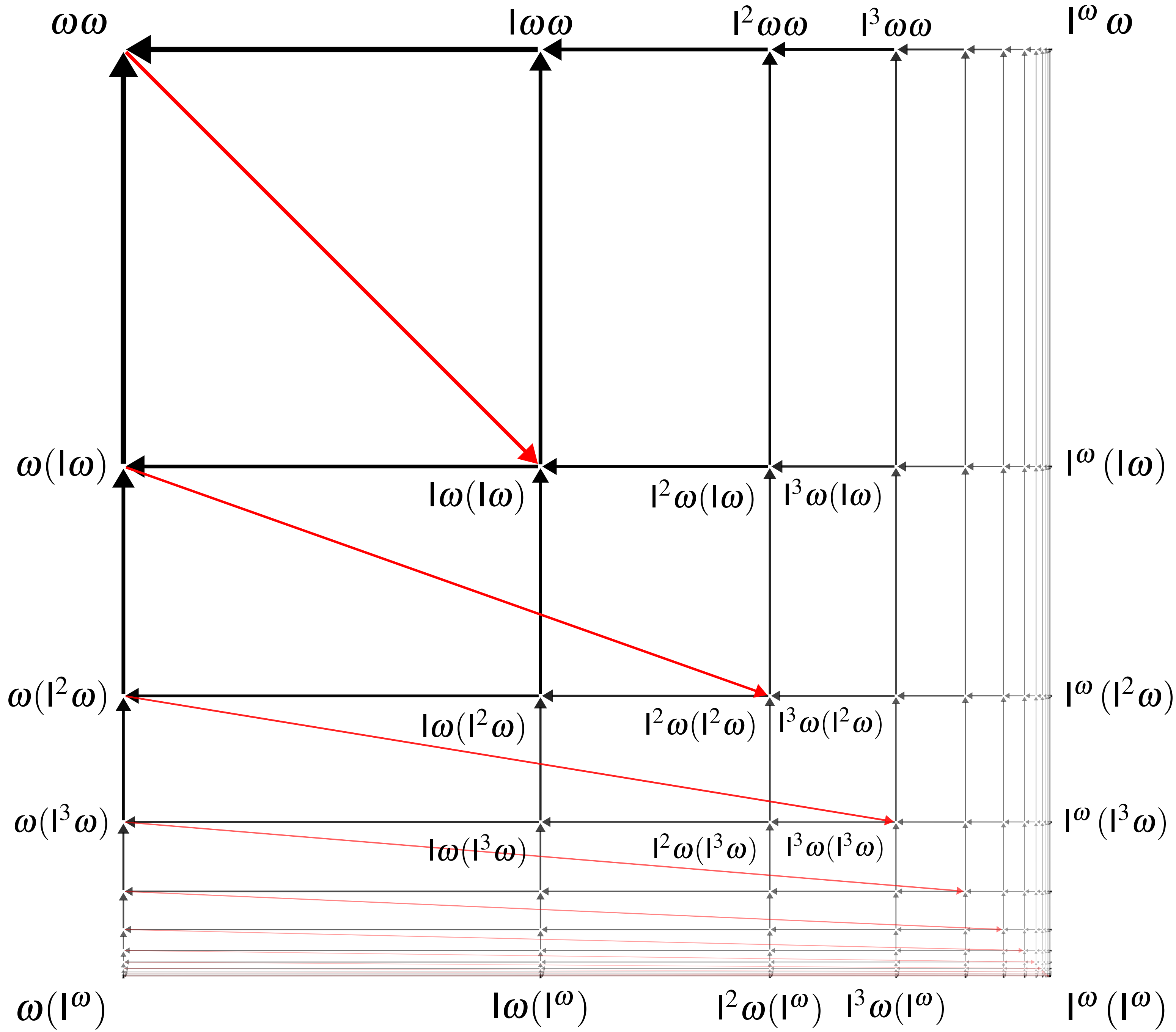
$\downarrow$

$\lambda c.Yc$

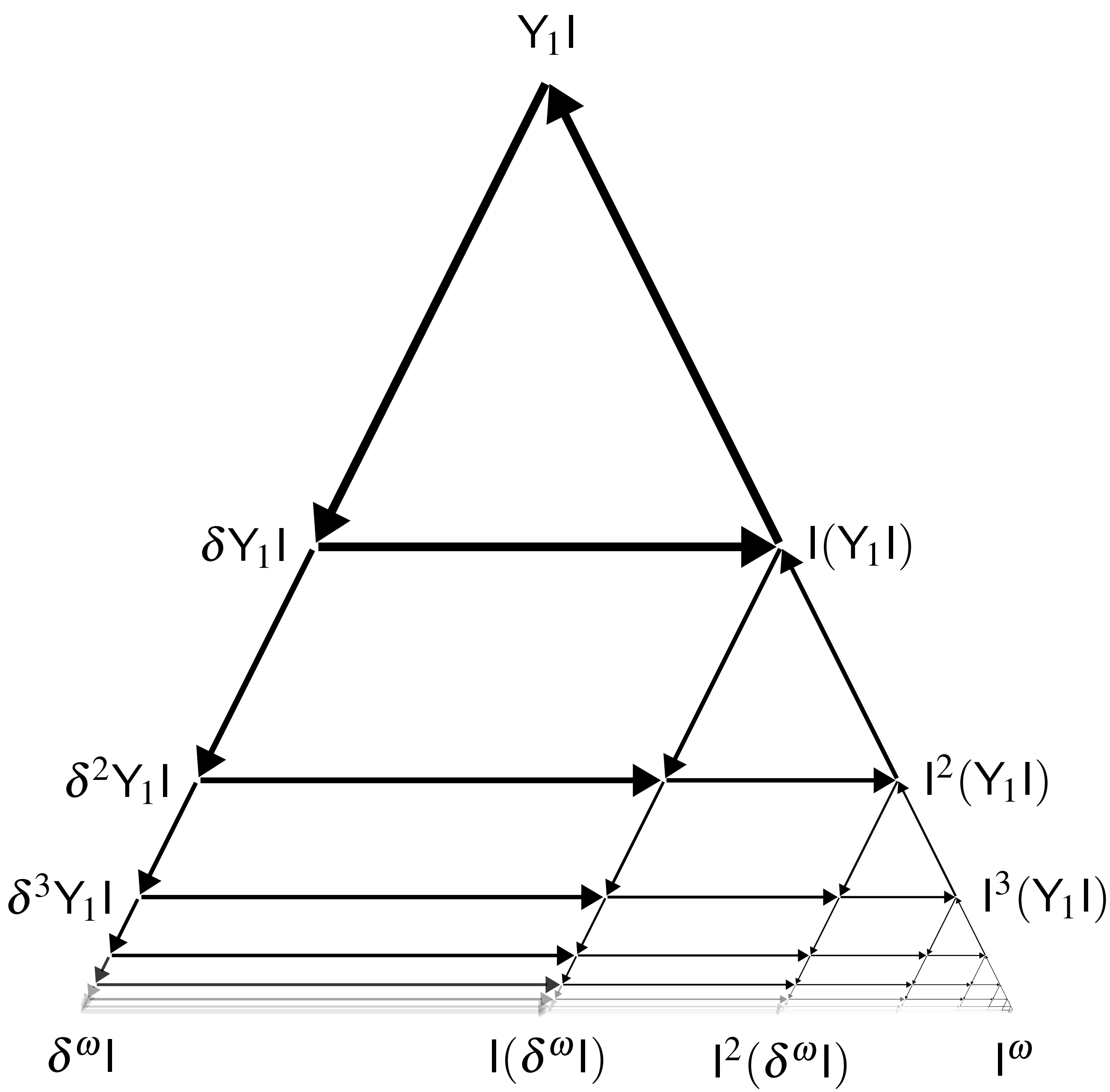
$\downarrow$

Y $\neq_\beta$! $Y(SI)$

Curry's fpc *Turing's fpc*



ωω
Iωω
I²ωω
I³ωω
I^ω ω
ω(Iω)
Iω(Iω)
I²ω(Iω)
I³ω(Iω)
I^ω (Iω)
ω(I²ω)
Iω(I²ω)
I²ω(I²ω)
I³ω(I²ω)
I^ω (I²ω)
ω(I³ω)
Iω(I³ω)
I²ω(I³ω)
I³ω(I³ω)
I^ω (I³ω)
ω(I^ω)
Iω(I^ω)
I²ω(I^ω)
I³ω(I^ω)
I^ω (I^ω)



$Y_1 I$
$\delta Y_1 I$
$I(Y_1 I)$
$\delta^2 Y_1 I$
$I^2(Y_1 I)$
$\delta^3 Y_1 I$
$I^3(Y_1 I)$
$\delta^\omega I$
$I(\delta^\omega I)$
$I^2(\delta^\omega I)$
I^ω

0. A few words on history

1. rewriting dictionary

2. two theorems in abstract rewriting

3. word rewriting: monoids and braids

tea, coffee

4. term rewriting: divide et impera; termination by stars

5. Lambda calculus and combinatory logic

6. Infinitary rewriting

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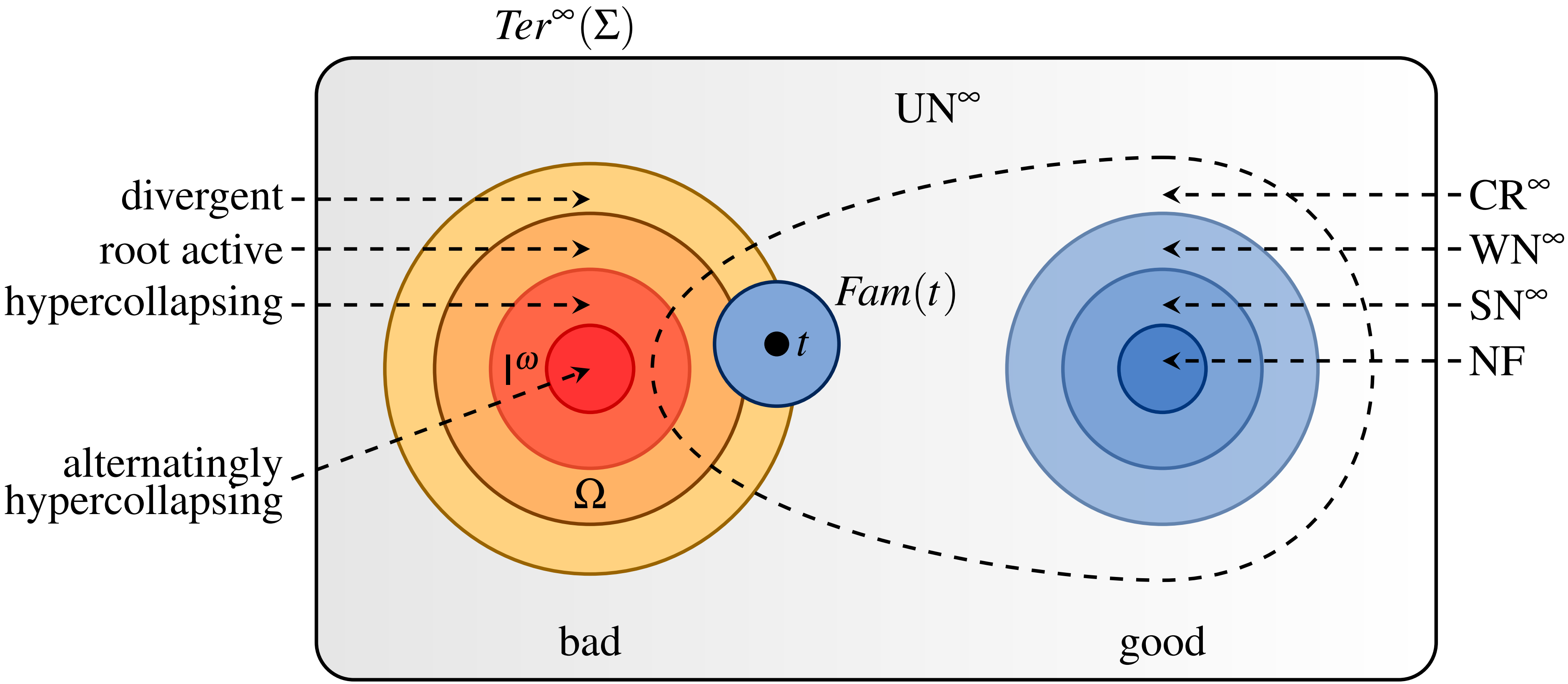
tea, coffee

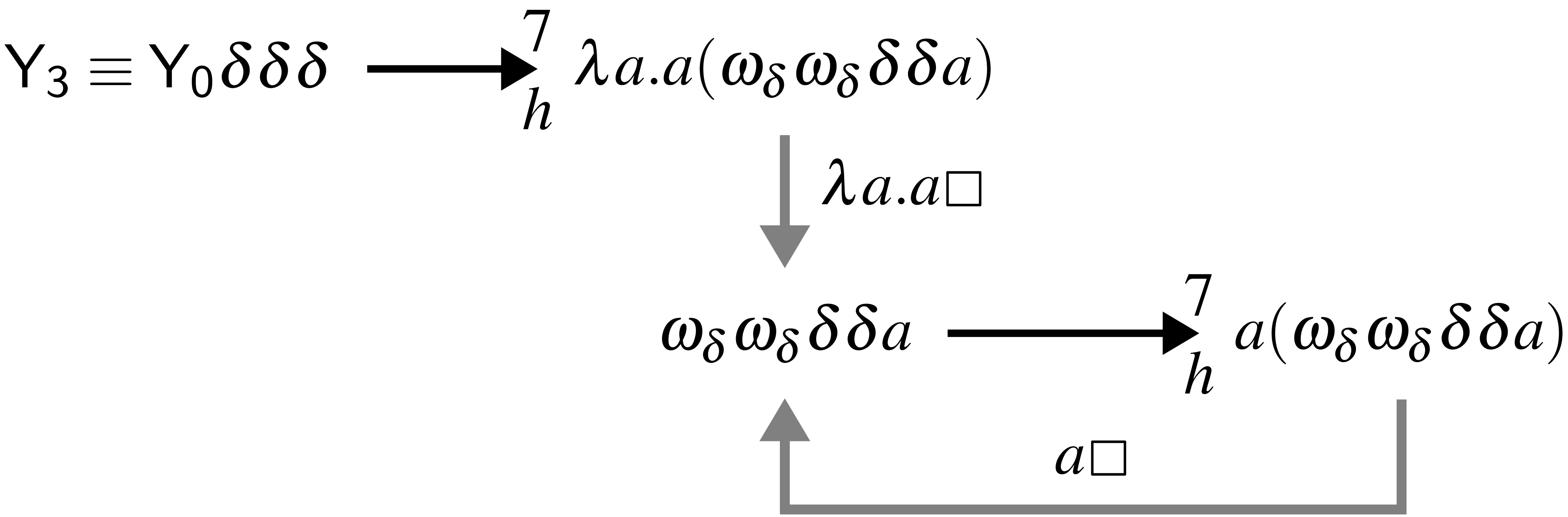
7.infinitary lambda calculus and the threefold path

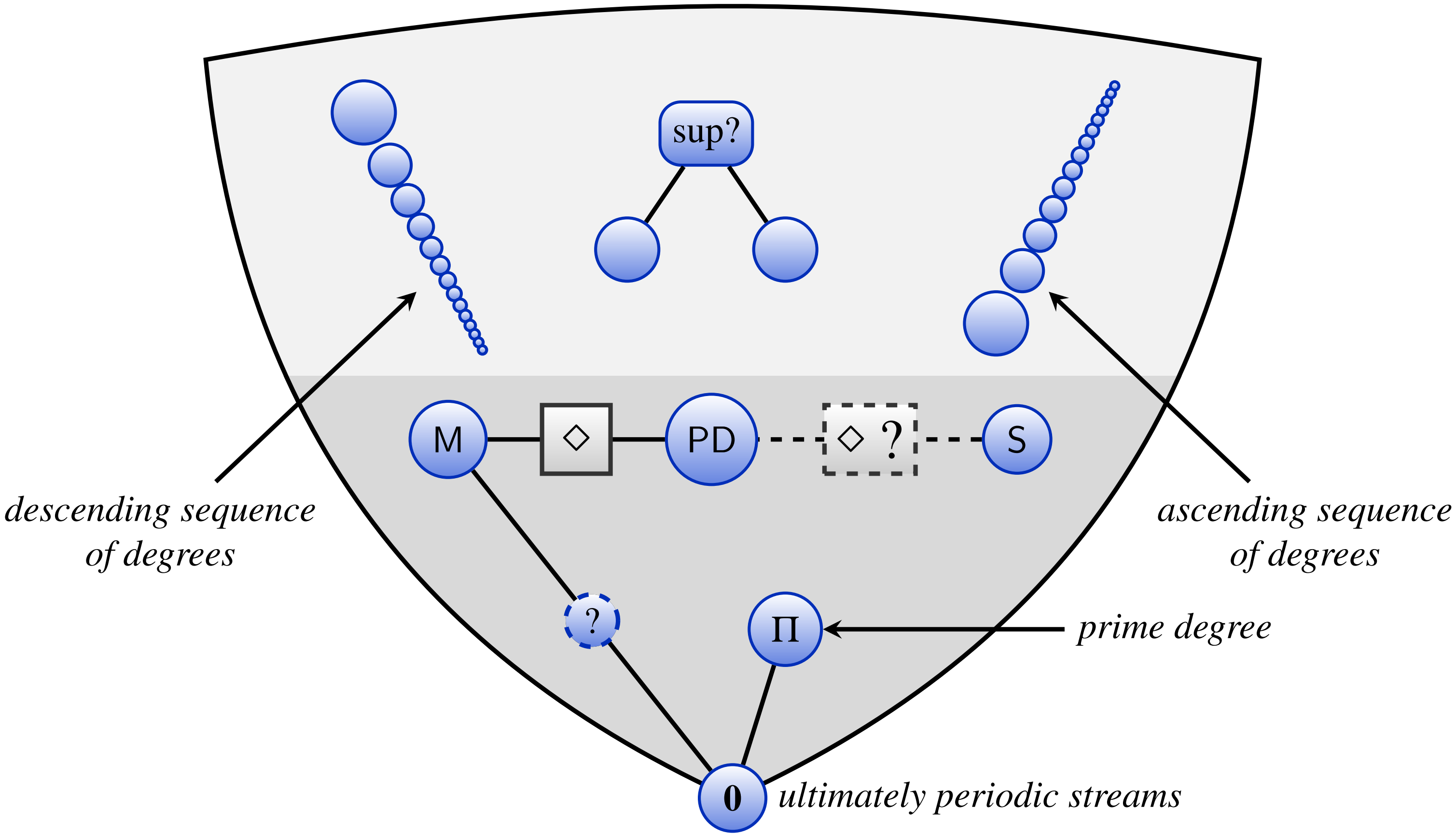
8.clocked semantics of lambda calculus

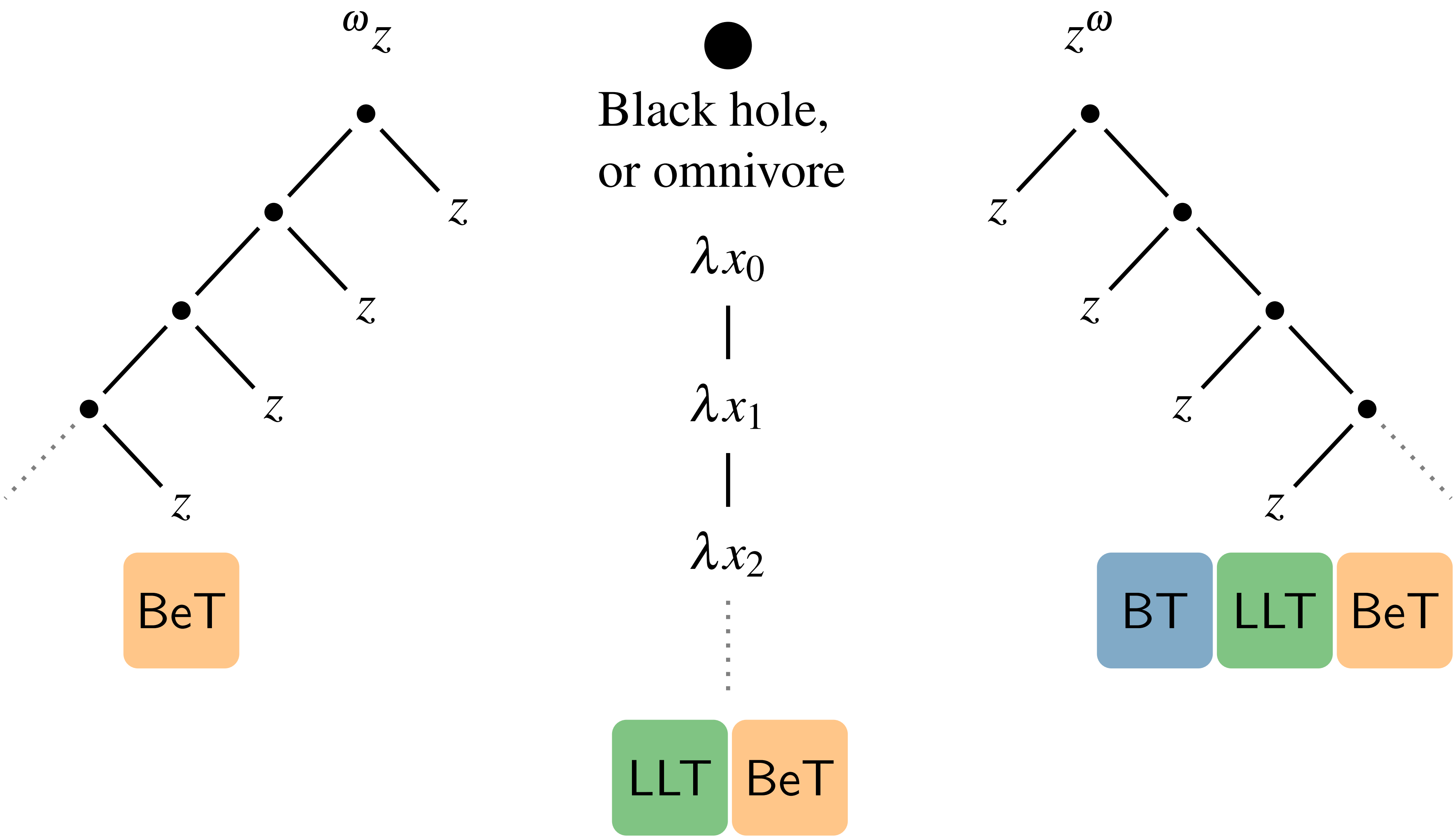
9.streams running forever



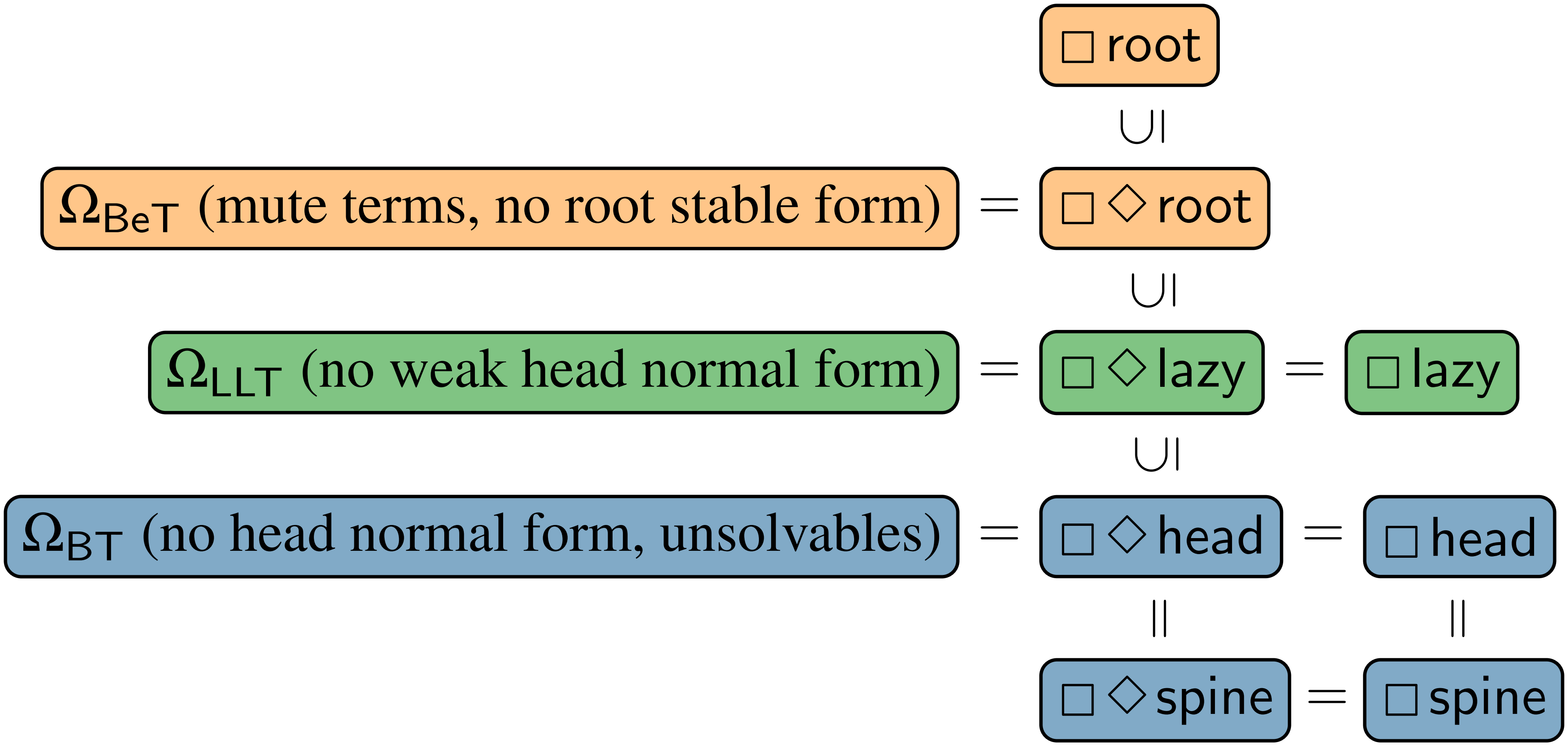
$Ter^{\infty}(\Sigma)$
UN^{∞}
divergent
root active
hypercollapsing
alternatingly hypercollapsing
I^{ω}
Ω
t
$Fam(t)$
CR^{∞}
WN^{∞}
SN^{∞}
NF
bad
good


Y3 ≡ Y0δδδ
7
h
λa.a(ωδωδδδa)
λa.a□
ωδωδδδa
7
h
a(ωδωδδδa)
a□

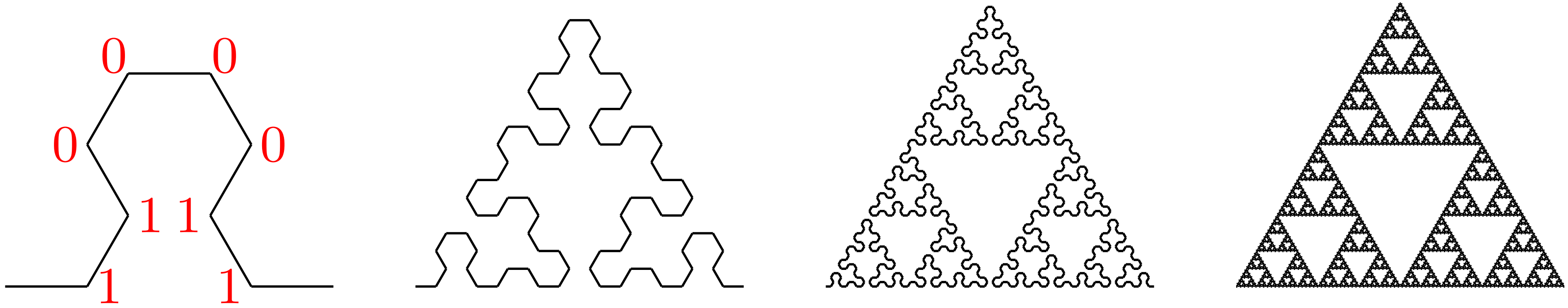

sup?
M
◇
PD
◇ ?
S
?
Π
0
descending sequence of degrees
ascending sequence of degrees
prime degree
ultimately periodic streams

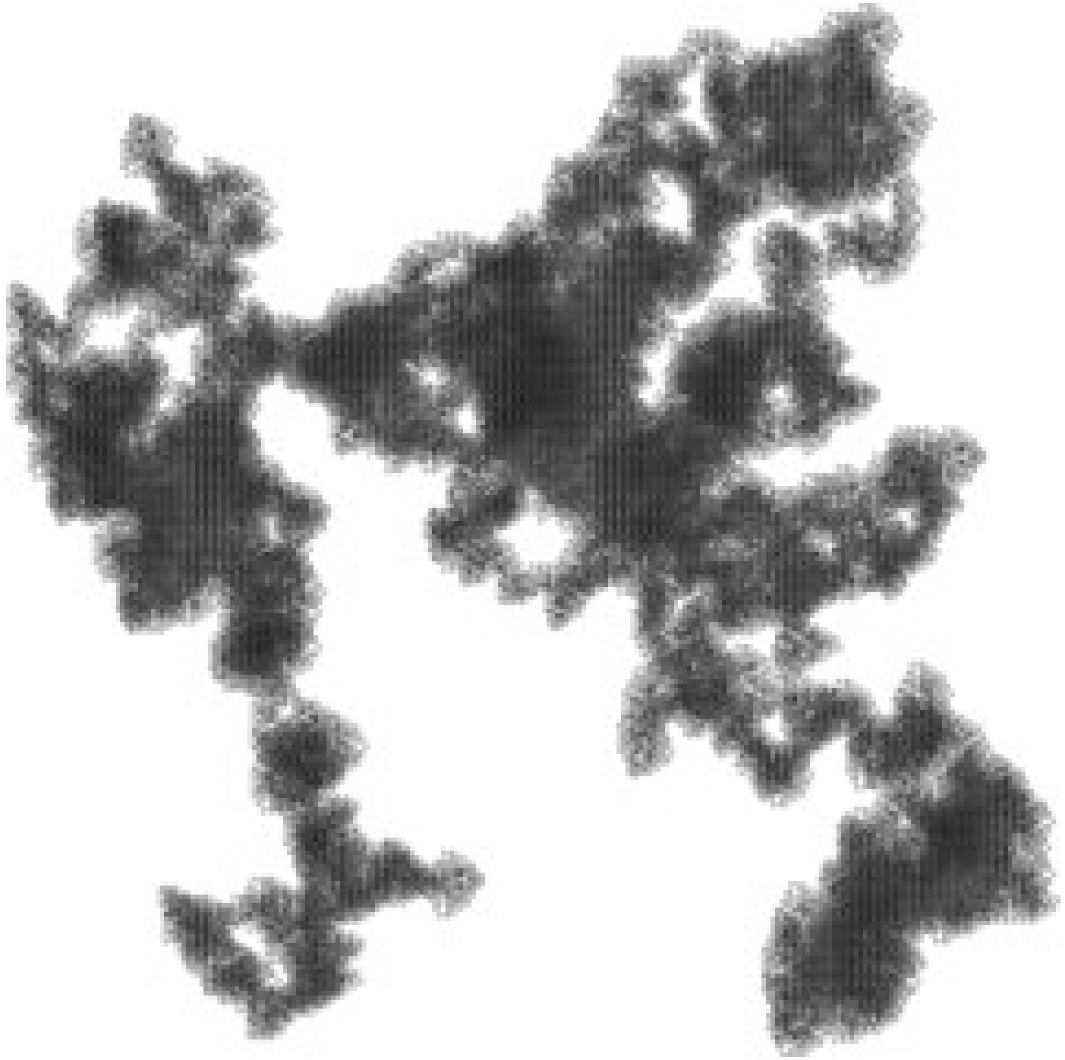


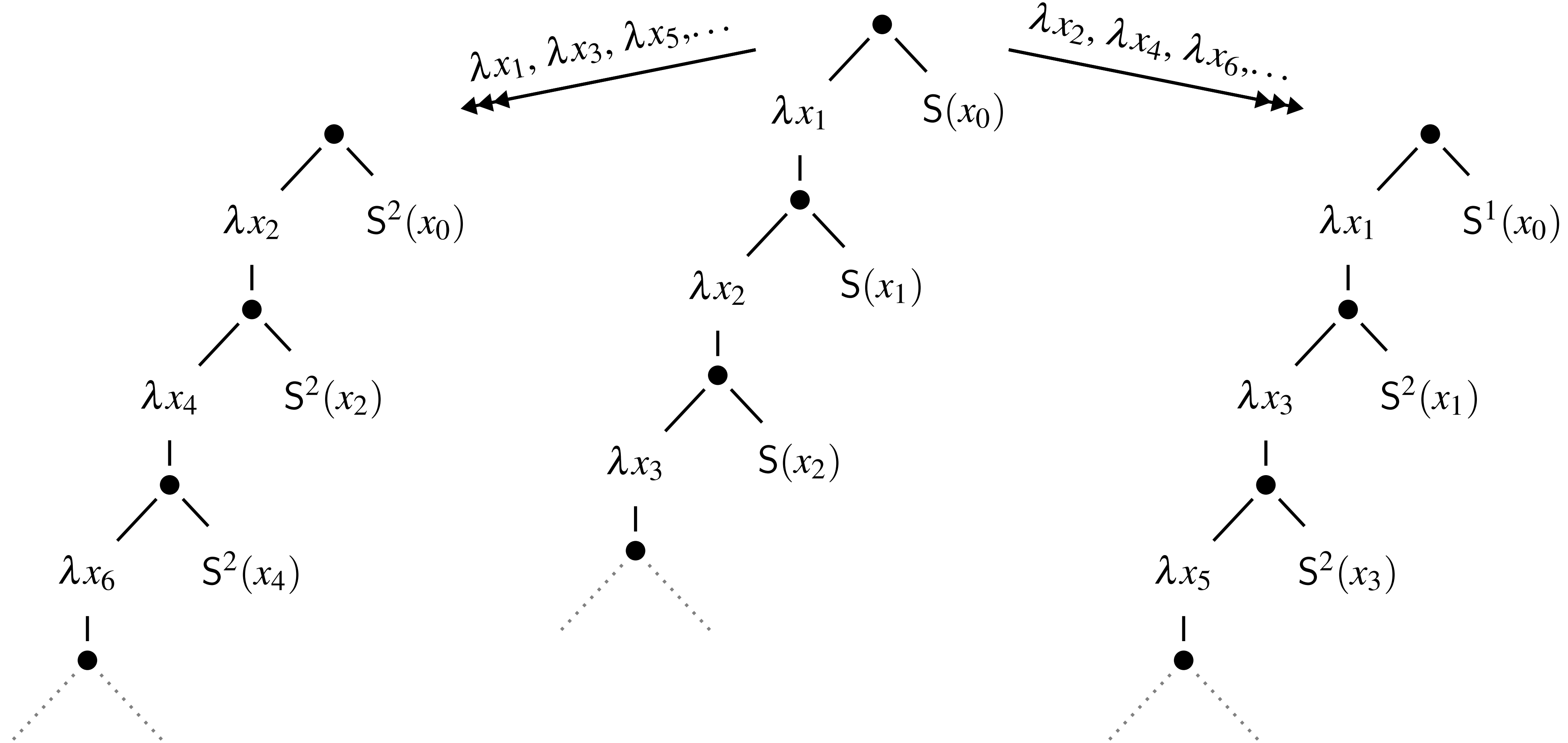
ωz
z
z
z
z
BeT
Black hole,
or omnivore
λx0
λx1
λx2
LLT
BeT
zω
z
z
z
z
BT
LLT
BeT



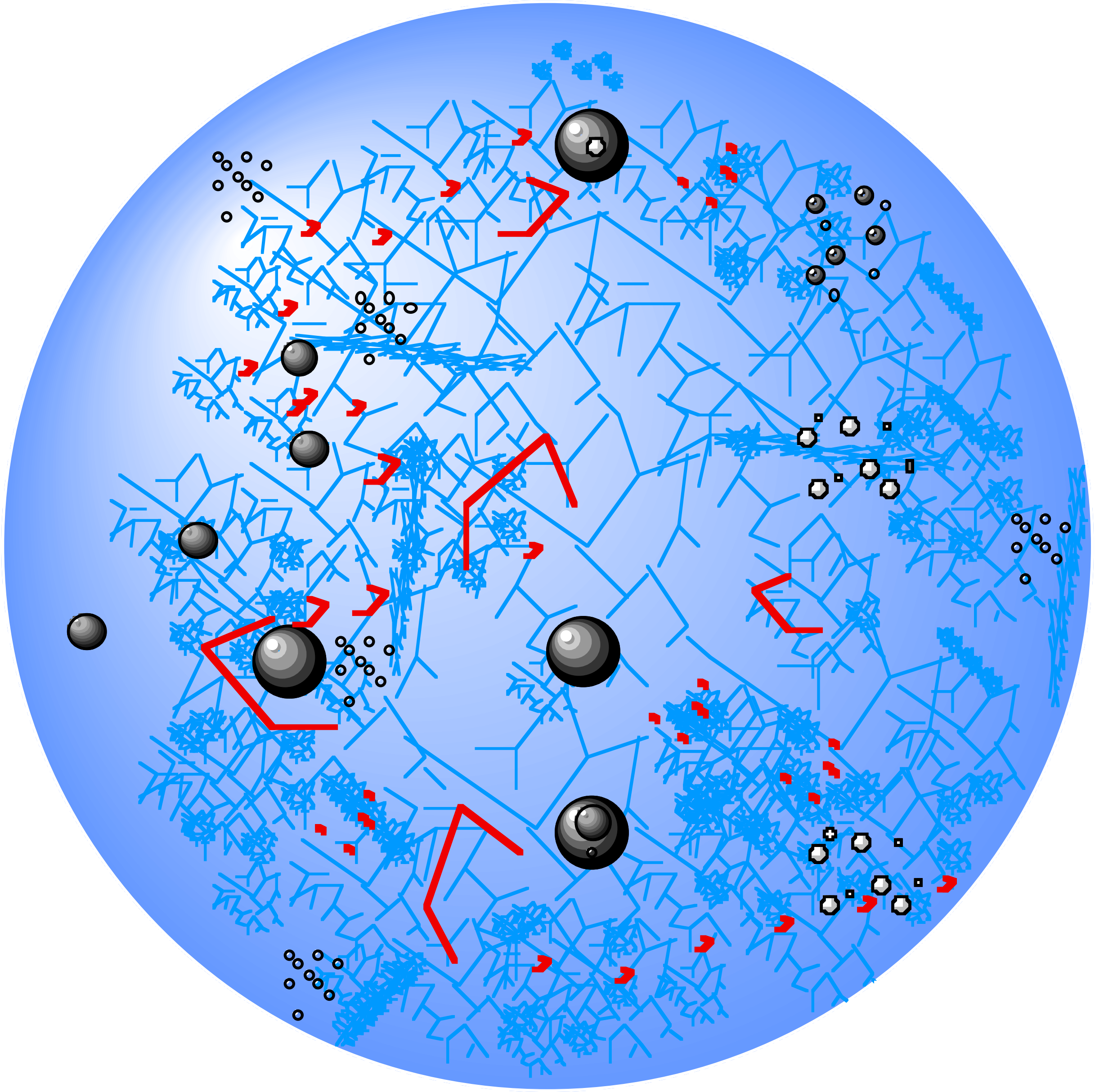
□ root
⊆
Ω_BeT (mute terms, no root stable form)
=
□◇ root
⊆
Ω_LLT (no weak head normal form)
=
□◇ lazy
=
□ lazy
⊆
Ω_BT (no head normal form, unsolvables)
=
□◇ head
=
□ head
||
||
□◇ spine
=
□ spine

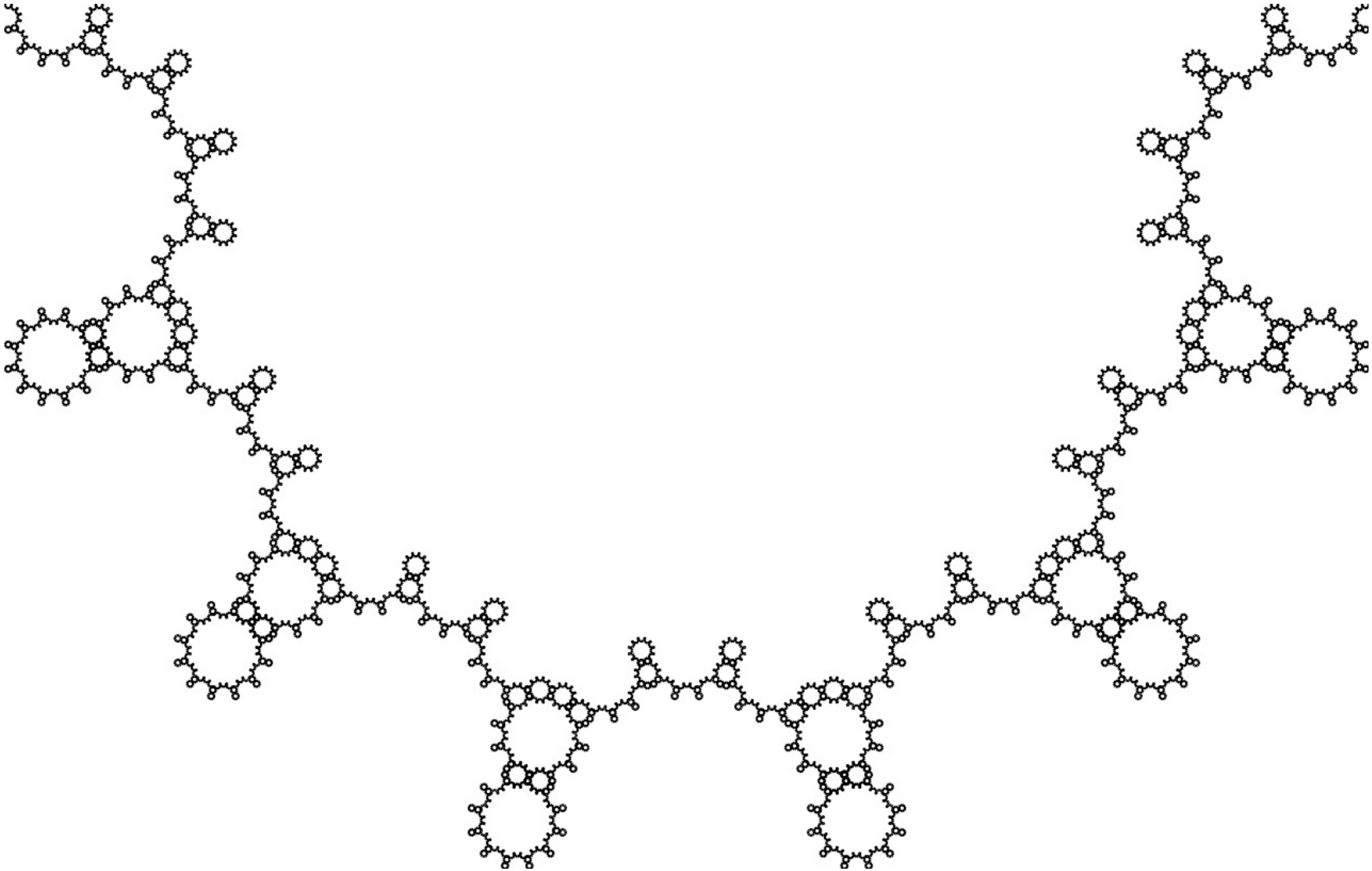

0
0
0
0
1
1
1
1

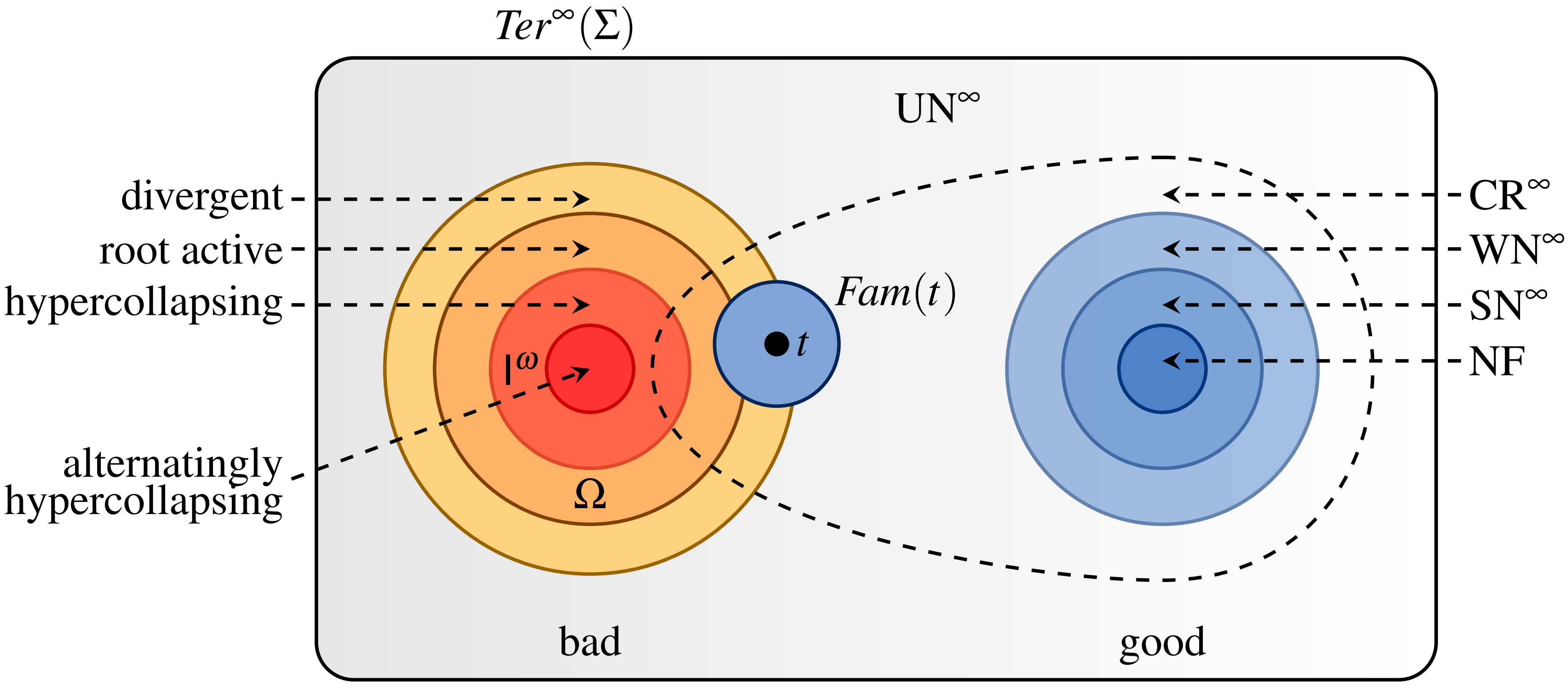




λx1, λx3, λx5,...
λx2, λx4, λx6,...
λx1
S(x0)
λx2
S²(x0)
S(x1)
λx1
S¹(x0)
λx4
S²(x2)
λx3
S(x2)
λx3
S²(x1)
λx6
S²(x4)
λx5
S²(x3)



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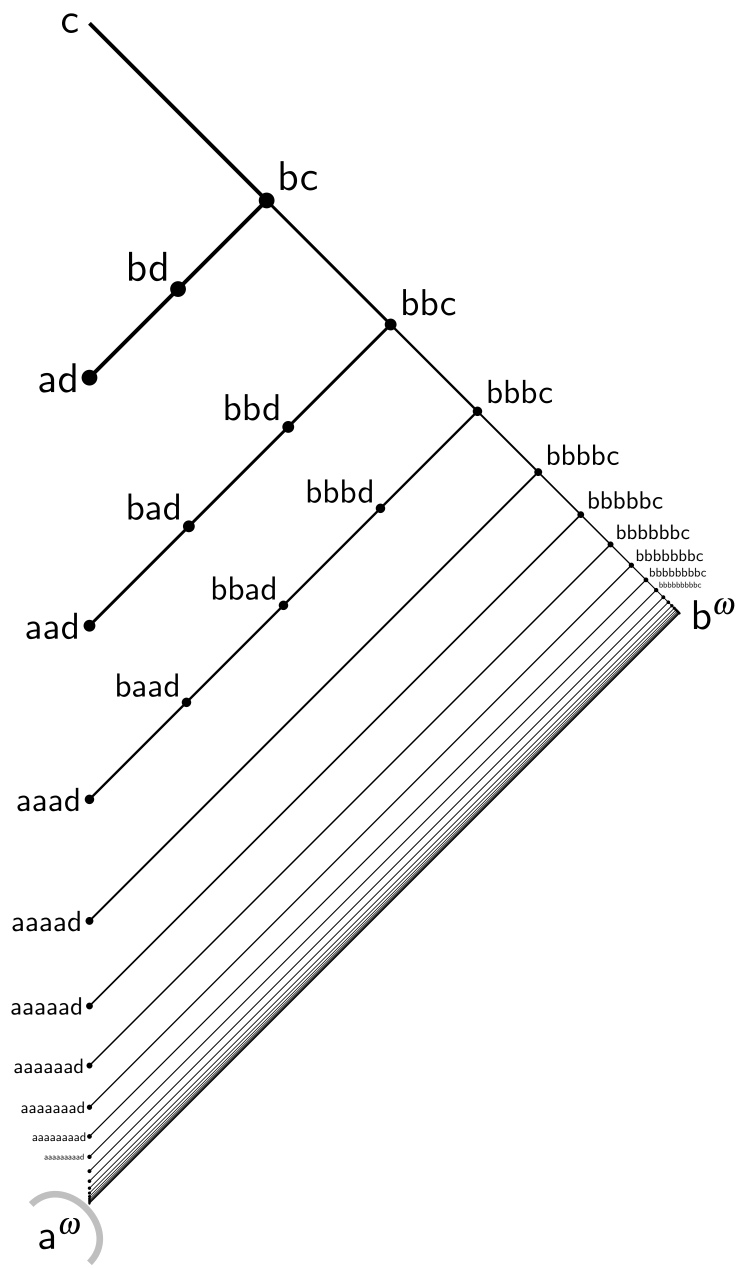

$$\mathsf{c} \to \mathsf{b}(\mathsf{c})$$
$$\mathsf{b}(\mathsf{c}) \to \mathsf{a}(\mathsf{d})$$
$$\mathsf{b}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{a}(x))$$